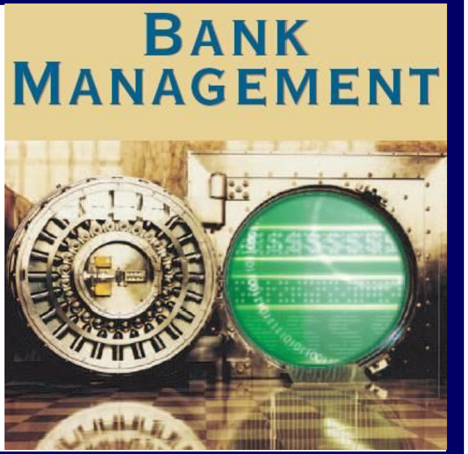
Pricing Fixed-Income Securities

Lecture 4



The Mathematics of Interest Rates

- **Future Value & Present Value: Single Payment**
  - **Terms**
    - **Present Value = PV**
      - **The value today of a single future cash flow.**
    - **Future Value = FV**
      - **The amount to which a single cash flow or series of cash flows will grow over a given period of time when compounded at a given interest rate.**

The Mathematics of Interest Rates

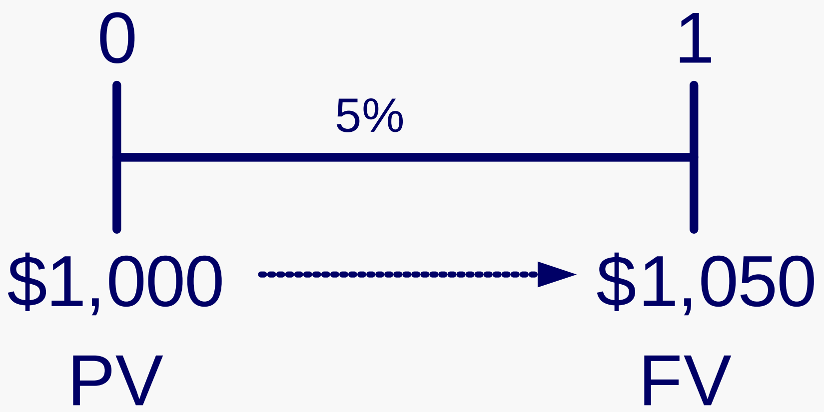
- Future Value & Present Value: Single Payment
  - Terms
    - Interest Rate Per Year = i
    - Number of Periods = n

The Mathematics of Interest Rates

- Future Value: Single Payment
  - Suppose you invest $1,000 for one year at 5% per year. What is the future value in one year?
    - Interest = $1,000(.05) = $50
    - Value in 1 year = Principal + Interest
      = $1,000 + $50
      = $1,050

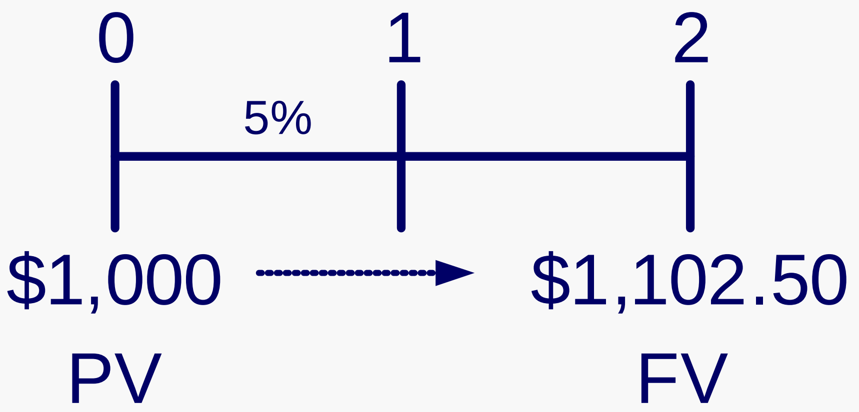
The Mathematics of Interest Rates

- **Future Value: Single Payment**
  - **$FV = PV(1 + i)^n$**
  - **$FV = \$1,000(1+.05)^1 = \$1,050$**



The Mathematics of Interest Rates

- **Future Value: Single Payment**
  - **Suppose you leave the money in for another year. How much will you have two years from now?**
    - **FV = $\$1000(1.05)(1.05)$**
    - **FV = $\$1000(1.05)^2 = \$1,102.50$**

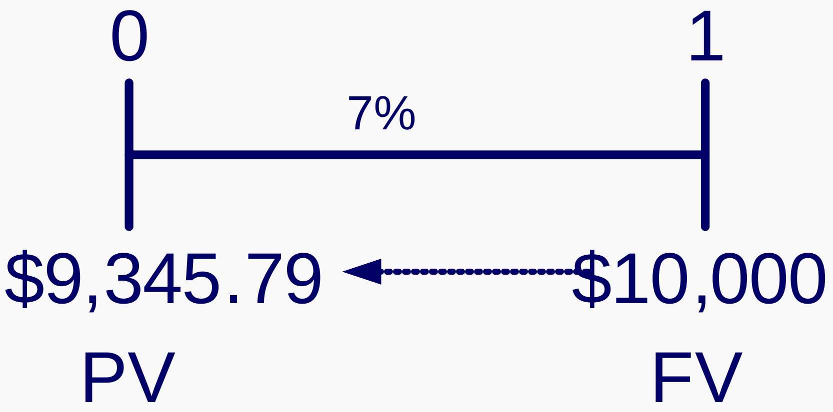


The Mathematics of Interest Rates

- **Present Value**
  - **How much do I have to invest today to have some amount in the future?**
    - **$FV = PV(1 + i)^n$**
    - **$PV = FV/(1 + i)^n$**

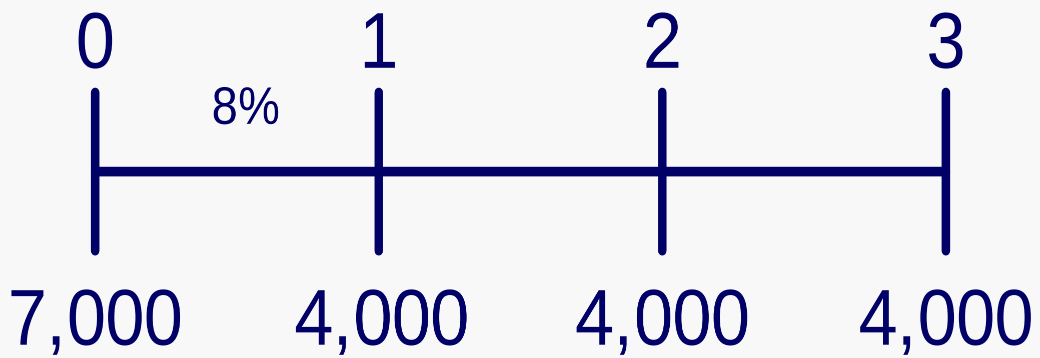
The Mathematics of Interest Rates

- **Present Value**
  - **Suppose you need $10,000 in one year. If you can earn 7% annually, how much do you need to invest today?**
    - **Present Value = $10,000/(1.07)^1$ = 9,345.79**



The Mathematics of Interest Rates

- Future Value: Multiple Payments
  - What is the Future Value of the cash flow stream at the end of year 3?

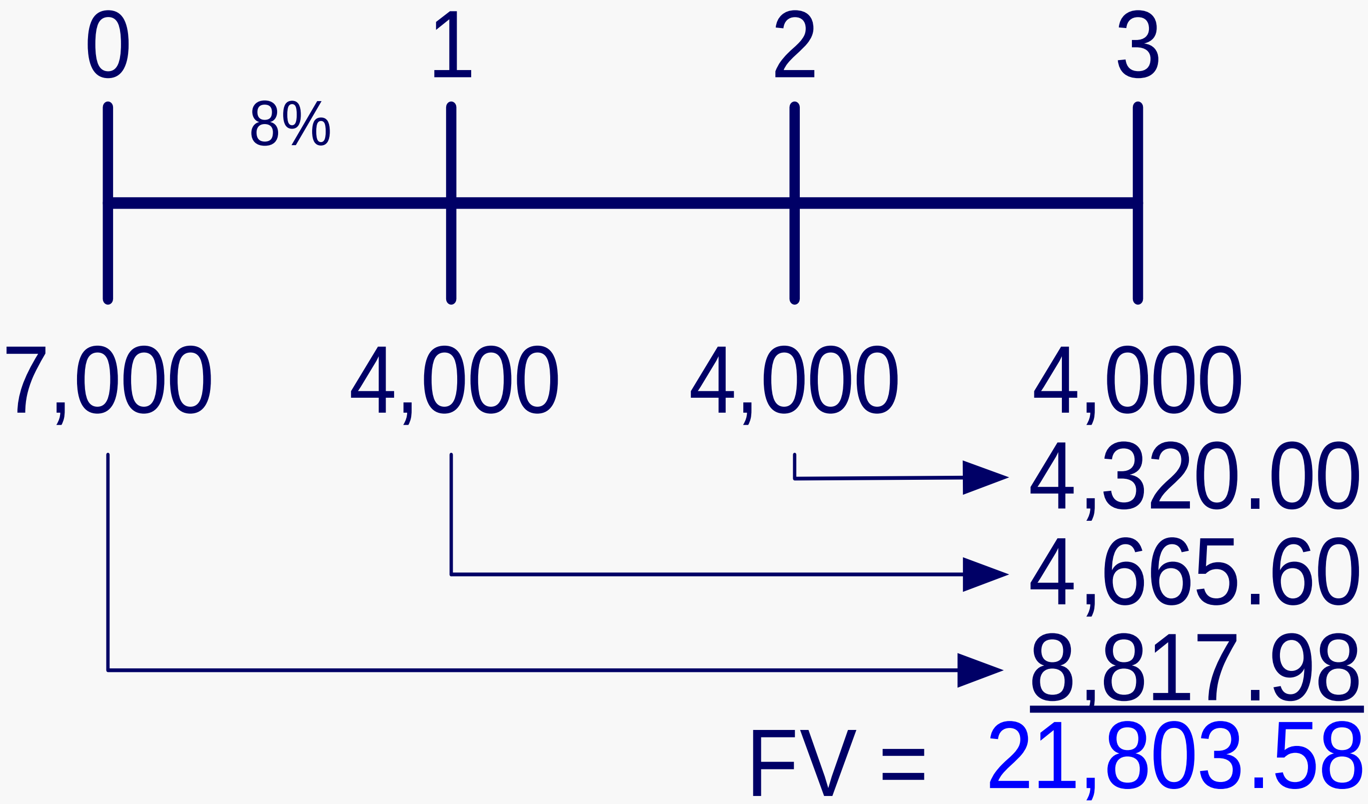


The Mathematics of Interest Rates

- **Future Value: Multiple Payments**
  - **Find the value at the end of Year 3 of each cash flow and add them together.**
    - **CF0**
      - **FV = 7,000$(1.08)^3$ = 8,817.98**
    - **CF1**
      - **FV = 4,000$(1.08)^2$ = 4,665.60**
    - **CF2**
      - **FV = 4,000(1.08) = 4,320**
    - **CF3**
      - **FV = 4,000**
    - **Total value in 3 years**
      - **8,817.98 + 4,665.60 + 4,320 + 4,000 = 21,803.58**

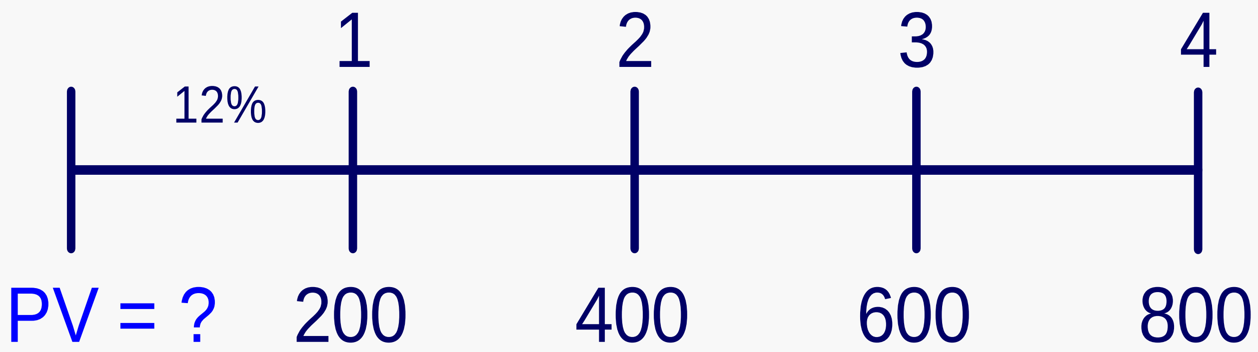
The Mathematics of Interest Rates

- **Future Value: Multiple Payments**



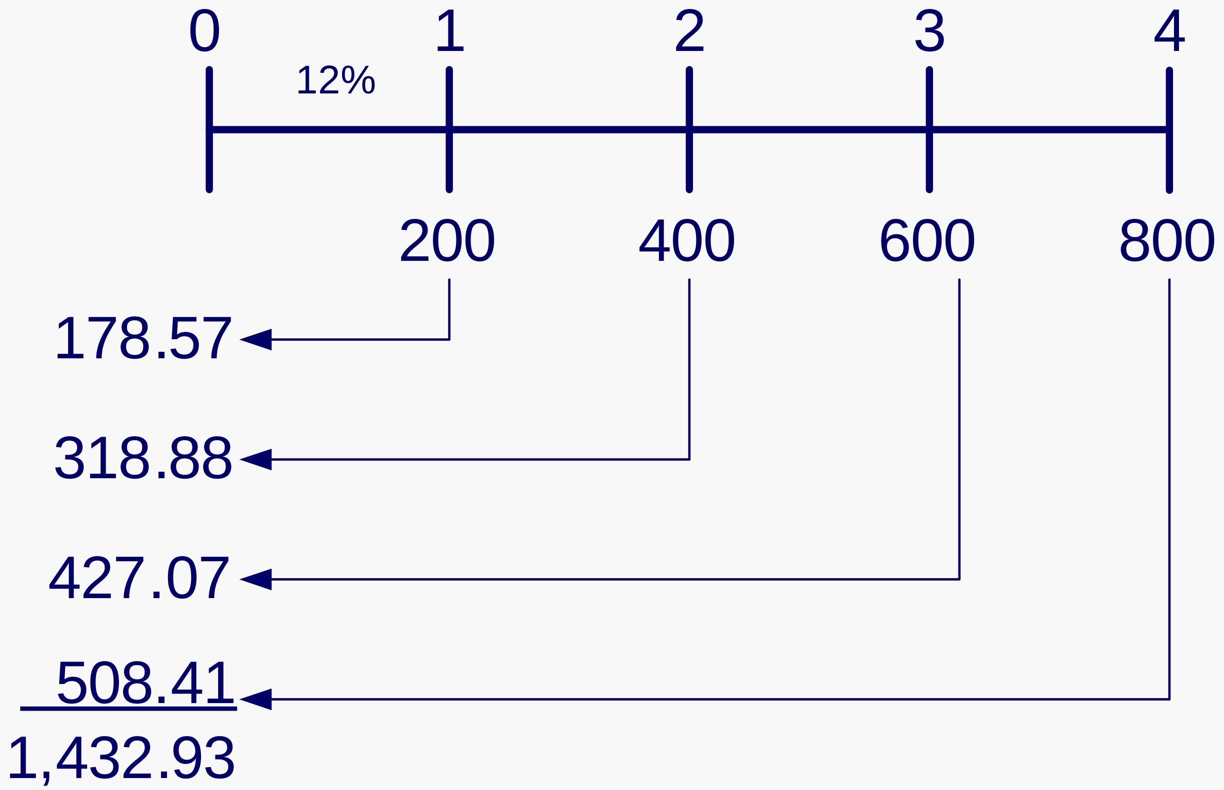
The Mathematics of Interest Rates

- **Present Value: Multiple Payments**
  - **What is the Present Value of the cash flow stream?**



The Mathematics of Interest Rates

- **Present Value: Multiple Payments**



The Mathematics of Interest Rates

- Simple versus Compound Interest
  - Compound Interest
    - Interest on Interest
  - Simple Interest
    - No Interest on Interest

The Mathematics of Interest Rates

- Simple versus Compound Interest
  - $1,000 deposited today at 5% for 2 years.
    - FV with Simple Interest
      - $1,000 + $50 + $50 = $1,100
    - FV with Compound Interest
      - $\$1000(1.05)^2 = \$1,102.50$
    - The extra $2.50 comes from the extra interest earned on the first $50 interest payment.
      - 5%* $50 = $2.50.

The Mathematics of Interest Rates

- **Compounding Frequency**
  - **i = Nominal Interest Rate**
  - **i* = Effective Annual Interest Rate**
  - **m = Number of Compounding Periods in a year**

$$i^* = \left[1 + \frac{i}{m}\right]^m - 1$$

The Mathematics of Interest Rates

- Compounding Frequency
  - Suppose you can earn 1% per month on $100 invested today.
    - How much are you effectively earning?
      - $i^* = (1 + .12/12)^{12} - 1$
      - $i^* = (1.01)^{12} - 1 = .1268 = 12.68\%$

The Effect of Compounding on Future Value and Present Value

A. What is the future value after 1 year of $1,000 invested at an 8% annual nominal rate?

Compounding Interval	Number of Compounding Intervals in 1 Year (m)	Future Value (FV1)*	Effective Interest Rate*
Year	1	$1,080.00	8.00%
Semiannual	2	1,081.60	8.16
Quarter	4	1,082.43	8.24
Month	12	1,083.00	8.30
Day	365	1,083.28	8.33
Continuous	†	1,083.29	8.33

B. What is the present value of $1,000 received at the end of 1 year with compounding at 8%?

Compounding Interval	Number of Compounding Intervals in 1 Year (m)	Present Value (PV)*	Effective Interest Rate*
Year	1	$925.93	8.00%
Semiannual	2	924.56	8.16
Quarter	4	923.85	8.24
Month	12	923.36	8.30
Day	365	923.12	8.33
Continuous	†	923.12	8.33

[†]Continuous compounding is based on Euler's e such that $\lim_{m=\infty}\left(1+\frac{1}{m}\right)^{in} = e^{i}$, where e = 2.71828. Thus, $FVn = PVe^{in}$, and $PV = \frac{FVn}{e^{in}}$.

The Relationship Between Interest Rates and Option-Free Bond Prices

- Bond Prices
  - A bond's price is the present value of the future coupon payments (CPN) plus the present value of the face (par) value (FV)

$$\text{Price} = \frac{CPN_1}{(1+r)^1} + \frac{CPN_2}{(1+r)^2} + \frac{CPN_3}{(1+r)^3} + \ldots + \frac{CPN_n + FV}{(1+r)^n}$$

$$\text{Price} = \sum_{t=1}^{n} \frac{CPN_t}{(1+i)^t} + \frac{FV}{(1+i)^n}$$

The Relationship Between Interest Rates and Option-Free Bond Prices

- **Bond Prices and Interest Rates are Inversely Related**
  - **Consider a bond which pays semi-annual interest payments of $470 with a maturity of 3 years.**
    - **If the market rate of interest is 9.4%, the price of the bond is:**

$$\text{Price} = \sum_{t=1}^{6} \frac{470}{(1.047)^t} + \frac{10{,}000}{(1.047)^6} = \$10{,}000$$

The Relationship Between Interest Rates and Option-Free Bond Prices

- **Bond Prices and Interest Rates are Inversely Related**
  - **If the market rates of interest increases to 10%, the price of the bond falls to:**

$$\text{Price} = \sum_{t=1}^{6} \frac{470}{(1.05)^t} + \frac{10{,}000}{(1.05)^6} = \$9{,}847.72$$

The Relationship Between Interest Rates and Option-Free Bond Prices

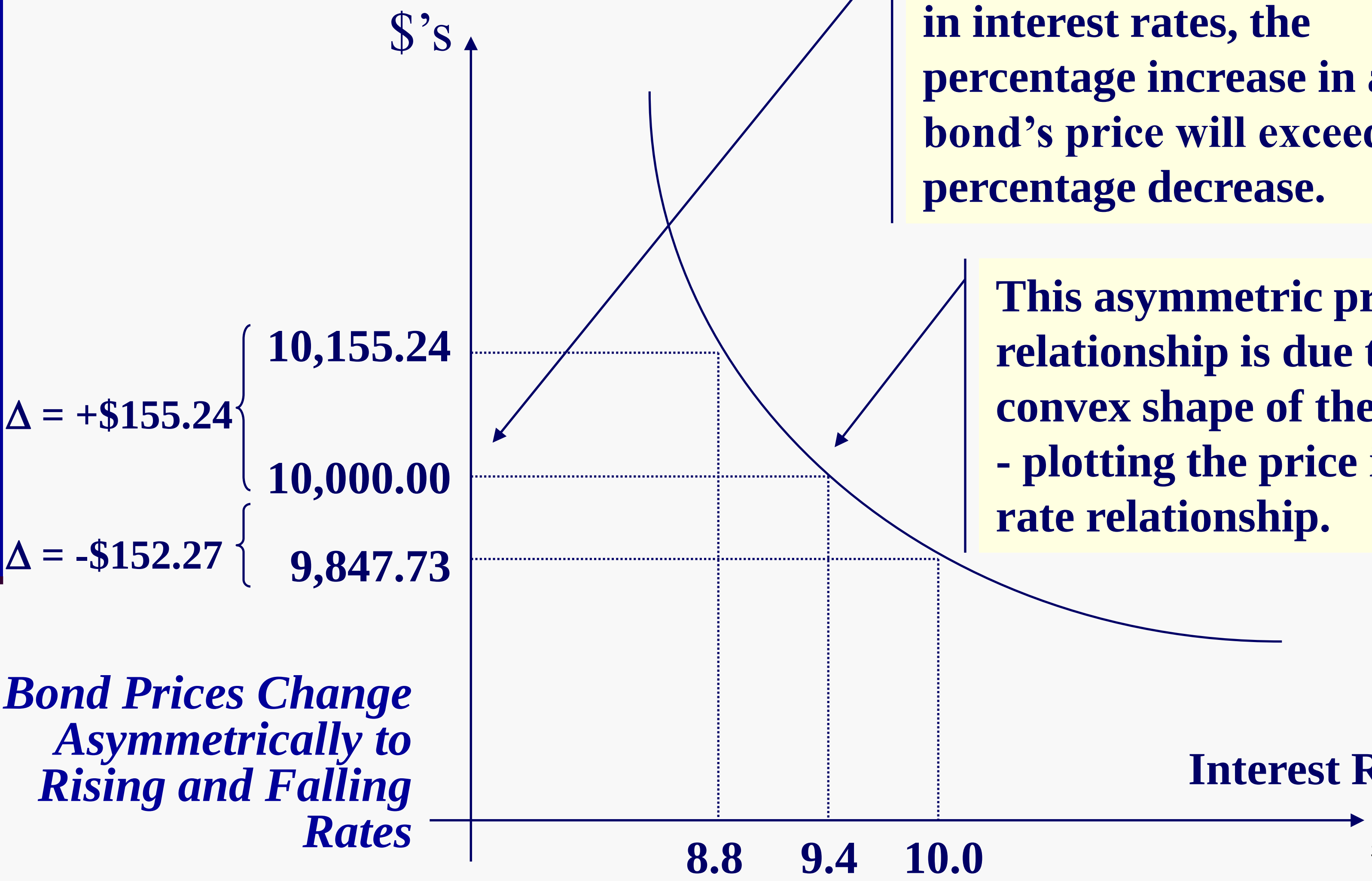
- **Bond Prices and Interest Rates are Inversely Related**
  - **If the market rates of interest decreases to 8.8%, the price of the bond rises to:**

$$\text{Price} = \sum_{t=1}^{6} \frac{470}{(1.044)^t} + \frac{10{,}000}{(1.044)^6} = \$10{,}155.24$$

The Relationship Between Interest Rates and Option-Free Bond Prices

- **Bond Prices and Interest Rates are Inversely Related**
  - **Par Bond**
    - **Yield to maturity = coupon rate**
  - **Discount Bond**
    - **Yield to maturity > coupon rate**
  - **Premium Bond**
    - **Yield to maturity < coupon rate**

Relationship between price and interest rate on a 3-year, $10,000 option-free par value bond that pays $270 in semiannual interest



For a given absolute change in interest rates, the percentage increase in a bond's price will exceed the percentage decrease.

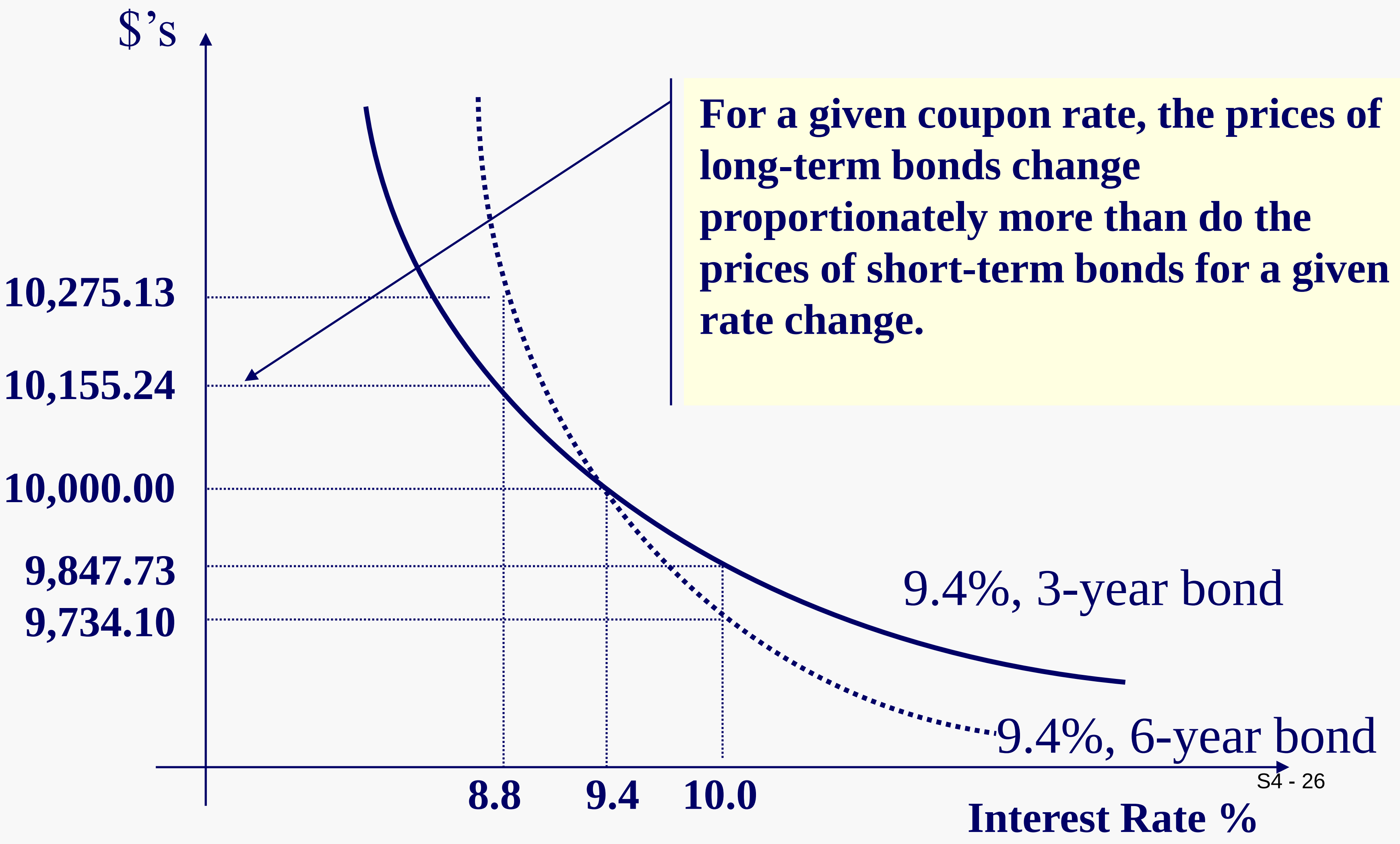
This asymmetric price relationship is due to the convex shape of the curve-- plotting the price interest rate relationship.

Bond Prices Change Asymmetrically to Rising and Falling Rates

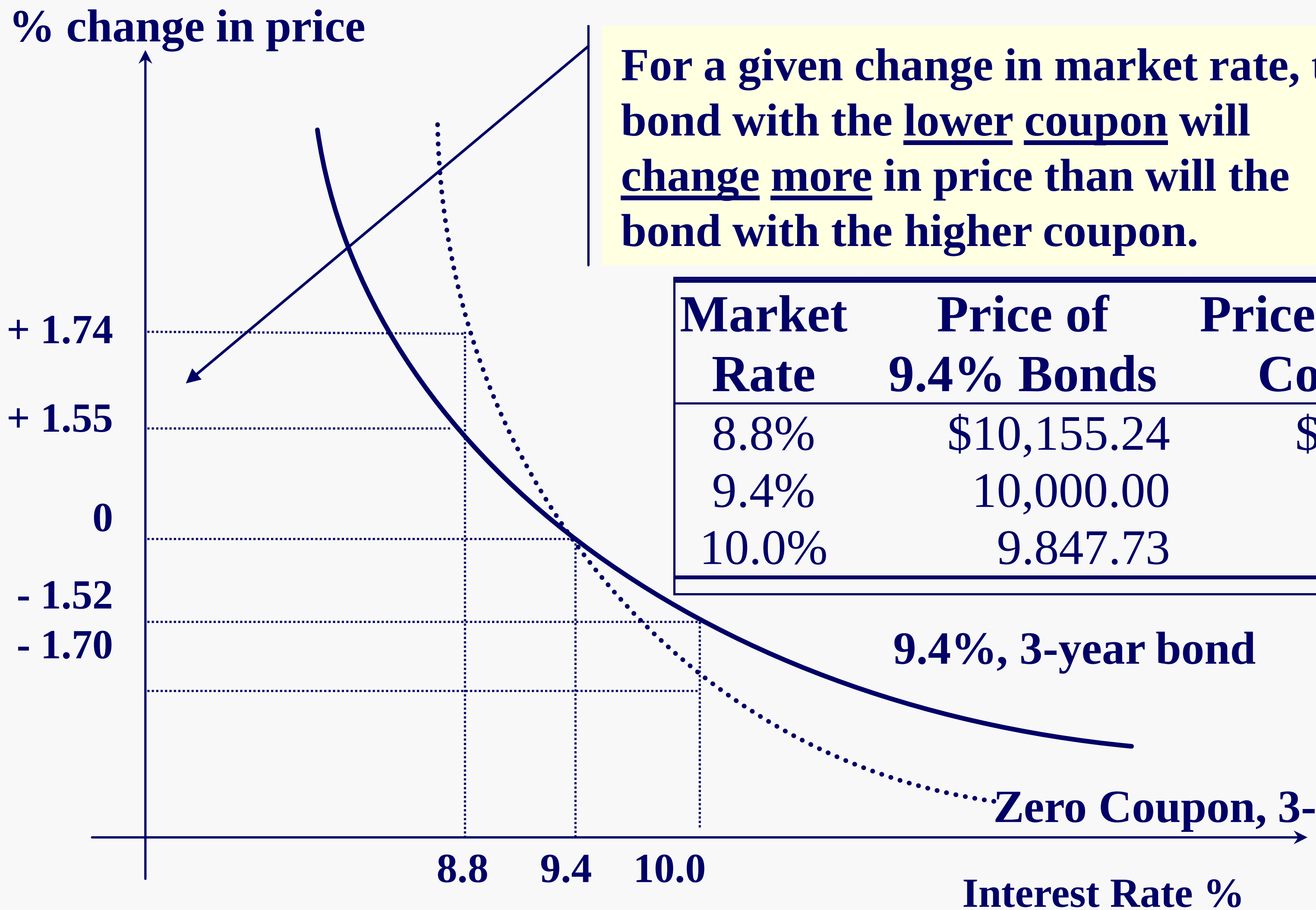
The Relationship Between Interest Rates and Option-Free Bond Prices

- Maturity Influences Bond Price Sensitivity
  - For bonds that pay the same coupon rate, long-term bonds change proportionally more in price than do short-term bonds for a given rate change.

The effect of maturity on the relationship between price and interest rate on fixed-income, option free bonds



The effect of <u>coupon</u> on the relationship between price and interest rate on fixed-income, option free bonds



For a given change in market rate, the bond with the lower coupon will change more in price than will the bond with the higher coupon.

Market Rate	Price of 9.4% Bonds	Price of Zero Coupon
8.8%	$10,155.24	$7,723.20
9.4%	10,000.00	7,591.37
10.0%	9.847.73	7,462.15

Duration and Price Volatility

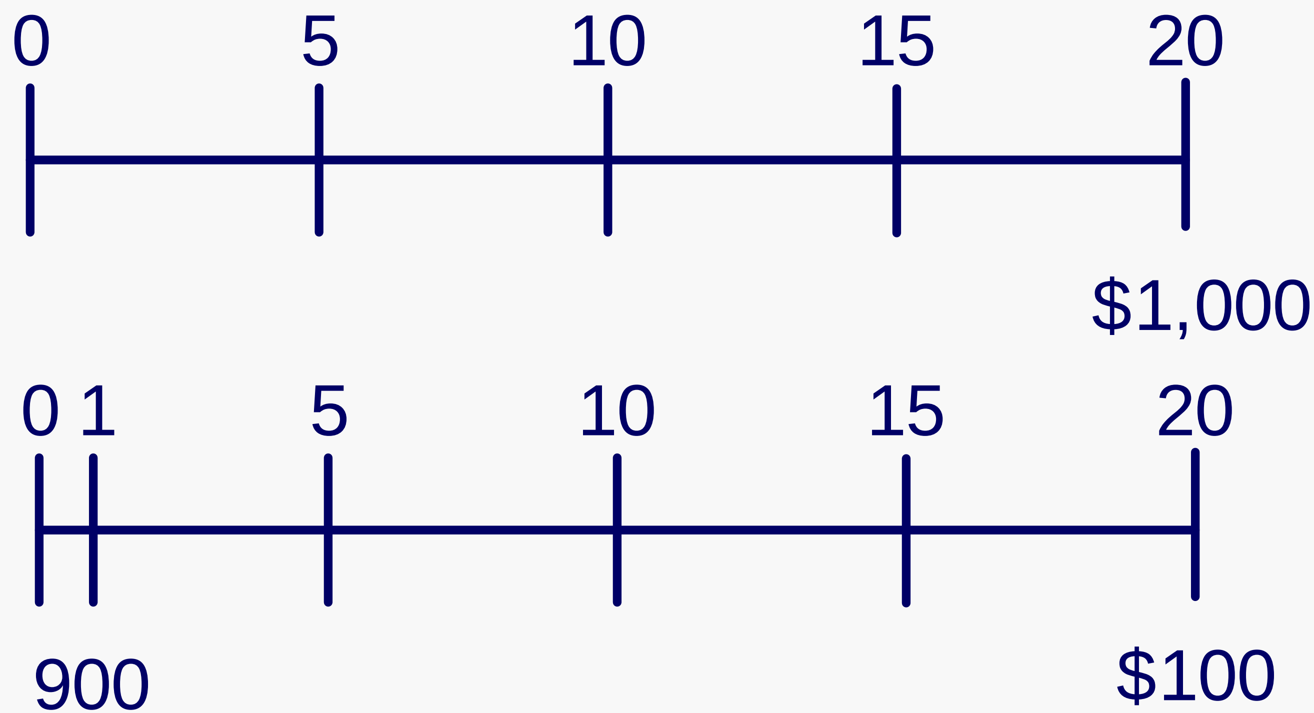
- Duration as an Elasticity Measure
  - Maturity simply identifies how much time elapses until final payment.
  - It ignores all information about the timing and magnitude of interim payments.

Duration and Price Volatility

- **Duration as an Elasticity Measure**
  - **Duration is a measure of the *effective* maturity of a security.**
    - **Duration incorporates the timing and size of a security's cash flows.**
    - **Duration measures how price sensitive a security is to changes in interest rates.**
      - **The greater (shorter) the duration, the greater (lesser) the price sensitivity.**

Duration and Price Volatility

- **Duration as an Elasticity Measure**
  - **Duration versus Maturity**
    - **Consider the cash flows for these two securities**



Duration and Price Volatility

- Duration as an Elasticity Measure
  - Duration versus Maturity
    - The maturity of both is 20 years
      - Maturity does not account for the differences in the timing of the cash flows

Duration and Price Volatility

- Duration as an Elasticity Measure
  - Duration versus Maturity
    - What is the effective maturity of both?
      - The effective maturity of the first security is: (1,000/1,000) x 1 = 20 years
      - The effective maturity of the second security is:
        - [(900/1,000) x 1]+[(100/1,000) x 20] = 2.9 years
    - Duration is similar, however, it uses a weighted average of the present values of the cash flows

Duration and Price Volatility

- **Duration as an Elasticity Measure**
  - **Duration is an approximate measure of the price elasticity of demand**

$$\text{Price Elasticity of Demand} = -\frac{\%\ \text{Change in Quantity Demanded}}{\%\ \text{Change in Price}}$$

Duration and Price Volatility

- **Duration as an Elasticity Measure**
  - **The longer the duration, the larger the change in price for a given change in interest rates.**

$$\text{Duration} \cong -\frac{\frac{\Delta P}{P}}{\frac{\Delta i}{(1+i)}}$$

$$\Delta P \cong -\text{Duration}\left[\frac{\Delta i}{(1+i)}\right]P$$

Duration and Price Volatility

- **Measuring Duration**
  - **Duration is a weighted average of the time until the expected cash flows from a security will be received, relative to the security's price**
- **Macaulay's Duration**

$$D = \frac{\sum_{t=1}^{k} \frac{CF_t(t)}{(1+r)^t}}{\sum_{t=1}^{k} \frac{CF_t}{(1+r)^t}} = \frac{\sum_{t=1}^{n} \frac{CF_t(t)}{(1+r)^t}}{\text{Price of the Security}}$$

Duration and Price Volatility

- **Measuring Duration**
  - **Example**
    - **What is the duration of a bond with a \$1,000 face value, 10% coupon, 3 years to maturity and a 12% YTM?**

$$D = \frac{\frac{100 \times 1}{(1.12)^1} + \frac{100 \times 2}{(1.12)^2} + \frac{100 \times 3}{(1.12)^3} + \frac{1,000 \times 3}{(1.12)^3}}{\sum_{t=1}^{3} \frac{100}{(1.12)^t} + \frac{1000}{(1.12)^3}} = \frac{2,597.6}{951.96} = 2.73 \text{ years}$$

Duration and Price Volatility

- **Measuring Duration**
  - **Example**
    - **What is the duration of a bond with a $1,000 face value, 10% coupon, 3 years to maturity but the YTM is 5%?**

$$D = \frac{\frac{100 * 1}{(1.05)^1} + \frac{100 * 2}{(1.05)^2} + \frac{100 * 3}{(1.05)^3} + \frac{1,000 * 3}{(1.05)^3}}{1136.16} = \frac{3,127.31}{1,136.16} = 2.75 \text{ years}$$

Duration and Price Volatility

- **Measuring Duration**
  - **Example**
    - **What is the duration of a bond with a $1,000 face value, 10% coupon, 3 years to maturity but the YTM is 20%?**

$$D = \frac{\frac{100 * 1}{(1.20)^1} + \frac{100 * 2}{(1.20)^2} + \frac{100 * 3}{(1.20)^3} + \frac{1{,}000 * 3}{(1.20)^3}}{789.35} = \frac{2{,}131.95}{789.35} = 2.68 \text{ years}$$

Duration and Price Volatility

- Measuring Duration
  - Example
    - What is the duration of a zero coupon bond with a $1,000 face value, 3 years to maturity but the YTM is 12%?

$$D = \frac{\frac{1{,}000 * 3}{(1.12)^3}}{\frac{1{,}000}{(1.12)^3}} = \frac{2{,}135.34}{711.78} = 3 \text{ years}$$

    - By definition, the duration of a zero coupon bond is equal to its maturity

Duration and Price Volatility

- **Comparing Price Sensitivity**
  - **The greater the duration, the greater the price sensitivity**

$$\frac{\Delta P}{P} \cong - \left[\frac{\text{Macaulay's Duration}}{(1+i)} \right] \Delta i$$

$$\text{Modified Duration} = \frac{\text{Macaulay's Duration}}{(1+i)}$$

Duration and Price Volatility

- **Comparing Price Sensitivity**
  - **With Modified Duration, we have an estimate of price volatility:**

$$\%\,\text{Change in Price} = \frac{\Delta P}{P} \cong -\,\text{Modified Duration} * \Delta i$$

Comparative price sensitivity indicated by duration

	Type of Bond			
	3-Yr. Zero	**6-Yr. Zero**	**3-Yr. Coupon**	**6-Yr. Coupon**
Initial market rate (annual)	9.40%	9.40%	9.40%	9.40%
Initial market rate (semiannual)	4.70%	4.70%	4.70%	4.70%
Maturity value	$10,000	$10,000	$10,000	$10,000
Initial price	$7,591.37	$5,762.88	$10,000	$10,000
Duration: semiannual periods	6.00	12.00	5.37	9.44
Modified duration	5.73	11.46	5.12	9.02
Rate Increases to 10% (5% Semiannually)				
Estimated ΔP	-$130.51	-$198.15	-$153.74	-$270.45
Estimated $\Delta P / P$	-1.72%	-3.44%	-1.54%	-2.70%
Initial elasticity	0.2693	0.5387	0.2406	0.4242

- **ΔP = - Duration $[\Delta i / (1 + i)]$ P**
- **$\Delta P / P$ = - [Duration / (1 + i)] Δi**
  where Duration equals Macaulay's duration.

Valuation of Fixed-Income Securities

- Traditional fixed-income valuation methods are too simplistic for three reasons:
  - Investors often do not hold securities until maturity
  - Present value calculations assume all coupon payments are reinvested at the calculated Yield to Maturity
  - Many securities carry embedded options, such as a call or put, which complicates valuation since it is unknown if the option will be exercised and at what price .

Valuation of Fixed-Income Securities

- **Fixed-Income securities should be priced as a package of cash flows with each cash flow discounted at the appropriate zero coupon rate.**

Valuation of Fixed-Income Securities

- Total Return Analysis
  - Sources of Return
    - Coupon Interest
    - Reinvestment Income
      - Interest-on-interest
    - Capital Gains or Losses

Valuation of Fixed-Income Securities

- Total Return Analysis
  - Example
    - What is the total return for a 9-year, 7.3% coupon bond purchased at $99.62 per $100 par value and held for 5-years?
      - Assume the semi-annual reinvestment rate is 3% and after five years a comparable 4-year maturity bond will be priced to yield 7% (3.5% semi-annually) to maturity

Valuation of Fixed-Income Securities

- **Total Return Analysis**
  - **Example**
    - **Coupon interest:**

      10 x \$3.65 = \$36.50
    - **Interest-on-interest:**

      $\$3.65\,[(1.03)^{10} - 1]/0.03 - \$36.50 = \$5.34$
    - **Sale price after five years:** $\sum_{t=1}^{8} \frac{\$3.65}{(1.035)^t} + \frac{\$100}{(1.035)^8} = \$101.03$
    - **Total future value:**

      \$36.50 + \$5.34 + \$101.03 = \$142.87
    - **Total return:**

      $[\$142.87 / \$99.62]^{1/10} - 1 = 0.0367$

      or 7.34% annually

Money Market Yields

- **Interest-Bearing Loans with Maturities of One Year or Less**
  - **The effective annual yield for a loan less than one year is:**

$$i^* = \left[1 + \frac{i}{365/h}\right]^{(365/h)} - 1$$

Money Market Yields

- Interest rates for most money market yields are quoted on a different basis.
  - Some money market instruments are quoted on a discount basis, while others bear interest.
  - Some yields are quoted on a 360-day year rather than a 365 or 366 day year.

Money Market Yields

- **Interest-Bearing Loans with Maturities of One Year or Less**
  - **Assume a 180 day loan is made at an annualized rate of 10%. What is the effective annual yield?**

$$i^* = \left[1 + \frac{.10}{365/180}\right]^{(365/180)} - 1 = \left[1 + \frac{.10}{2.0278}\right]^{(2.0278)} - 1 = .1025 = 10.25\%$$

Money Market Yields

- 360-Day versus 365-Day Yields
  - Some securities are reported using a 360 year rather than a full 365 day year.
    - This will mean that the rate quoted will be 5 days too small on a standard annualized basis of 365 days.

Money Market Yields

- 360-Day versus 365-Day Yields
  - To convert from a 360-day year to a 365-day year:
    - $i_{365} = i_{360}\ (365/360)$
  - Example
    - One year instrument at an 8% nominal rate on a 360-day year is actually an 8.11% rate on a 365-day year:
      - $i_{365} = 0.08\ (365/360) = 0.0811$

Money Market Yields

- Discount Yields
  - Some money market instruments, such as Treasury Bills, are quoted on a <u>discount</u> basis.
    - This means that the purchase price is always below the par value at maturity.
    - The difference between the purchase price and par value at maturity represents interest.

Money Market Yields

- Discount Yields
  - The pricing equation for a discount instrument is:

$$i_{dr} = \left[\frac{Pf - Po}{Pf}\right]\left[\frac{360}{h}\right]$$

where:

i_{dr} = discount rate
Po = initial price of the instrument
Pf = final price at maturity or sale,
h = number of days in holding period.

Money Market Yields

- Two Problems with the Discount Rate
  - The return is based on the final price of the asset, rather than on the purchase price
  - It assumes a 360-day year
  - One solution is the Bond Equivalent Rate: i_{be}

$$i_{be} = \left[\frac{Pf - Po}{Po}\right]\left[\frac{365}{h}\right]$$

Money Market Yields

- **A problem with the Bond Equivalent Rate is that it does not incorporate compounding. The Effective Annual Rate addresses this issue.**

$$i^* = \left[1 + \frac{Pf - Po}{Po}\right]^{\left[\frac{365}{h}\right]} - 1 = \left[1 + \frac{i_{be}}{365/h}\right]^{\left[\frac{365}{h}\right]} - 1$$

Money Market Yields

- **Example:**
  - **Consider a $1 million T-bill with 182 days to maturity and a price of $964,500.**

$$i_{dr} = \left[\frac{1{,}000{,}000 - 964{,}500}{1{,}000{,}000}\right]\left[\frac{360}{182}\right] = 0.0720 = 7.20\%$$

$$i_{be} = \left[\frac{1{,}000{,}000 - 964{,}500}{964{,}500}\right]\left[\frac{365}{182}\right] = 0.0738 = 7.38\%$$

$$i^* = \left[1 + \frac{1{,}000{,}000 - 964{,}500}{964{,}500}\right]^{\left[\frac{365}{182}\right]} - 1 = \left[1 + \frac{0.0738}{365/182}\right]^{\left[\frac{365}{182}\right]} - 1 = 0.0752 = 7.52\%$$

Money Market Yields

- **Yields on Single-Payment, Interest-Bearing Securities**
  - **Some money market instruments, such as large negotiable CD’s, Eurodollars, and federal funds, pay interest calculated against the par value of the security and make a single payment of interest and principal at maturity.**

Money Market Yields

- **Yields on Single-Payment, Interest-Bearing Securities**
  - **Example: consider a 182-day CD with a par value of $1,000,000 and a quoted rate of 7.02%.**
    - **Actual interest paid at maturity is:**
      - **(0.0702)(182 / 360) $1,000,000 = $35,490**
    - **The 365 day yield is:**
      - **i_{365} = 0.0702(365 / 360) = 0.0712**
    - **The effective annual rate is:**

$$i^* = \left[1 + \frac{0.0712}{365/182}\right]^{\left[\frac{365}{182}\right]} - 1 = 0.0724 = 7.25\%$$

Summary of money market yield quotations and calculations

- Simple Interest i_s:

$$i_s = \frac{p_f - p_o}{p_o}$$

- Discount Rate i_{dr}:

$$i_{dr} = \frac{p_f - p_o}{p_f}\left[\frac{360}{h}\right]$$

- Money Mkt 360-day rate, i_{360}

$$i_{360} = \frac{p_f - p_o}{p_o}\left[\frac{360}{h}\right]$$

- Bond equivalent 365 day rate, i_{365} or i_{be}:

$$i_{be} = \frac{p_f - p_o}{p_o}\left[\frac{365}{h}\right]$$

- Effective ann. interest rate,

$$i^* = \left[1 + \frac{i}{365/h}\right]^{365/h} - 1$$

Definitions
- **P_f = final value**
- **P_o = initial value**
- **h=# of days in holding period**

Discount Yield quotes:
- **Treasury bills**
- **Repurchase agreements**
- **Commercial paper**
- **Bankers acceptances**

Interest-bearing, Single Payment:
- **Negotiable CDs**
- **Federal funds**