Introduction to Quantitative Finance

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Chapter 1

Financial Derivatives

Assume that the price of a stock is given, at time t, by S_t. We want to study the so called market of options or derivatives.

Definition 1.0.1 *An option is a contract that gives the right (but not the obligation) to buy (CALL) or shell (PUT) the stock at price K (strike) at time T (maturity of the contract).*

The profit or payoff of this contract is:

$$(S_T - K)_+$$

in the case of a CALL or

$$(K - S_T)_+$$

for a PUT.

Problem 1: ¿How much should the buyer pay for the option? This is called *the pricing problem.*

Problem 2: How the seller of the contract can guarantee the quantity $(S_T - K)_+$ (in the case of a CALL) from the price charged. This is the *hedging problem.*

Assumption: we are going to assume that the financial market is free of making profit without risk or free of *arbitrage* opportunities. We also assume that there is a continuous interest rate r in such a way that one euro becomes e^{rT} euros at time T. We have the following result.

Proposition 1.0.1 *(PUT-CALL parity) If the market is free of arbitrage opportunities and C_t is the price of a CALL, at time t, with strike K and maturity T and P_t the put price, with the same strike and maturity, we have*

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \text{ for all } 0 \leq t \leq T$$

Proof. We shall see that otherwise there will be arbitrage. Assume for instance that

$$C_t - P_t > S_t - Ke^{-r(T-t)}.$$

Then at time t we buy a unit of stock, one PUT and we sell one CALL. The profit we obtain by this trade is

$$C_t - P_t - S_t.$$

If this quantity is positive we can put it in a bank account until time T with interest rate r. If it is negative we can borrow it with the same interest rate At time T we can have two situations: 1) If $S_T > K$ the owner of the CALL will exercise the option, then we will give him the stock by K, in total we will have

$$\begin{aligned}&(C_t - P_t - S_t)\, e^{r(T-t)} + K \\ &= \left(C_t - P_t - S_t + Ke^{-r(T-t)}\right) e^{r(T-t)} > 0.\end{aligned}$$

2) If $S_T \leq K$, we will exercise the PUT and we will sell the stock by K, we will have again $(C_t - P_t - S_t)\, e^{r(T-t)} + K$ that is positive. So there will be an arbitrage opportunity. An analogous situation happens if

$$C_t - P_t < S_t - Ke^{-r(T-t)}.$$

■

1.1 Discrete time models

The values of the stocks (shares, commodities or other stocks) will be random variables defined in a certain probability space $(\Omega, \mathcal{F}, P)$. We will consider an increasing sequence of σ-fields (filtration) : $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_N \subseteq \mathcal{F}$. $\mathcal{F}_n$ represents the available information at the instant n. The horizon N, will correspond with the maturity of the options. We shall assume that Ω is finite, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_N = \mathcal{F} = \mathcal{P}(\Omega)$ and that $P(\{\omega\}) > 0$, for all $\omega \in \Omega$.

The financial market will consist on $(d+1)$ stocks whose prices at instant n will be given by positive random variables $S_n^0, S_n^1, \ldots, S_n^d$ measurable with respect to $\mathcal{F}_n$ (that is, the prices depend on what has been observed so far, there is not privilege information). In many cases we shall assume that $\mathcal{F}_n = \sigma(S_k^1, \ldots, S_k^d, 0 \leq k \leq n)$, in such a way that whole the information will be in the prices observed until this moment.

The super-index zero corresponds to the riskless stock (a bank account) and by convention we take $S_0^0 = 1$. If the relative profit of the riskless stock is constant:

$$\frac{S_{n+1}^0 - S_n^0}{S_n^0} = r \geq 0$$

we will have

$$S_{n+1}^0 = S_n^0(1+r) = S_0^0(1+r)^{n+1}.$$

The factor $\beta_n = \frac{1}{S_n^0} = (1+r)^{-n}$ will be called the dicount factor.

1.1.1 Strategies of investment

A strategy of investment is a stochastic processes (a sequence or random variables in the discrete time setting) $\phi = ((\phi_n^0, \phi_n^1, ..., \phi_n^d))_{0 \le n \le N}$ in R^{d+1}. ϕ_n^i indicates the number of stocks of i kind in the portfolio at the instant n. ϕ es *predictable* that is:

$$\begin{cases} \phi_0^i & \text{is } \mathcal{F}_0\text{-measurable} \\ \phi_n^i & \text{es } \mathcal{F}_{n-1}\text{-measurable, for all } 1 \le n \le N. \end{cases}$$

This means that the positions in the portfolio at n were decided at $n-1$. In other words, during the period $(n-1, n]$ the quantity of stocks of i kind is ϕ_n^i. The value of the portfolio at n is given by the scalar product

$$V_n(\phi) = \phi_n \cdot S_n = \sum_{i=0}^{d} \phi_n^i S_n^i,$$

and its discounted value

$$\tilde{V}_n(\phi) = \beta_n V_n(\phi) = \phi_n \cdot \tilde{S}_n$$

with

$$\tilde{S}_n = (1, \beta_n S_n^1, ..., \beta_n S_n^d) = (1, \tilde{S}_n^1, ..., \tilde{S}_n^d)$$

Definition 1.1.1 *An investment strategy is said to be self-financing if*

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n, 0 \le n \le N-1$$

Remark 1.1.1 *The meaning is tat at n, once the new prices S_n are announced, the investors relocate their portfolio without add or take out wealth: if there is an increment $\phi_{n+1} - \phi_n$ of stocks the cost of this trade is $(\phi_{n+1} - \phi_n) \cdot S_n$, and we want to do this without any cost so $\phi_n \cdot S_n = \phi_{n+1} \cdot S_n, 0 \le n \le N-1$.*

Proposition 1.1.1 *An investment strategy is self-financing iff:*

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n), 0 \le n \le N-1$$

Proposition 1.1.2 *The following statements are equivalent: (i) the strategy ϕ is self-financing, (ii) for all $1 \le n \le N$*

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (S_j - S_{j-1}) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j = V_0(\phi) + \sum_{j=1}^{n} \sum_{i=0}^{d} \phi_j^i \Delta S_j^i$$

(iii) for all $1 \le n \le N$

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j = V_0(\phi) + \sum_{j=1}^{n} \sum_{i=1}^{d} \phi_j^i \Delta \tilde{S}_j^i$$

Proof. (i) is equivalent to (ii):

$$\begin{aligned} V_n(\phi) &= V_0(\phi) + \sum_{j=1}^{n}(V_j(\phi) - V_{j-1}(\phi)) \\ &= V_0(\phi) + \sum_{j=1}^{n}\phi_j \cdot (S_j - S_{j-1}) \text{ ,(previous proposition)} \end{aligned}$$

(i) is equivalent to (iii): the self-financing condition can be written as $\phi_n \cdot \tilde{S}_n = \phi_{n+1} \cdot \tilde{S}_n, 0 \leq n \leq N-1$, so

$$\tilde{V}_{n+1}(\phi) - \tilde{V}_n(\phi) = \phi_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n), 0 \leq n \leq N-1$$

and

$$\begin{aligned} \tilde{V}_n(\phi) &= \tilde{V}_0(\phi) + \sum_{j=1}^{n}(\tilde{V}_j(\phi) - \tilde{V}_{j-1}(\phi)) \\ &= V_0(\phi) + \sum_{j=1}^{n}\phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) \end{aligned}$$

■

The previous proposition tell us that any self-financing strategy is defined by its initial value V_0 and for the positions in the risky stocks. More precisely:

Proposition 1.1.3 *For any predictable process* $\hat{\phi} = ((\phi_n^1, ..., \phi_n^d))_{0 \leq n \leq N}$ *and any random variable* V_0 $\mathcal{F}_0$*-measurable, there exists a unique predictable process* $\left(\phi_n^0\right)$ *such that the strategy* $\phi = ((\phi_n^0, \phi_n^1, ..., \phi_n^d))_{0 \leq n \leq N}$ *is self-financing with initial value* V_0.

Proof.

$$\begin{aligned} \tilde{V}_n(\phi) &= V_0(\phi) + \sum_{j=1}^{n}\phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) \\ &= V_0(\phi) + \sum_{j=1}^{n}\phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) \\ &= \ \phi_n \cdot \tilde{S}_n = \phi_n^0 + \sum_{i=1}^{d}\phi_n^i \tilde{S}_n^i. \end{aligned}$$

Therefore

$$\phi_n^0 = V_0(\phi) + \sum_{j=1}^{n-1}\phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) - \sum_{i=1}^{d}\phi_n^i \tilde{S}_{n-1}^i \in \mathcal{F}_{n-1}$$

■

1.1.2 Admissible strategies and arbitrage

First of all note that we are not doing any assumption about the sign of the quantities ϕ_n^i. $\phi_n^i < 0$ means that we borrowed this number of stocks and converted in cash (*short-selling*) or, if $i = 0$, we borrowed this number of monetary units and converted in stocks (a loan to buy stocks). We assume that any unit of cash at 0 becomes $(1+r)^n$ at n. We shall assume that loans and short-selling are allowed provided the value of the portfolio is always positive.

Definition 1.1.2 *A strategy ϕ is admissible if it is self-financing and $V_n(\phi) \geq 0$, for all $0 \leq n \leq N$.*

Definition 1.1.3 *A strategy of arbitrage is an admissible strategy with zero initial value and with final value different from zero.*

Remark 1.1.2 *Note that if there is an arbitrage we can get a strictly positive wealth with a null initial investment. Most of the models of prices exclude arbitrage opportunities. A market without arbitrage opportunities is said to be* viable. *The next purpose will be to characterize viable markets with the aid of the notion of martingale.*

Exercise 1.1.1 *Consider a portfolio with initial value $V_0 = 1000a$ and formed by the following quantities of risky stocks:*

	Stock 1	*Stock 2*
$n > 0$	200	100
$n > 1$	150	120
$n > 2$	500	60

The prices of he stocks are

	Stock 1	*Stock 2*
$n = 0$	3.4	2.3
$n = 1$	3.5	2.1
$n = 2$	3.7	1.8.

To find out, at any time, the amount invested in the riskless stock in the portfolio assuming that $r = 0.05$ and that the portfolio is self-financing.

Solution 1.1.1 *Assuming that the value at time $t = 0$ is $V_0 = 1000$, we can calculate the initial composition of the portfolio according with the positions in the risky stocks $\phi_1 = (200, 100)$ and leaving the remainder of the 1000 euros in the bank account 1 y 2. Later we calculate how the value of the portfolio change in terms of change of prices between instants 0 and 1. We rebuilt our portfolio according with e positions $\phi_2 = (150, 120)$, in the bank account we leave the remainder after buying the indicated quantities of stocks 1 and 2. Later we calculate again how the value of the portfolio evolves.*

Stock	N^o shares	Price $t=0$	Value $t=0$	Precio $t=1$	Valor $t=1$
0	90	1	90	1,05	94,5
1	200	3,4	680	3,5	700
2	100	2,3	230	2,1	210
Total			1000		1004,5

Stock	N^o assets	Price $t=1$	Value $t=1$	Price $t=2$	Value $t=2$
0	216,67	1,05	2227,5	1,103	238,88
1	150	3,5	525	3,7	555
2	120	2,1	252	1,8	216
Total			1004,5		1009,88

Exercise 1.1.2 *Consider a financial market with one single period, with interest rate r and one stock S. Suppose that $S_0 = 1$ and, for $n = 1$, S_1 can take two different values: 2, 1/2. ¿For which values of r the market is viable viable (free of arbitrage opportunities)? ¿what if S_1 can also take the value 1?*

Solution 1.1.2 *We want to calculate the values of r such that there is an arbitrage opportunity. We take a portfolio with zero initial value $V_0 = 0$. Then we invest the amount q in the stock without risk, we have to invest $-q$ in the risky stock (q can be negative or positive). We calculate the value of this portfolio in the time 2.*
$V_1(\omega_1) = q(r-1)$
$V_1(\omega_2) = q(r+1/2)$
So, if $r > 1$ there is an arbitrage oppportunity taking q positive (money in the bank account and short position in the risky stock) and if $r < -1/2$ we have an arbitrage opportunity with q positive (borrowing money and investing in the risky stock). The situation does not change if S_1 can take the value 1.

Exercise 1.1.3 *Consider a financial market with two risky stocks ($d = 2$) and such that the values at $t = 0$ are $S_0^1 = 9.52$ Eur. and $S_0^2 = 4.76$ Euros. The simple interest rate is 5% during the period $[0, 1]$. We also assume that at time 1, S_1^1 and S_1^2 can take three different values, depending of the market state: $\omega_1, \omega_2, \omega_3$: $S_1^1(\omega_1) = 20$ Eur., $S_1^1(\omega_2) = 15$ Eur. and $S_1^1(\omega_3) = 7.5$ Eur, and $S_1^2(\omega_1) = 6$ Eur, $S_1^2(\omega_2) = 6$ Eur. and $S_1^2(\omega_3) = 4$. ¿Is that a viable market?*

Solution 1.1.3 *To know if he market is viable we have to check if there are arbitrage opportunities. We take a portfolio with initial value equal to zero and we see if the yield can be non-negative in all states of time 1 with some of them strictly positive yield. Let q_1 and q_2 be the amounts invested in the stocks 1 and 2 respectively. Since the initial value of the portfolio es zero, we should have $-9.52q_1 - 4.76q_2$ in the bank account. Then we calculate the value of our portfolio at time 1 for all possible states.*
$V_1(\omega_1) = 10.004q_1 + 1.002q_2$
$V_1(\omega_2) = 5.004q_1 + 1.002q_2$
$V_1(\omega_3) = -2.4964q_1 - 0.998q_2$.

It is easy to see that there is a region of the plane where the three expressions are positive at the same time (see Figure 1), therefore there are arbitrage opportunities.

Figura 1

1.1.3 Martingales and opportunities of arbitrage

et $(\Omega, \mathcal{F}, P)$ a finite probability space. With $\mathcal{F} = \mathcal{P}(\Omega)$ y $P(\{\omega\}) > 0$, for all todo ω. Consider a filtration $(\mathcal{F}_n)_{0\leq n\leq N}$.

Definition 1.1.4 *We say that a sequence of random variables $X = (X_n)_{0\leq n\leq N}$ are adapted if X_n es $\mathcal{F}_n$-measurable, $0 \leq n \leq N$.*

Definition 1.1.5 *An adapted sequence $(M_n)_{0\leq n\leq N}$, is said to be a*

$$
\begin{array}{ll}
\textit{submartingale if} & E(M_{n+1}|\mathcal{F}_n) \geq M_n \\
\textit{martingale if} & E(M_{n+1}|\mathcal{F}_n) = M_n \\
\textit{supermartingale if} & E(M_{n+1}|\mathcal{F}_n) \leq M_n
\end{array}
$$

for all $0 \leq n \leq N - 1$

Remark 1.1.3 *This definition can be extended to the multi-dimensional case in a component-wise fashion. If $(M_n)_{0\leq n\leq N}$ is a martingale is easy to see that $E(M_{n+j}|F_n) = M_n, j \geq 0; E(M_n) = E(M_0), n \geq 0$ and that if (N_n) is another martingale, $(aM_n + bN_n)$ is also a martingale. We shall omit the sub-index.*

Proposition 1.1.4 *Let (M_n) be a martingale and (H_n) a predictable sequence, let $\Delta M_n = M_n - M_{n-1}$. Then, the sequence defined by*

$$
\begin{aligned}
X_0 &= H_0 M_0 \\
X_n &= H_0 M_0 + \sum_{j=1}^{n} H_j \Delta M_j, n \geq 1 \textit{ is a martingale}
\end{aligned}
$$

Proof. It is enough to see that for all $n \geq 0$

$$E(X_{n+1} - X_n|\mathcal{F}_n) = E(H_{n+1}\Delta M_{n+1}|\mathcal{F}_n) = H_{n+1}E(\Delta M_{n+1}|\mathcal{F}_n) = 0$$

■

Remark 1.1.4 *The previous transform is called martingale transform of (M_n) by (H_n). Remind that*

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta\tilde{S}_j$$

with (ϕ_i) predictable. Then if $(\tilde{S}_i)$ is a martingale, we will have that $(\tilde{V}_n)$ is a martingale and in particular $E(\tilde{V}_n(\phi)) = E(V_0(\phi)) = V_0(\phi)$.

Proposition 1.1.5 *An adapted process (M_n) is a martingale iff for all predictable process (H_n) we have*

$$E(\sum_{j=1}^{N} H_j \Delta M_j) = 0 \tag{1.1}$$

Proof. Assume that (M_n) is a martingale Then (**??**mart) follows by the previous proposition. Assume that (1.1) is satisfied, then we can take $H_n = 0, 0 \leq n \leq j, H_{j+1} = 1_A$ with $A \in \mathcal{F}_j$, $H_n = 0, n > j$. So

$$E(\mathbf{1}_A(M_{j+1} - M_j)) = 0.$$

Since this is true for all A this is equivalent to $E(M_{j+1} - M_j | F_j) = 0$. But this is true for all j. ■

Theorem 1.1.1 *A financial market is viable (free of arbitrage opportunities) if and only if there exists P^* equivalent to P such that the discounted prices of the stocks $((\tilde{S}_n^j), j = 1, ..., d)$ are P^*- martingales.*

Proof. Assume there exists P^* and let φ and admissible strategy with zero initial value, then

$$\tilde{V}_n = \sum_{i=1}^{n} \varphi_i \cdot \Delta \tilde{S}_i$$

es una P^* martingale and consequently

$$E_{P^*}(\tilde{V}_N) = 0$$

and since $\tilde{V}_N \geq 0$ we have $\tilde{V}_N = 0$ (because $P^*(\omega) > 0$ for all ω). So, there is not arbitrage.

We identify each random variable X to a vector in $\mathbb{R}^{Card(\Omega)}$ $(X(\omega_1), ..., X(\omega_{Card(\Omega)}))$. Suppose now that there is not arbitrage and let Γ be the set of random variables strictly positive define in Ω (that is random non-negative variables such that for some $\omega \in \Omega$ their value is strictly greater than zero). Consider the subset, S, compact and convex of the random variables in Γ such that $\sum X(\omega_i) = 1$. Let $L = \{V_N(\varphi), \varphi$ be a *self-financing strategy*, $V_0(\varphi) = 0\}$ (it is clear that L is a vectorial of $\mathbb{R}^{Card(\Omega)}$). Also, (we shall see it later) $L \cap S = \phi$. As a result of the hyperplane separation theorem there exists a linear map A such that $A(Y) > 0$ for all $Y \in S$ and $A(Y) = 0$ if $Y \in L$. $A(Y) = \sum \lambda_i Y(\omega_i)$. Then all $\lambda_i > 0$ (since $A(Y) > 0$ for all $Y \in S$) and we can define

$$P^*(\omega_i) = \frac{\lambda_i}{\sum \lambda_i}$$

and for all ϕ predictable

$$E_{P^*}(\sum_{i=1}^{N} \phi_i \cdot \Delta \tilde{S}_i) = E_{P^*}(\tilde{V}_N) = \frac{A(\tilde{V}_N)}{\sum \lambda_i} = 0.$$

So, by the previous proposition, $\tilde{S}$ is a P^*-martingale (proposition anterior).

See know that $L \cap \Gamma = \phi$ (and a fortiori $L \cap S = \phi$). Assume it is not true in such a way that there exists φ self-financing with $V_N(\varphi) \in \Gamma$. Then, from φ, you can built an arbitrage strategy: let

$$n = \sup\{k, V_k(\varphi) \ngeq 0 \}$$

note that $n \leq N-1$ since $V_N(\varphi) \geq 0$. Let $A = \{V_n(\varphi) < 0\}$, define the self-financing strategy such that for all $i = 1, ..., d$

$$\theta_j^i = \begin{cases} 0 & j \leq n \\ \mathbf{1}_A \varphi_j^i & j > n \end{cases}$$

Then, for all $k > n$

$$\begin{aligned} \tilde{V}_k(\theta) &= \sum_{j=n+1}^{k} \mathbf{1}_A \varphi_j \cdot \Delta\tilde{S}_j = \mathbf{1}_A \left(\sum_{j=1}^{k} \varphi_j \cdot \Delta\tilde{S}_j - \sum_{j=1}^{n} \varphi_j \cdot \Delta\tilde{S}_j \right) \\ &= \mathbf{1}_A \left(\tilde{V}_k(\varphi) - \tilde{V}_n(\varphi) \right) \end{aligned}$$

so θ is admissible and $\tilde{V}_N(\theta) > 0$ in A. ■

Remark 1.1.5 *P^* is named martingale measure or neutral probability.*

Exercise 1.1.4 *Consider a sequence $\{X_n\}_{n\geq 1}$ of independent random variables with law $N(0,\ \sigma^2)$. Define the sequence $Y_n = \exp\left(a \sum_{i=1}^{n} X_i - n\sigma^2\right)$, $n \geq 1$, for a a real parameter, and $Y_0 = 1$. Find the values of a such that the sequence $\{Y_n\}_{n\geq 0}$ is a martingale (supermartingale) (submartingale).*

Exercise 1.1.5 *Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent, identically distributed random variables*

$$P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}.$$

Set $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$ if $n \geq 1$.

Check if the following sequences are martingales:

$$\begin{aligned} M_n^{(1)} &= \frac{e^{\theta S_n}}{(\cosh\theta)^n},\ n \geq 0 \\ M_n^{(2)} &= \sum_{k=1}^{n} \operatorname{sign}(S_{k-1}) Y_k,\ n \geq 1,\ M_0^{(2)} = 0 \\ M_n^{(3)} &= S_n^2 - n \end{aligned}$$

Exercise 1.1.6 *Consider a discrete-time financial market, with two periods, interest rate $r \geq 0$, and a single risky stock, S. Suppose that S evolves as:*

$$\begin{array}{ccc}
n=0 & n=1 & n=2 \\
 & & 9 \\
 & 8\begin{smallmatrix}\nearrow^{p_2} \\ \searrow_{1-p_2}\end{smallmatrix} & \\
5\begin{smallmatrix}\nearrow^{p_1} \\ \searrow_{1-p_1}\end{smallmatrix} & & 6 \\
 & 4\begin{smallmatrix}\nearrow^{p_3} \\ \searrow_{1-p_3}\end{smallmatrix} & \\
 & & 3
\end{array}$$

a) Find p_1, p_2 i p_3, in terms of r such that the probability is neutral. b) Assuming that $r = 0.1$, give the initial value of a derivative with maturity $N = 2$ and payoff $\frac{S_1+S_2}{2}$. Construct first the portfolio that covers the risk of the derivative and see its initial value. Check that this value coincides with the expectation, with respect to the neutral probability, of the discounted payoff.

Exercise 1.1.7 *Find the neutral probabilities in the market model of Exercise 1.1.3 assuming that only stock 1 is tradable.*

Theorem 1.1.2 *(Hyperplane Separation Theorem) Let L a subspace of $\mathbb{R}^n$ and K a convex and compact subset of $\mathbb{R}^n$ without intersection with L. Then there exists a linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(x) = 0$ for all $x \in L$ and $\phi(x) > 0$ for all $x \in K$.*

The proof is based in the following lemma:

Lemma 1.1.1 *Let C be a closed convex set of $\mathbb{R}^n$ not containing the origin, then there exists $\phi : \mathbb{R}^n \to \mathbb{R}$, linear, such that $\phi(x) > 0$ for all $x \in C$.*

Proof. *Let $B(0, r)$ a ball of radius r and centered at the origin, take r sufficiently big in such a way that $B(0, r) \cap C \neq \phi$. The map*

$$\begin{aligned} B(0,r) \cap C &\to \mathbb{R}_+ \\ x \longmapsto ||x|| &= \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \end{aligned}$$

is continuous and since it is defined in a compact set there will exist $z \in B(0, r) \cap C$ such that $||z|| = \inf_{x \in B(0,r) \cap C} ||x||$ and it satisfies $||z|| > 0$ since C does not contain the origin. Let $x \in C$, since C is convex $\lambda x + (1 - \lambda)z \in C$ for all $0 \leq \lambda \leq 1$. It is obvious that

$$\|\lambda x + (1 - \lambda)z\| \geq ||z|| > 0,$$

then

$$\lambda^2 x \cdot x + 2\lambda(1 - \lambda)x \cdot z + (1 - \lambda)^2 z \cdot z \geq z \cdot z,$$

equivalently

$$\lambda^2(x \cdot x + z \cdot z) + 2\lambda(1 - \lambda)x \cdot z - 2\lambda z \cdot z \geq 0.$$

Take $\lambda > 0$, then

$$\lambda(x \cdot x + z \cdot z) + 2(1-\lambda)x \cdot z \geq 2z \cdot z$$

and taking the limit when $\lambda \to 0$ we have

$$x \cdot z \geq z \cdot z > 0.$$

Then it is enough to take $\phi\ (x) = x \cdot z$. ∎

Proof. *(of the theorem) $K - L = \{u \in \mathbb{R}^n, u = k - l, k \in K, l \in L\}$ is closed and convex. In fact, let $0 \leq \lambda \leq 1$ and $u, \tilde{u} \in K - L$*

$$\begin{aligned} \lambda u + (1-\lambda)\tilde{u} &= \lambda k + (1-\lambda)\tilde{k} - (\lambda l + (1-\lambda)\tilde{l}) \\ &= \bar{k} - \bar{l} \end{aligned}$$

where $\bar{k} \in K$ (by convexity of K) and $\bar{l} \in L$ (since it is a vectorial space), then it is convex. Furthermore, if we take a sequence $(u_n) \in K - L$ converging to u, we have that $u_n = k_n - l_n$ with $k_n \in K, l_n \in L$, that is $l_n = k_n - u_n$. But since K is compact, there exists a subsequence k_{n_r} that converges to a certain $k \in K$, so l_{n_r} will converge to $k - u$, and since l_{n_r} is a convergent sequence in a closed vectorial space ($\mathbb{R}^d$ is for all d) we will have $k - u = l \in L$, in such a way that $u = k - l \in K - L$. Now $K - L$ does not contain the origin and by the previous proposition there exists ϕ linear such that

$$\phi(k) - \phi(l) > 0, \text{ para todo } k \in K \text{ y todo } l \in L.$$

Moreover, since L is a vectorial space $\phi(l)$ has to be zero. In fact if we assume for instance that $\phi(l) > 0$, then $\lambda l \in L$ for all $\lambda > 0$ arbitrary big and we will have that

$$\phi(k) > \lambda\phi(l),$$

but this is impossible if $\phi(k)$ es finite. Finally, since $\phi(l) = 0$ we have that $\phi(k) > 0$ for all $k \in K$. ∎

1.1.4 Complete markets and option pricing

We define a *European option, derivative or contingent claim* as a contract with maturity N and with a *payoff* $h \geq 0$, where h is $\mathcal{F}_N$- measurable.

For instance a *call* is a European option with *payoff* $h = (S^1_N - K)_+$, and a *put* $h = (K - S^1_N)_+$, and an *Asian option* is a European one! with $h = (\frac{1}{N}\sum_{j=0}^N S^1_j - K)_+$

Definition 1.1.6 *A derivative defined by h is said to replicable if there exists an admissible strategy ϕ such that replicates h that is $V_N(\phi) = h$.*

Proposition 1.1.6 *If ϕ is a self-financing strategy that replicates h and the market is viable then it is admissible.*

Proof. $\tilde{V}_N(\phi) = \tilde{h}$ and since there exists P^* such that $E_{p^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$, we have $\tilde{V}_n(\phi) \geq 0$. ■

Definition 1.1.7 *A market is said to be complete if any derivative is replicable.*

Theorem 1.1.3 *A viable market is complete if and only if there is a unique probability P^* equivalent to P under which the discounted prices are martingales*

Proof. Assume that the market is viable and complete, then, given h $\mathcal{F}_N$-measurable there exists ϕ admissible, such that $V_N(\phi) = h$ that is:

$$\tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j = \frac{h}{S_N^0}.$$

Assume there exist P_1 and P_2 martingale measures, then

$$E_{P_1}(\frac{h}{S_N^0}) = V_0(\phi)$$
$$E_{P_2}(\frac{h}{S_N^0}) = V_0(\phi)$$

and since this is true for all h $\mathcal{F}_N$-measurable both probability are the same in $\mathcal{F}_N = \mathcal{F}$.

Assume now that the market is viable but incomplete, we shall see that we can built more than on e neutral probability. Let H be the subset of random variables of the form

$$V_0 + \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j$$

with V_0 $\mathcal{F}_0$-measurable and $\phi = ((\phi_n^1, ..., \phi_n^d))_{0 \leq n \leq N}$ predictable. H is a vectorial subspace of the vectorial space, E, formed by all random variables. Moreover it is not trivial, in fact since the market is incomplete there will exist h such that $\frac{h}{S_n^0} \notin H$. Let P^* be a neutral probability in E, we can define the scalar product $\langle X, Y\rangle = E_{P^*}(XY)$. Let X be an random variable orthogonal to H and define

$$P^{**}(\omega) = (1 + \frac{X(\omega)}{2||X||_\infty})P^*(\omega).$$

Then we have an equivalent probability to P^* :

$$P^{**}(\omega) = (1 + \frac{X(\omega)}{2||X||_\infty})P^*(\omega) > 0$$

$$\sum P^{**}(\omega) = \sum P^*(\omega) + \frac{E_{P^*}(X)}{2||X||_\infty} = 1$$

since $1 \in H$ and X is orthogonal to H. Also, by this orthogonality

$$E_{P^{**}}(\sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j) = E_{P^*}(\sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j) + \frac{E_{P^*}(X \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j)}{2||X||_\infty} = 0$$

in such a way that $\tilde{S}$ is a P^{**}-martingale by Proposition 1.1.5. ■

Pricing and hedging in complete markets

Assume we have a derivative with payoff $h \geq 0$ and that the market is viable and complete. We know that there exists ϕ admissible, such that $V_N(\phi) = h$ and if P^* is the neutral probability neutral we have that

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$$

is a P^*-martingale, in particular

$$E_{P^*}(\frac{h}{S_N^0}|\mathcal{F}_n) = E_{P^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$$

that is

$$V_n(\phi) = S_n^0 E_{P^*}(\frac{h}{S_N^0}|\mathcal{F}_n) = E_{P^*}(\frac{h}{(1+r)^{N-n}}|\mathcal{F}_n)$$

so, the value of the replicating portfolio of h is given by the previous formula and this gives us the price of the derivative at time n that we shall denote by C_n, that is $C_n = V_n(\phi)$. Note that if we have a single risky stock ($d = 1$) then

$$\frac{\tilde{C}_n - \tilde{C}_{n-1}}{\Delta \tilde{S}_n} = \phi_n$$

and we can calculate the hedging portfolio if we have an expression of C as a function of S.

The binomial model of Cox-Ross-Rubinstein (CRR)

Assume a model with one risky stock that evolves as:

$$S_n(\omega) = S_0(1+b)^{U_n(\omega)}(1+a)^{n-U_n(\omega)}$$

where

$$U_n(\omega) = \xi_1(\omega) + \xi_2(\omega) + ... + \xi_n(\omega)$$

and where ξ_i are random variables wit values 0 or 1, that is Bernoulli random variables, and $a < r < b$:

$$\begin{array}{ccc}
n=0 & n=1 & n=2... \\
 & & S_0(1+b)^2 \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} \\
 & S_0(1+b) \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & \\
S_0 \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & S_0(1+b)(1+a) \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} \\
 & S_0(1+a) \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & \\
 & & S_0(1+a)^{\ 2} \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix}.
\end{array}$$

We can also write

$$S_n = S_{n-1}(1+b)^{\xi_n}(1+a)^{1-\xi_n(\omega)},$$

then

$$\tilde{S}_n = S_0 \left(\frac{1+b}{1+r}\right)^{U_n} \left(\frac{1+a}{1+r}\right)^{n-U_n} = \tilde{S}_{n-1} \left(\frac{1+b}{1+r}\right)^{\xi_n} \left(\frac{1+a}{1+r}\right)^{1-\xi_n}.$$

For $\tilde{S}_n$ to be a martingale with respect to P^* we need

$$E_{P^*}(\tilde{S}_n|\mathcal{F}_{n-1}) = \tilde{S}_{n-1}$$

and if we take $\mathcal{F}_n = \sigma(S_0, S_1, ..., S_n)$ we have that the previous condition is equivalent to

$$E_{P^*}(\left(\frac{1+b}{1+r}\right)^{\xi_n} \left(\frac{1+a}{1+r}\right)^{1-\xi_n} |\mathcal{F}_{n-1}) = 1$$

that is

$$\left(\frac{1+b}{1+r}\right) P^*(\xi_n = 1|\mathcal{F}_{n-1}) + \left(\frac{1+a}{1+r}\right) P^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1$$

and consequently

$$P^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{r-a}{b-a},$$
$$P^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1 - P^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{b-r}{b-a}$$

Note that this conditional probability is deterministic and does not depends on n, so *under it* $\xi_i, i = 1, ..., N$ are independent, identically distributed random variables with common distribution Bernoulli(p), for $p = \frac{r-a}{b-a}$. P^* is unique as well, so the market is viable and complete. So, under the neutral probability P^*

$$\begin{aligned} S_N &= S_n(1+b)^{\xi_{n+1}+\ldots+\xi_N}(1+a)^{N-n-(\xi_{n+1}+\ldots+\xi_N)} \\ &= S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}} \end{aligned}$$

with $W_{n,N} \sim$Bin$(N-n, p)$ independent of $S_n, S_{n-1}, ...S_1$. Since we have the neutral probability we can calculate the price of a *call* at time n

$$\begin{aligned} C_n &= E_{P*}\left(\frac{(S_N-K)_+}{(1+r)^{N-n}}|\mathcal{F}_n\right) \\ &= E_{P*}\left(\frac{(S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}}-K)_+}{(1+r)^{N-n}}|\mathcal{F}_n\right) \\ &= \sum_{k=0}^{N-n} \frac{(S_n(1+b)^k(1+a)^{N-n-k}-K)_+}{(1+r)^{N-n}} \binom{N-n}{k} p^k (1-p)^{N-n-k} \\ &= S_n \sum_{k=k^*}^{N-n} \binom{N-n}{k} \frac{(p(1+b))^k((1-p)(1+a))^{N-n-k}}{(1+r)^{N-n}} \\ &- K(1+r)^{n-N} \sum_{k=k^*}^{N-n} \binom{N-n}{k} p^k (1-p)^{N-n-k} \end{aligned}$$

where

$$\begin{aligned} k^* &= \inf\{k, S_n(1+b)^k(1+a)^{N-n-k} > K\} \\ &= \inf\{k, k > \frac{\log \frac{K}{S_n} - (N-n)\log(1+a)}{\log(\frac{1+b}{1+a})}\} \end{aligned}$$

Note that

$$\frac{p(1+b)}{1+r} + \frac{(1-p)(1+a)}{1+r} = 1,$$

so, if we define

$$\bar{p} = \frac{p(1+b)}{1+r}$$

we can write

$$\begin{aligned} C_n = S_n \sum_{k=k^*}^{N-n} \binom{N-n}{k} \bar{p}^k(1-\bar{p})^{N-n-k} \\ - K(1+r)^{n-N} \sum_{k=k^*}^{N-n} \binom{N-n}{k} p^k\,(1-p)^{N-n-k} \\ = S_n \Pr\{\mathrm{Bin}(N-n,\bar{p}) \geq k^*\} - K(1+r)^{n-N} \Pr\{\mathrm{Bin}(N-n,p) \geq k^*\} \end{aligned}$$

Hedging portfolio in the CRR model

We have that

$$V_n = \phi_n^0(1+r)^n + \phi_n^1 S_n.$$

Fixed S_{n-1}, S_n can take two value $S_n^u = S_{n-1}(1+b)$ ó $S_n^d = S_{n-1}(1+a)$ and analogously V_n. Then

$$\phi_n^1 = \frac{V_n^u - V_n^d}{S_{n-1}(b-a)}. \tag{1.2}$$

and

$$\phi_n^0 = \frac{V_n^u - \phi_n^1 S_n^u}{(1+r)^n}$$

In the case of a call, if we take $n = N$ we have:

$$\phi_N^1 = \frac{V_N^u - V_N^d}{S_{N-1}(b-a)} = \frac{(S_{N-1}(1+b)-K)_+ - (S_{N-1}(1+a)-K)_+}{S_{N-1}(b-a)}.$$

Now we can calculate by the self-financing condition the value of the portfolio at $N-1$:

$$V_{N-1} = \phi_N^0(1+r)^{N-1} + \phi_N^1 S_{N-1}$$

and from here ϕ_{N-1}^1 using (1.2) again.

Example 1.1.1 *The following example is a compute program written in* Mathematica *to calculate the value of a* call *and* put *for a CRR mode with the following data:* $S_0 = 100$ *eur.,* $K = 100$ *eur.* $b = 0.2$, $a = -0.2$, $r = 0.02$, $n = 4$ *periods.*

```
Clear[s, call, pu];
s[0] = Table[100, {1}];
a = -0.2; b = 0.2; r = 0.02; n = 4;
p = (r - a)/(b - a);
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i], {i, 0, n}], Center]
pp[x_] := Max[x, 0]
call[n] = Map[pp, s[n] - 100]; pu[n] = Map[pp, 100 - s[n]];
call[x_] := call[x] =
          Drop[p*call[x + 1]/(1 + r) + (1 - p)*RotateLeft[call[x +
  1], 1]/(1 + r), -1]
ColumnForm[Table[call[i], {i, 0, n}], Center]
pu[x_] := pu[x] = Drop[p*pu[x + 1]/(1 +
r) + (1 - p)*RotateLeft[pu[x + 1], 1]/(1 + r), -1]
ColumnForm[Table[pu[i], {i, 0, n}], Center]
```

Example 1.1.2 *Consider a CRR model with 91 periods $a = -b$. We want to calculate the initial value of a European call where the underlying is a share of Telefónica.*

- *Maturity: 3 months (91 days= n) ($T = 91/365$).*
- *Current price of the share of Telefónica 15.54 euros.*
- *Strike 15.54 euros.*
- *Interest rate 4.11 % annual.*
- *Annual volatility: 23,20% (b^2=volatility$^2 \times T/n$)*

```
Clear[s, c];
n = 91;
so = 15.54;
K = 15.54;
vol = 0.232;
T = 91/365;
r = 0.0411*T/n;
b =vol*Sqrt[T/n];
a =-b;
p = (r - a)/(b - a);
q = 1 - p;
s[0] = Table[so, {1}];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
pp[x_] := Max[x, 0];
c[n] = Map[pp, s[n] - K];
c[x_] := c[x] = Drop[p*c[x + 1]/(1 + r) +
     q*RotateLeft[c[x + 1], 1]/(1 + r), -1];
c[0][[1]]
```

Exercise 1.1.8 *Consider a financial market with two periods, interest rate* $r = 0$*, and a single risky asset* S^1*. Suppose that* $S_0^1 = 1$ *and for* $n = 1, 2$*,* $S_n^1 = S_{n-1}^1 \xi_n$*, where the random variables* ξ_1*,* ξ_2 *are independent, and take two different values:* $2, \frac{3}{4}$*, with the same a probability. a) Is that a viable market? Is it complete? Find the price of a European option with maturity* $N = 2$ *and payoff* $\max_{0 \le n \le 2} S_n^1$*. Find the hedging portfolio of this option.*

Assume now that we have a second risky asset in this market with S_n^2 *such that* $S_0^2 = 1$ *and for* $n = 1, 2$

$$S_n^2 = S_{n-1}^2 \eta_n,$$

where the random variables η_n *take three different values s* $2, 1, \frac{1}{2}$*,* η_1 *y* η_2 *are independent and*

$$P(\eta_n = 2 | \xi_n = 2) = 1,$$
$$P(\eta_n = 1 | \xi_n = \frac{3}{4}) = \frac{1}{3},$$
$$P(\eta_n = \frac{1}{2} | \xi_n = \frac{3}{4}) = \frac{2}{3},$$

in such a way that the vector (ξ_n, η_n) *takes only the values* $(2, 2)$*,* $(\frac{3}{4}, 1)$*,* $(\frac{3}{4}, \frac{1}{2})$ *with probabilities* $\frac{1}{2}, \frac{1}{6}, \frac{1}{3}$*. b) Prove that these two assets* S_n^1*,* S_n^2 *form a viable and complete market and calculate the neutral probability. Is it possible to know the value of the European option mentioned in a) without doing any calculation? Why?*

Exercise 1.1.9 *Prove that if* $X_n \xrightarrow{\mathcal{L}} X$*,* X *absolutely continuous, and* $a_n \to a \in \bar{R}$*, then* $P\{X_n \le a_n\} \to P\{X \le a\}$*.*

Exercise 1.1.10 *Let* $\{X_{nj},\ j = 1, ..., k_n,\ n \geqq 1\}$*, where* $k_n \xrightarrow{n} \infty$ *, a triangular system of centered and independent random variables, fixed* n*, with* $X_{nj} = O(k_n^{-1/2})$*, and such that* $\sum_{j=1}^{k_n} E(X_{nj}^2) \to \sigma^2 > 0$*, prove that* $S_n = \sum_{j=1}^{k_n} X_{nj} \xrightarrow{\mathcal{L}} N(0, \sigma^2)$*.*

Exercise 1.1.11 *Assume now a sequence of CRR binomial models where the number of periods depends of* n *and such that*

$$1 + r(n) = e^{\frac{rT}{n}},$$
$$1 + b(n) = e^{\sigma\sqrt{\frac{T}{n}}},$$
$$1 + a(n) = e^{-\sigma\sqrt{\frac{T}{n}}},$$

Prove that for n *big enough the markets are viable. Calculate the limit of the price of a call at the initial time when* $n \to \infty$*.*

Exercise 1.1.12 *Consider the analogous situation as in the previous exercise but with*

$$1 + b(n) = e^{\tau},$$
$$1 + a(n) = e^{\lambda \frac{T}{n}},$$

where $\tau > 0$ *y* $0 < \lambda < r$.

1.1.5 American options

An American option can be exercised in any time between 0 and N, and we shall define an $(\mathcal{F}_n)$-adapted positive sequence (Z_n) to indicate the immediate payoff when it is exercised at time n. In the case of an American *call* $Z_n = (S_n - K)_+$ and in the case of an American *put* $Z_n = (K - S_n)_+$. To obtain the price, U_n, at time n, we proceed by doing a backward induction. Define $U_N = Z_N$. At time $N-1$, the owner of the option can choose between receiving Z_{N-1} or the *equivalent* amount to a Z_N at time $N-1$ that is the amount to replicate Z_N at $N-1$ given by $S^0_{N-1} E_{P^*}(\tilde{Z}_N|\mathcal{F}_{N-1})$ (we are assuming that the market is viable and complete and that P^* is the neutral probability). Obviously he will choose the maximum of the two amounts, so we have

$$U_{N-1} = \max(Z_{N-1}, S^0_{N-1} E_{P^*}(\tilde{Z}_N|\mathcal{F}_{N-1}))$$

and by backward induction

$$U_n = \max(Z_n, S^0_n E_{P^*}(\tilde{U}_{n+1}|\mathcal{F}_n))$$

or analogously

$$\tilde{U}_n = \max(\tilde{Z}_n, E_{P^*}(\tilde{U}_{n+1}|\mathcal{F}_n)), 0 \leq n \leq N-1$$

Proposition 1.1.7 *The sequence $(\tilde{U}_n)$ is the smallest a P^*-supermartingale that dominates the sequence$(\tilde{Z}_n)$*

Proof. $(\tilde{U}_n)$ is adapted and by construction

$$E_{P^*}(\tilde{U}_{n+1}|\mathcal{F}_n) \leq \tilde{U}_n.$$

Let (T_n) be another supermartingale that dominates $(\tilde{Z}_n)$, then $T_N \geq \tilde{Z}_N = \tilde{U}_N$. Assume that $T_{n+1} \geq \tilde{U}_{n+1}$. Then, by the monotony of the expectation and since (T_n) is a supermartingale

$$T_n \geq E_{P^*}(T_{n+1}|\mathcal{F}_n) \geq E_{P^*}(\tilde{U}_{n+1}|\mathcal{F}_n)$$

moreover (T_n) dominates $(\tilde{Z}_n)$, so

$$T_n \geq \max(\tilde{Z}_n, E_{P^*}(\tilde{U}_{n+1}|\mathcal{F}_n)) = \tilde{U}_n$$

■

Remark 1.1.6 *If we exercise the option at time n, we receive Z_n and the initial value of this is*

$$V_0 = E_{P^*}(\tilde{Z}_n|\mathcal{F}_0),$$

since we can exercise the American option at any time $\{0, 1, .., N\}$ one wonders if

$$U_0 = \sup_\nu E_{P^*}(\tilde{Z}_\nu|\mathcal{F}_0),$$

where ν is a random time, where the decision on stopping at time n is made according with the information we have till this time n. That is $\{\nu = n\} \in \mathcal{F}_n$. The answer, as we shall see later, is positive.

1.1.6 The optimal stopping problem

Definition 1.1.8 *A random variable ν taking values in $\{0, 1, ..., N\}$ is a stopping time if*

$$\{\nu = n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$$

Remark 1.1.7 *Equivalently ν is a stooping time if $\{\nu \leq n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$, definition that can be extended to the continuous case.*

Now we introduce the concept of a stochastic process "stopped" by a stopping time. Let (X_n) be an adapted stochastic process and ν a stopping time, then we define

$$X_n^\nu = X_{n\wedge\nu} \text{ para todo } n.$$

Note that

$$X_n^\nu(\omega) = \begin{cases} X_n & \text{si } n \leq \nu(\omega) \\ X_{\nu(\omega)} & \text{si } n > \nu(\omega) \end{cases}$$

Proposition 1.1.8 *Let (X_n) adapted, then (X_n^ν) is adapted and if (X_n) is a martingale (sup, super), then (X_n^ν) is a martingale (sub, super).*

$$\begin{aligned} X_n^\nu = X_{n\wedge\nu} &= X_0 + \sum_{j=1}^{n\wedge\nu}(X_j - X_{j-1}) \\ &= X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j\leq\nu\}}(X_j - X_{j-1}), \end{aligned}$$

but $\{j \leq \nu\} = \overline{\{\nu \leq j-1\}} \in \mathcal{F}_{j-1}$ con lo que $1_{\{j\leq\nu\}}$ es $\mathcal{F}_{j-1}$-measurable and the sequence (ϕ_j) con $\phi_j = 1_{\{j\leq\nu\}}$ is predictable. Obviously X_n^ν es $\mathcal{F}_n$-measurable and

$$\begin{aligned} E(X_{n+1}^\nu - X_n^\nu|\mathcal{F}_n) &= E(\mathbf{1}_{\{n+1\leq\nu\}}(X_{n+1} - X_n)|\mathcal{F}_n) \\ &= \mathbf{1}_{\{n+1\leq\nu\}} E(X_{n+1} - X_n|\mathcal{F}_n) \lesseqgtr 0 \text{ if } (X_n) \text{ es } \begin{array}{l} \text{super} \\ \text{martingala} \\ \text{sub} \end{array} \end{aligned}$$

The Snell envelope

Let (Y_n) an adapted process (to $(\mathcal{F}_n)$), define

$$\begin{aligned} &X_N = Y_N \\ &X_n = \max(Y_n, E(X_{n+1}|\mathcal{F}_n)), \quad 0 \leq n \leq N-1, \end{aligned}$$

we say that (X_n) is the Snell envelope of (Y_n).

Remark 1.1.8 *Note that $(\tilde{U}_n)$, the sequence of the discounted prices of the American options is the Snell envelope of discounted payoffs $(\tilde{Z}_n)$.*

Remark 1.1.9 *By Proposition 1.1.7 the Snell envelope of an adapted process is the smallest supermartingale that dominates it.*

Remark 1.1.10 *Fixed ω if X_n is strictly greater than Y_n, $X_n = E(X_{n+1}|\mathcal{F}_n)$ so X_n behaves, until this n as a martingale, this indicates that if we "stop" X_n properlywe can have a martingale.*

Proposition 1.1.9 *The random variable*

$$\nu = \inf\{n \geq 0, X_n = Y_n\}$$

is a stopping time and (X_n^ν) is a martingale.

Proof.

$$\{\nu = n\} = \{X_0 > Y_0\} \cap ... \cap \{X_{n-1} > Y_{n-1}\} \cap \{X_n = Y_n\} \in \mathcal{F}_n.$$

And

$$X_n^\nu = X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j \leq \nu\}}(X_j - X_{j-1})$$

therefore

$$X_{n+1}^\nu - X_n^\nu = \mathbf{1}_{\{n+1 \leq \nu\}}(X_{n+1} - X_n)$$

and

$$E(X_{n+1}^\nu - X_n^\nu|\mathcal{F}_n) = E(\mathbf{1}_{\{n+1 \leq \nu\}} X_{n+1} - \mathbf{1}_{\{n+1 \leq \nu\}} X_n|\mathcal{F}_n),$$

but

$$\begin{aligned}
\mathbf{1}_{\{n+1 \leq \nu\}} X_n &= \mathbf{1}_{\{n+1 \leq \nu\}} \max(Y_n, E(X_{n+1}|\mathcal{F}_n)) \\
&= \max(\mathbf{1}_{\{n+1 \leq \nu\}} Y_n, E(\mathbf{1}_{\{n+1 \leq \nu\}} X_{n+1}|\mathcal{F}_n)) \\
&= E(\mathbf{1}_{\{n+1 \leq \nu\}} X_{n+1}|\mathcal{F}_n)),
\end{aligned}$$

since

$$\mathbf{1}_{\{n+1 \leq \nu\}} X_n > \mathbf{1}_{\{n+1 \leq \nu\}} Y_n.$$

■

We denote $\tau_{n,N}$ stopping times with values in $\{n, n+1, ..., N\}$.

Corollary 1.1.1

$$X_0 = E(Y_\nu|\mathcal{F}_0) = \sup_{\tau \in \tau_{0,N}} E(Y_\tau|\mathcal{F}_0)$$

Proof. (X_n^ν) is a martingale and consequently

$$\begin{aligned}
X_0 &= E(X_N^\nu|\mathcal{F}_0) = E(X_{N \wedge \nu}|\mathcal{F}_0) \\
&= E(X_\nu|\mathcal{F}_0) = E(Y_\nu|\mathcal{F}_0).
\end{aligned}$$

On the other hand (X_n) is supermartingale and then (X_n^τ) as well for all $\tau \in \tau_{0,N}$, so

$$X_0 \geq E(X_N^\tau|\mathcal{F}_0) = E(X_\tau|\mathcal{F}_0) \geq E(Y_\tau|\mathcal{F}_0),$$

therefore

$$E(Y_\nu|\mathcal{F}_0) \geq E(Y_\tau|\mathcal{F}_0), \quad \forall \tau \in \tau_{0,N}$$

■

Remark 1.1.11 *Analogously we could prove*

$$X_n = E(Y_{\nu_n}|\mathcal{F}_n) = \sup_{\tau \in \tau_{n,N}} E(Y_\tau|\mathcal{F}_n),$$

where

$$\nu_n = \inf\{j \geq n, X_j = Y_j\}$$

Definition 1.1.9 *A stopping time ν is said to be optimal for the sequence (Y_n) if*

$$E(Y_\nu|\mathcal{F}_0) = \sup_{\tau \in \tau_{0,N}} E(Y_\tau|\mathcal{F}_0).$$

Remark 1.1.12 *The stopping time $\nu = \inf\{n, X_n = Y_n\}$ (where X is the Snell envelope of Y) is then an optimal stopping time for Y. We shall see the it is the smallest optimal stopping time.*

The following theorem characterize the optimal stopping times.

Theorem 1.1.4 *τ is an optimal stopping time if and only if*

$$\begin{cases} X_\tau = Y_\tau & \\ (X_n^\tau) & \textit{is a martingale} \end{cases}$$

Proof. If (X_n^τ) is a martingale and $X_\tau = Y_\tau$

$$\begin{aligned} X_0 &= E(X_N^\tau|\mathcal{F}_0) = E(X_{N\wedge\tau}|\mathcal{F}_0) \\ &= E(X_\tau|\mathcal{F}_0) = E(Y_\tau|\mathcal{F}_0). \end{aligned}$$

On the other hand for all stopping time π, (X_n^π) is a supermartingale, so

$$X_0 \geq E(X_N^\pi|\mathcal{F}_0) = E(X_\pi|\mathcal{F}_0) \geq E(Y_\pi|\mathcal{F}_0).$$

Reciprocally, we know, by the previous corollary, that $X_0 = \sup_{\tau \in \tau_{0,N}} E(Y_\tau|F_0)$. Then, if τ is optimal

$$X_0 = E(Y_\tau|\mathcal{F}_0) \leq E(X_\tau|\mathcal{F}_0) \leq X_0,$$

where the last inequality is due to the fact that (X_n^τ) is a supermartingale. So, we have

$$E(X_\tau - Y_\tau|\mathcal{F}_0) = 0$$

and since $X_\tau - Y_\tau \geq 0$, we conclude that $X_\tau = Y_\tau$.

Now we can also see that (X_n^τ) is a martingale. We know that it is a supermartingale, then

$$X_0 \geq E(X_n^\tau|\mathcal{F}_0) \geq E(X_N^\tau|\mathcal{F}_0) = E(X_\tau|\mathcal{F}_0) = X_0$$

as we saw before. Then, for all n

$$E(X_n^\tau - E(X_\tau|\mathcal{F}_n)|\mathcal{F}_0) = 0,$$

and since (X_n^τ) is supermartingale,

$$X_n^\tau \geq E(X_N^\tau|\mathcal{F}_n) = E(X_\tau|\mathcal{F}_n)$$

therefore $X_n^\tau = E(X_\tau|\mathcal{F}_n)$. ■

Decomposition of supermartingales

Proposition 1.1.10 *Any supermartingale (X_n) has a unique decomposition:*

$$X_n = M_n - A_n$$

where (M_n) is a martingale and (A_n) is non-decreasing predictable with $A_0 = 0$.

Proof. It is enough to write

$$X_n = \sum_{j=1}^{n}(X_j - E(X_j|\mathcal{F}_{j-1})) - \sum_{j=1}^{n}(X_{j-1} - E(X_j|\mathcal{F}_{j-1})) + X_0$$

and to identify

$$M_n = \sum_{j=1}^{n}(X_j - E(X_j|\mathcal{F}_{j-1})) + X_0,$$
$$A_n = \sum_{j=1}^{n}(X_{j-1} - E(X_j|\mathcal{F}_{j-1}))$$

where we define $M_0 = X_0$ and $A_0 = 0$. So (M_n) is a martingale:

$$M_n - M_{n-1} = X_n - E(X_n|\mathcal{F}_{n-1}), \quad 1 \leq n \leq N$$

in such a way that

$$E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \quad 1 \leq n \leq N.$$

Finally since (X_n) is supermartingale

$$A_n - A_{n-1} = X_{n-1} - E(X_n|\mathcal{F}_{n-1}) \geq 0, \quad 1 \leq n \leq N.$$

Now we can see the uniqueness. If

$$M_n - A_n = M'_n - A'_n, \quad 0 \le n \le N$$

we have

$$M_n - M'_n = A_n - A'_n, \quad 0 \le n \le N,$$

but then since (M_n) y (M'_n) are martingales and (A_n) y (A'_n) predictable, it turns out that

$$\begin{aligned} A_{n-1} - A'_{n-1} &= M_{n-1} - M'_{n-1} = E(M_n - M'_n|\mathcal{F}_{n-1}) \\ &= E(A_n - A'_n|\mathcal{F}_{n-1}) = A_n - A'_n, \quad 1 \le n \le N, \end{aligned}$$

that is

$$A_N - A'_N = A_{N-1} - A'_{N-1} = ... = A_0 - A'_0 = 0,$$

since by hypothesis $A_0 = A'_0 = 0$. ■

This decomposition is known as the Doob decomposition.

Proposition 1.1.11 *The biggest optimal stopping time for (Y_n) is given by*

$$\nu_{\max} = \begin{cases} N & si\ A_N = 0 \\ \inf\{n, A_{n+1} > 0\} & si\ A_N > 0 \end{cases},$$

where (X_n), Snell envelope of (Y_n), has a Doob decomposition $X_n = M_n - A_n$.

Proof. $\{\nu_{\max} = n\} = \{A_1 = 0, A_2 = 0, ..., A_n = 0, A_{n+1} > 0\} \in \mathcal{F}_n$, $0 \le n \le N-1$, $\{\nu_{\max} = N\} = \{A_N = 0\} \in \mathcal{F}_{N-1}$. So, it is a stopping time.

$$X_n^{\nu_{\max}} = X_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}} - A_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}}$$

since $A_{n\wedge\nu_{\max}} = 0$. Therefore $(X_n^{\nu_{\max}})$ is a martingale. So, to see that this stopping time is optimal we have to prove that

$$X_{\nu_{\max}} = Y_{\nu_{\max}}$$

$$\begin{aligned} X_{\nu_{\max}} &= \sum_{j=1}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} X_j + \mathbf{1}_{\{\nu_{\max}=N\}} X_N \\ &= \sum_{j=1}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} \max(Y_j, E(X_{j+1}|\mathcal{F}_j)) + \mathbf{1}_{\{\nu_{\max}=N\}} Y_N, \end{aligned}$$

but in $\{\nu_{\max} = j\}$, $A_j = 0, A_{j+1} > 0$ so

$$E(X_{j+1}|\mathcal{F}_j) = E(M_{j+1}|\mathcal{F}_j) - A_{j+1} < E(M_{j+1}|\mathcal{F}_j) = M_j = X_j$$

therefore $X_j = Y_j$ en $\{\nu_{\max} = j\}$ and consequently $X_{\nu_{\max}} = Y_{\nu_{\max}}$. Finally we see that is the biggest optimal stopping time. Let $\tau \ge \nu_{\max}$ and $P\{\tau > \nu_{\max}\} > 0$. Then

$$\begin{aligned} E(X_\tau) &= E(M_\tau) - E(A_\tau) = E(M_0) - E(A_\tau) \\ &= X_0 - E(A_\tau) < X_0 \end{aligned}$$

so $(X_{\tau\wedge n})$ cannot be a martingale. ■

1.1.7 Application to American options

Another expression for the price of American options

We already saw that the price of an American option with payoffs (Z_n) was given by

$$\begin{cases} U_N = Z_N \\ U_n = \max(Z_n, S_n^0 E_{P^*}(U_{n+1}|\mathcal{F}_n)) \quad \text{si } n \leq N-1. \end{cases}$$

In other words, the sequence of discounted prices $(\tilde{U}_n)$ is the Snell envelope of the discounted payoffs $(\tilde{Z}_n)$. The previous results allow us to say that

$$\tilde{U}_n = \sup_{\tau \in \tau_{n,N}} E_{P^*}(\tilde{Z}_\tau|\mathcal{F}_n),$$

or equivalently

$$U_n = S_n^0 \sup_{\tau \in \tau_{n,N}} E_{P^*}(\frac{Z_\tau}{S_\tau^0}|\mathcal{F}_n).$$

Hedging of American options

By the previous results we know that we can decompose

$$\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$$

where $(\tilde{M}_n)$ is a P^*-martingale and $(\tilde{A}_n)$ is an increasing and predictable with zero value at $n = 0$. If we receive the amount U_0 we can built the self-financing portfolio replicating M_N In fact, since the market is complete, any positive *payoff* (we assume that $(Z_n) \geq 0$), can be replicated, so there will exist ϕ such that

$$V_N(\phi) = M_N$$

or what is the same

$$\tilde{V}_N(\phi) = \tilde{M}_N$$

but $(\tilde{V}_n(\phi))$ and $(\tilde{M}_n)$ are P^*-martingales in such away that $\tilde{V}_n(\phi) = \tilde{M}_n, 0 \leq n \leq N$. Note that then we have

$$U_n = M_n - A_n = V_n(\phi) - A_n$$

and therefore

$$V_n(\phi) = U_n + A_n \geq U_n.$$

In other words with the money we receive we can super-hedge the derivative.

Optimal exercise of the American option

Assume we buy an American option and we want when to exercise the option. That is, we want to know which stopping time τ to use. If τ is such that $U_{\tau(\omega)}(\omega) > Z_{\tau(\omega)}(\omega)$ it is not worth to exercise the option since its value $U_{\tau(\omega)}(\omega)$

is greater than we would obtain if we exercised: $Z_{\tau(\omega)}(\omega)$. So, we will look for τ such that $\tilde{U}_\tau = \tilde{Z}_\tau$. On the other hand we will look for $A_n = 0$, for all $1 \leq n \leq \tau$, (or equivalently $A_\tau = 0$) otherwise, from certain time it would be better to exercise the option and to built a portfolio with the strategy ϕ. In such a way that $V_{\tau\wedge n}(\phi) = U_{\tau\wedge n}$, but then $(\tilde{U}_n^\tau)$ is a P^*-martingale and this together with $\tilde{U}_\tau = \tilde{Z}_\tau$ are the two conditions for τ to be an optimal stopping time for $(\tilde{Z}_n)$.

Note that from the point of view of a seller, if the buyer does not exercise the option at an optimal stopping time then or $U_\tau > Z_\tau$ or $A_\tau > 0$ and in both cases, since the seller has invested the prime to built a portfolio with the strategy ϕ, he will have the profit

$$V_\tau(\phi) - Z_\tau = U_\tau + A_\tau - Z_\tau > 0.$$

Example 1.1.3 *Here it is shown how to calculate the premium of an American put option with maturity of 3 months on stocks whose current value is 60 euros, the strike price is also 60 euros (* at the money *) , the annual interest rate is 10% and the annual volatility 45%. We assume a CRR model with 12 periods. It is also analyzed in which nodes is convenient to exercise the option.*

```
Clear[s, pa, vc, vi];
T = 1/4;n =12;so = 60;K = 60;vol = 0.45;ra = 0.10;
r = ra*T/n;b =vol*Sqrt[T/n];a=-b;
p = (r - a)/(b - a);
q = 1 - p;
pp[x_] := Max[x, 0]
s[0] = Table[so, {1}];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i], {i, 0, n}], Center]
pa[n] = Map[pp, K - s[n]];
pa[x_] := pa[x] = K - s[
x] + Map[pp, Drop[p*pa[x + 1]/(1 + r) + q*RotateLeft[pa[x + 1],
1]/(1 + r), -1] - K + s[x]]
ColumnForm[Table[pa[i], {i, 0, n}], Center]
vc[n] = Map[pp, K - s[n]];
vc[x_] := Drop[p*pa[x + 1]/(1 + r) + q*RotateLeft[pa[x +
1], 1]/(1 + r), -1]
vi[i_] := Map[pp, K - s[i]]
ColumnForm[Table[vc[i] - vi[i], {i, 0, n}], Center]
ColumnForm[Table[pa[i] - vi[i], {i, 0, n}], Center]
```

Exercise 1.1.13 *Obtain the following bounds for the call prices (C) and for the put ones (P) European (E) and American (A):*

$$\max(S_n - K, 0) \leq C_n(E) \leq C_n(A);$$
$$\max(0, (1+r)^{-(N-n)}K - S_n) \leq P_n(E) \leq (1+r)^{-(N-n)}K$$

Exercise 1.1.14 *Consider a viable and complete market with N periods of trading. Show that, with the usual notations,*

$$\sup_{\tau,\ stopping\ time} E_Q\left(\frac{(S_\tau - K)_+}{(1+r)^\tau}\right) = E_Q\left(\frac{(S_N - K)_+}{(1+r)^N}\right)$$

where Q is the risk neutral probability.

Exercise 1.1.15 *Let $\{C_n^E\}_{n=0}^N$ be the price of a European option with payoff Z_N and let $\{Z_n\}_{n=0}^N$ be the payoffs of an American option. Demonstrate that if $C_n^E \geq Z_n, n = 0, 1, ..., N-1$, then $\{C_n^A\}_{n=0}^N$ (the prices of the American option) coincide with $\{C_n^E\}_{n=0}^N$.*

Exercise 1.1.16 *Let $X_n = \xi_1 + \xi_2 + ... + \xi_n, n \geq 1$, where the ξ_i are i.i.d. such that $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Find the Doob decomposition of $|X|$.*

1.2 Continuous-time models

We are going to consider now continuous-time models and even thought the basic ideas are the same, the technical aspects are more delicate.

The main reason to consider such models is not necessary to fix the time between trades, the models are more realistic and we can get close formulas for pricing derivatives. It was Louis Bachelier in 1900 with his "Théorie de la spéculation" the first in considering the Brownian motion to describe stock prices and in obtaining formulas to price options. However his work was not understood at that time and consequently undervalued.

We start by giving some definitions and basic results to understand the new framework. In particular we define the Brownian motion, which is the basic ingredient of the Black-Scholes model. Later we introduce the concept of continuous-time martingale and the differential calculus associated with the Brownian motion, that is the Itô calculus, and finally we apply all these tools to study the Black-Scholes model.

Definition 1.2.1 *An stochastic process is a family of real random variables $(X_t)_{t \in \mathbb{R}_+}$ define in a probability space $(\Omega, \mathcal{F}, P)$.*

Remark 1.2.1 *Usually, index t indicates time and it takes values between 0 and T.*

Remark 1.2.2 *A stochastic process can be also seen as a random map: for all $\omega \in \Omega$ we can associate the map from $\mathbb{R}_+$ to $\mathbb{R}$: $t \longmapsto X_t(\omega)$ named* trajectory *of the process. If the trajectories are continuous then the process is said to be continuous.*

Remark 1.2.3 *Moreover a stochastic process can be also described as a map from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$. We shall assume that in $\mathbb{R}_+ \times \Omega$ we have the σ-field $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ and that the map is measurable (measurable process). This condition is a bit* *stronger that the condition of being simply a process. Nevertheless if the process has continuous trajectories on the left or right sides, then there is always a measurable* version*, say* Y*. That is, there exists* Y *measurable such that* $P(X_t = Y_t) = 1$*, for all* t*.*

Definition 1.2.2 *Let* $(\Omega, \mathcal{F}, P)$ *a probability space, a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is an increasing family of sub-*σ*-fields of* $\mathcal{F}$*. We say that* (X_t) *is an* adapted *process if for all* t*,* X_t *es* $\mathcal{F}_t$*-measurable.*

Remark 1.2.4 *W shall work with filtrations satisfying the property*

$$\text{If } A \in \mathcal{F} \text{ y } P(A) = 0 \text{ then } A \in \mathcal{F}_t \text{ para todo } t.$$

That is $\mathcal{F}_0$ *contents all the* P*-null sets of* $\mathcal{F}$*. The importance of this is that if* $X_t = Y_t$ *a.s. and* X_t *is* $\mathcal{F}_t$*-measurable then* Y_t *is* $\mathcal{F}_t$*-measurable. Then if a process* (X_t) *is adapted and* (Y_t) *is a version of it then* (Y_t) *is adapted.*

Remark 1.2.5 *We can built the filtration generated by a process* (X_t) *and write* $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$*. In general this filtration does not satisfy the previous condition and we shall substitute for* $\mathcal{F}_t$ *by* $\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{N}$ *where* $\mathcal{N}$ *is the collection of null sets of* $\mathcal{F}$*. We call it the natural filtration generated by* (X_t)*.*

The Brownian motion describes the random movement that is possible to observe in some microscopic particles in a fluid mean (for instance pollen in a water drop). This name is due to the botanist Robert Brown who first observed this phenomenon en 1828.

The zigzagging of these particles is due to the fact that they are buffeted by the molecules of the fluid in an intense way depending of the temperature of the fluid.

The mathematical description of this phenomenon was elaborated by Albert Einstein in 1905. Lately around the twenties 20 Norbert Wiener gave a characterization of the Brownian motion as an stochastic process and this is the reason why Wiener process is also used to name the Brownian motion. We consider the one-dimensional case.

Definition 1.2.3 *We say that* $(X_t)_{t\geq 0}$ *is a process with independent increments if for all* $0 \leq t_1 < \ldots < t_n$*,* $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

Definition 1.2.4 *A Brownian motion is a continuous process with independent and stationary increments. That is:*

P*-c.s* $s \longmapsto X_s(\omega)$ *is continuous.*
$s \leq t$*,* $X_t - X_s$ *is independent of* $\mathcal{F}_s = \sigma(X_u, 0 \leq u \leq s)$*.*
$s \leq t$*,* $X_t - X_s \sim X_{t-s} - X_0$*.*

We deduce that the law of $X_t - X_0$ is Gaussian:

Theorem 1.2.1 *If (X_t) is a Brownian motion then*

$$X_t - X_0 \sim N(rt, \sigma^2 t)$$

Proposition 1.2.1 *If (X_t) is a process with independent increments, continuous and $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{nn} \leq t$ is a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, then for all $\varepsilon > 0$*

$$\lim_{n\to\infty} \sum_{i=1}^{n} P\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\} = 0.$$

Proof. We have that for all $\varepsilon > 0$

$$\lim_{n\to\infty} P\{\sup_i |X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\} = 0,$$

but

$$\begin{aligned}
P\left\{\sup_i |X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\right\} &= 1 - \prod_{i=1}^{n} P\{|X_{t_{in}} - X_{t_{i-1,n}}| \leq \varepsilon\} \\
&= 1 - \prod_{i=1}^{n} (1 - P\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\}) \\
&\geq 1 - \exp\{-\sum_{i=1}^{n} P\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\}\} \geq 0
\end{aligned}$$

■

Proposition 1.2.2 *Let $\{Y_{kn}, k = 1, ..., n\}$ be independent random variables such that $|Y_{kn}| \leq \varepsilon_n$ con $\varepsilon_n \downarrow 0$. Then if $\liminf Var(\sum_{k=1}^{n} Y_{kn}) > 0$*

$$\frac{\sum_{k=1}^{n} Y_{kn} - E(\sum_{k=1}^{n} Y_{kn})}{\sqrt{Var(\sum_{k=1}^{n} Y_{kn})}} \to N(0,1)$$

Proof. Write $X_{kn} = Y_{kn} - E(Y_{kn})$ and $v_n^2 = Var(\sum_{k=1}^{n} Y_{kn})$

$$\begin{aligned}
&\log E(\exp\{it\frac{1}{v_n}\sum_{k=1}^{n} X_{kn}\}) \\
&= \log(\prod_{i=1}^{n} E(\exp it\frac{X_{kn}}{v_n})) = \sum_{i=1}^{n} \log(E(\exp it\frac{X_{kn}}{v_n})) \\
&= -\frac{1}{2}t^2 \frac{\sum_{k=1}^{n} E(X_{kn}^2)}{v_n^2} - \frac{i}{3!}t^3 \frac{\sum_{k=1}^{n} E(X_{kn}^3)}{v_n^3} + ... \\
&= -\frac{1}{2}t^2 + O(\frac{\varepsilon_n}{v_n}),
\end{aligned}$$

since

$$\left|\frac{\sum_{k=1}^{n} E(X_{kn}^3)}{v_n^3}\right| \leq \left|\frac{2\varepsilon_n \sum_{k=1}^{n} E(X_{kn}^2)}{v_n^3}\right|.$$

■

Remark 1.2.6 *Note that if* $\liminf Var(\sum_{k=1}^{n} Y_{kn}) = 0$ *we will have that e* $\sum_{k=1}^{n_r} Y_{kn} - E(\sum_{k=1}^{n_r} Y_{kn}) \xrightarrow{P} 0$ *for certain subsequence.*

Proof. (Theorem) Given the partition $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{nn} \leq t$ define

$$Y_{nk} = (X_{t_{kn}} - X_{t_{k-1,n}})\mathbf{1}_{\{|X_{t_{kn}} - X_{t_{k-1,n}}| \leq \varepsilon_n\}},$$

then, by a slight extension of the previous proposition (here ε depends on n),

$$P(X_t - X_0 \neq \sum_{k=1}^{n} Y_{nk}) \leq \sum_{k=1}^{n} P(|X_{t_{kn}} - X_{t_{k-1,n}}| > \varepsilon_n) \stackrel{n \to \infty}{\to} 0.$$

So $\sum_{k=1}^{n} Y_{nk} \xrightarrow{P} X_t - X_0$. On the other hand, by the second proposition, if $\liminf Var(\sum_{k=1}^{n} Y_{kn}) > 0$,

$$\frac{\sum_{k=1}^{n} Y_{kn} - E(\sum_{k=1}^{n} Y_{kn})}{\sqrt{Var(\sum_{k=1}^{n} Y_{kn})}} \to N(0,1)$$

consequently $X_t - X_0$ has a normal law (or it is a constant). We have then that the law of all increments are normal. If we take as definition of r, σ^2 that $X_1 - X_0 \sim N(r, \sigma^2)$, since increments are homogeneous and independent, and from the continuity we obtain that $X_t - X_0 \sim N(rt, \sigma^2 t)$:

$$X_1 - X_0 = \sum_{i=1}^{p} (X_{i/p} - X_{(i-1)/p}),$$

then $X_{1/p} - X_0 \sim N(r/p, \sigma^2/p)$. Analogously $X_{q/p} - X_0 \sim N(qr/p, q\sigma^2/p)$. Now we can approximate any real time t by a rational one and to apply the continuity of X. ■

Definition 1.2.5 *We say that a Brownian motion is standard if* $X_0 = 0$ P *a.s.* $r = 0$ *and* $\sigma^2 = 1$. *We shall always assume that it is standard.*

In a discrete-time model, with a single risky stock S, the discounted value of a self-financing portfolio ϕ is given by

$$\tilde{V}_n = V_0 + \sum_{j=1}^{n} \phi_j \Delta \tilde{S}_j,$$

the analogous in a continuous-time model will be

$$V_0 + \int_0^t \phi_s d\tilde{S}_s.$$

We will see that these differentials (or integrals) will be well defined whenever we have a definition of

$$\int_0^t \phi_s dW_s$$

where (W_s) is a Brownian motion. In a first glance we can think in a definition ω to ω (path-wise) but though $W_s(\omega)$ is continuous in s, it is not a function with bounded variation and we cannot associate a measure with the increments of the path to build a Lebesgue-Stieltjes integral.

Proposition 1.2.3 *The trajectories of a Brownian motion has not bounded variation with probability one.*

Proof. Given the partition $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{nn} \leq t$ de $[0,t]$ con $\lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, we have:

$$\Delta_n = \sum_{i=1}^{n} (W_{t_{in}} - W_{t_{i-1,n}})^2 \xrightarrow{L^2} t.$$

In fact:

$$\begin{aligned} E((\Delta_n - t)^2) &= E(\Delta_n^2 - 2t\Delta_n + t^2) \\ &= E(\Delta_n^2) - 2t^2 + t^2, \end{aligned}$$

but

$$\begin{aligned} E(\Delta_n^2) &= E\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(W_{t_{in}} - W_{t_{i-1,n}})^2(W_{t_{jn}} - W_{t_{j-1,n}})^2\right) \\ &= \sum_{i=1}^{n} E((W_{t_{in}} - W_{t_{i-1,n}})^4) + 2\sum_{i=1}^{n}\sum_{j<i} E((W_{t_{in}} - W_{t_{i-1,n}})^2(W_{t_{jn}} - W_{t_{j-1,n}})^2) \\ &= 3\sum_{i=1}^{n}(t_{in} - t_{i-1,n})^2 + 2\sum_{i=1}^{n}\sum_{j<i}(t_{in} - t_{i-1,})((t_{jn} - t_{j-1,n}) \\ &= t^2 + 2\sum_{i=1}^{n}(t_{in} - t_{i-1,n})^2 \end{aligned}$$

so

$$E((\Delta_n - t)^2) = 2\sum_{i=1}^{n}(t_{in} - t_{i-1,n})^2 \leq 2t \sup |t_{in} - t_{i-1,n}| \to 0.$$

Then

$$P\{|\Delta_n - t| > \varepsilon\} \leq \frac{2t \sup |t_{in} - t_{i-1,n}|}{\varepsilon^2},$$

and if the sequence of partitions is such that $\sum_{n=1}^{\infty} \sup |t_{in} - t_{i-1,n}| < \infty$, by applying the Borel-Cantelli Lemma, we have $\Delta_n \xrightarrow{a.s.} t$.

$$\sum_{i=1}^{n} |W_{t_{in}} - W_{t_{i-1n}}| \geq \frac{\sum_{i=1}^{n} |W_{t_{in}} - W_{t_{i-1,n}}|^2}{\sup_i |W_{t_{i,n}} - W_{t_{i-1,n}}|} = \frac{\Delta_n}{\sup_i |W_{t_{in}} - W_{t_{i-1,n}}|} \xrightarrow{c.s.} \frac{t}{0}.$$

■

Proposition 1.2.4 *If* (X_t) *is a Brownian motion and* $0 < t_1 < ... < t_n$ *, then* $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ *is a Gaussian vector.*

Proof. $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is a linear transformation of $(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}})$ and this is a vector of independent normal random variables. ■

Proposition 1.2.5 *If* (X_t) *is a Brownian motion then* $Cov(X_t, X_s) = s \wedge t$.

Proof. $Var(X_t - X_s) = Var(X_t) + Var(X_s) - 2Cov(X_t, X_s)$. That is $t - s = t + s - 2Cov(X_t, X_s)$. ■

Definition 1.2.6 *A continuous process* (X_t) *is an* $(\mathcal{F}_t)$*- Brownian motion if*

- X_t *es* $\mathcal{F}_t$*-measurable.*
- $X_t - X_s$ *es independent of* $\mathcal{F}_s$*,* $\quad s \leq t$.
- $X_t - X_s \sim X_{t-s} - X_0$

Example 1.2.1 *Let* (X_t) *be Brownian motion. Fixed* $T > 0$*, define* $\mathcal{F}_t = \sigma(X_s, T - t \leq s \leq T), 0 \leq t < T$ *then*

$$Y_t = X_{T-t} - X_T + \int_{T-t}^{T} \frac{X_s}{s} ds, \quad 0 \leq t < T$$

is an$(\mathcal{F}_t)$*-Brownian motion.*

Proof. It is obvious that Y is $(\mathcal{F}_t)$-adapted, continuous, Gaussian and that $Y_0 = 0$. It has independent increments, in fact, let $0 \leq u < v < T$

$$Y_v - Y_u = X_{T-v} - X_{T-u} + \int_{T-v}^{T-u} \frac{X_s}{s} ds,$$

then $E(Y_v - Y_u) = 0$ and

$$\begin{aligned}
Var(Y_v - Y_u) &= v - u + 2\int_{T-v}^{T-u} \frac{E\left(\left(X_{T-v} - X_{T-u}\right) X_s\right)}{s} ds \\
&+ 2\int_{T-v}^{T-u} \left(\int_{T-v}^{r} \frac{E\left(X_s X_r\right)}{sr} ds\right) dr \\
&= v - u + 2\int_{T-v}^{T-u} \frac{T - v - s}{s} ds \\
&+ 2\int_{T-v}^{T-u} \int_{T-v}^{r} \frac{1}{r} ds dr \\
&= v - u + 2\int_{T-v}^{T-u} \frac{T - v - s}{s} ds \\
&+ 2\int_{T-v}^{T-u} \frac{r - (T - v)}{r} dr \\
&= v - u.
\end{aligned}$$

Finally, $Y_v - Y_u$ is independent of $\mathcal{F}_u$. Since the random variables are Gaussian, it is enough to see that $E(Y_v - Y_u|\mathcal{F}_u) = 0$, but

$$E(Y_v - Y_u|\mathcal{F}_u) = E(Y_v - Y_u|X_{T-u}) = 0,$$

since

$$\begin{aligned} E((Y_v - Y_u)\, X_{T-u}) &= T - v - (T-u) + \int_{T-v}^{T-u} \frac{E(X_{T-u}X_s)}{s} du \\ &= u - v + v - u = 0. \end{aligned}$$

■

1.2.1 Continuous-time Martingales

Definition 1.2.7 *Let (M_t) be a family of adapted random variables to $(\mathcal{F}_t)$ with moments of first order, then it is:*

- *a martingale if $E(M_t|\mathcal{F}_s) = M_s$, for all $s \leq t$*
- *a submartingale if $E(M_t|\mathcal{F}_s) \geq M_s$, for all $s \leq t$*
- *a supermartingale if $E(M_t|\mathcal{F}_s) \leq M_s$, for all $s \leq t$.*

In the previous definition equalities and inequalities are *almost surely*.

Proposition 1.2.6 *If (X_t) is an $(\mathcal{F}_t)$-Brownian motion then:*

- (X_t) is an $(\mathcal{F}_t)$-martingale.
- $(X_t^2 - t)$ is an $(\mathcal{F}_t)$-martingale.
- $(\exp(\sigma X_t - \frac{\sigma^2}{2}t))$ is an $(\mathcal{F}_t)$-martingale.

Proof.

$$\begin{aligned} E(X_t|\mathcal{F}_s) &= E(X_t - X_s + X_s|\mathcal{F}_s) \\ &= E(X_t - X_s|\mathcal{F}_s) + X_s \\ &= E(X_t - X_s) + X_s = X_s, \end{aligned}$$

$$\begin{aligned} E(X_t^2 - t|\mathcal{F}_s) &= E((X_t - X_s + X_s)^2|\mathcal{F}_s) - t \\ &= E((X_t - X_s)^2 + X_s^2 + 2(X_t - X_s)|\mathcal{F}_s) - t \\ &= t - s + X_s^2 - t \\ &= X_s^2 - s, \end{aligned}$$

$$
\begin{aligned}
E(\exp(\sigma X_t - \frac{\sigma^2}{2}t)|\mathcal{F}_s) &= \exp(\sigma X_s - \frac{\sigma^2}{2}t)E(\exp(\sigma(X_t - X_s))|\mathcal{F}_s)\\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)E(\exp(\sigma(X_t - X_s))\\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\exp(\frac{\sigma^2}{2}(t-s)) \text{ (since } X_t - X_s \sim N(0,t-s))\\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}s)
\end{aligned}
$$

■

Exercise 1.2.1 *Prove that the following stochastic processes, defined from a a Brownian motion B, are martingales, respect to $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$,*

$$
\begin{aligned}
X_t &= t^2 B_t - 2\int_0^t sB_s ds\\
X_t &= e^{t/2}\cos B_t\\
X_t &= e^{t/2}\sin B_t\\
X_t &= (B_t + t)\exp(-B_t - \frac{1}{2}t)\\
X_t &= B_t^1 B_t^2.
\end{aligned}
$$

In the last case B_t^1 y B_t^2 are two independent Brownian motion and $\mathcal{F}_t = \sigma(B_s^1, B_s^2, 0 \leq s \leq t)$.

Exercise 1.2.2 *Let $c > 0$ and let $(B_t)_{t\geq 0}$ be a Brownian motion. Prove that:*

(1) $(B_{c+t} - B_c)_{\ t\geq 0}$ *is a Brownian motion.*

(2) $(cB_{t/c^2})_{\ t\geq 0}$ *is a Brownian motion.*

Exercise 1.2.3 *Let (X_t) be a Brownian motion, prove that*

$$
X_t - \int_0^t \frac{X_T - X_s}{T-s} ds, \quad 0 \leq t < T
$$

is an $(\mathcal{F}_t)$-Brownian motion between 0 and T with $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t, X_T)$.

1.2.2 Stochastic Integration

Let (W_t) be a Brownian motion, and (τ_n) a sequence of partitions: $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{nn} = t$, with $d_n := \lim_{n\to\infty} \sup|t_{in} - t_{i-1,n}| = 0$, such that for all $0 \leq s \leq t$

$$
\lim_{n\to\infty} \sum_{\substack{t_{i,n}\in\tau_n\\ t_{i,n}\leq s}} |W_{t_{in}} - W_{t_{i-1,n}}|^2 \overset{c.s.}{=} s. \tag{1.3}
$$

Let f a C^2 map in $\mathbb{R}$. Then, fixed ω,

$$
f(W_{t_{in}}) - f(W_{t_{i-1,n}}) = f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}) + \frac{1}{2}f''(W_{\tilde{t}_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})^2,
$$

where $\tilde{t}_{i-1,n} \in (t_{i-1,n}, t_{in})$. Since $f^{''}$ is uniformly continuous in a the compact set $(W_s(\omega))_{0\leq s\leq t}$, we have

$$\sum_{i=1}^{n}|f^{''}(W_{\tilde{t}_{i-1,n}})-f^{''}(W_{t_{i-1,n}})|(W_{t_{in}}-W_{t_{i-1,n}})^2 \leq \varepsilon_n \sum_{i=1}^{n}(W_{t_{in}}-W_{t_{i-1,n}})^2 \underset{n\to\infty}{\to} 0,$$

For each n, $\mu_n(A)(\omega) := \sum_{i=1}^{n} |W_{t_{in}}(\omega) - W_{t_{i-1,n}}(\omega)|^2 \mathbf{1}_A(t_{i-1,n})$ defines a measure in $[0,t]$ that converges, by (1.3), to the Lebesgue measure in $[0,t]$. So

$$\begin{aligned}\sum_{i=1}^{n} f^{''}(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})^2 &= \int_0^t f^{''}(W_s)\mu_n(ds)\\ &\underset{n\to\infty}{\to} \int_0^t f^{''}(W_s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}f(W_t) - f(0) = \lim_{n\to\infty}\sum(f(W_{t_{in}}) - f(W_{t_{i-1,n}})) &= \lim_{n\to\infty}\sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})\\ &+ \frac{1}{2}\int_0^t f^{''}(W_s)ds.\end{aligned}$$

Consequently

$$\lim_{n\to\infty}\sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})$$

is well defined since it coincides with $f(W_t) - f(0) - \frac{1}{2}\int_0^t f^{''}(W_s)ds$ and then we can define

$$\int_0^t f'(W_s)\mathrm{d}W_s = \lim_{n\to\infty}\sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}).$$

The drawback of this construction is that this integral depends on the sequences of partitions. Nevertheless if we get that our Riemann sums converge in probability or L^2) independently of the partitions we choose, the limit will be the same by the uniqueness of the limit in probability. In this way we have established that

$$\int_0^t f'(W_s)\mathrm{d}W_s = f(W_t) - f(0) - \frac{1}{2}\int_0^t f^{''}(W_s)\mathrm{d}s$$

and this result modifies the chain rule of the *classical analysis.*

Example 1.2.2

$$\int_0^t W_s\mathrm{d}W_s = \frac{1}{2}W_t^2 - \frac{1}{2}t,$$

$$\int_0^t \exp\{W_s\}\mathrm{d}W_s = \exp\{W_t\} - 1 - \frac{1}{2}\int_0^t \exp\{W_s\}ds$$

It is straightforward to see that we can extend the previous result to integrands that are $C^{1,2}$-functions $f : [0,t] \times R \to R$ in such a way that

$$f(t,W_t) = f(0,0) + \int_0^t f_t(s,W_s)\mathrm{d}s + \int_0^t f_x(s,W_s)\mathrm{d}W_s + \frac{1}{2}\int_0^t f_{xx}(s,W_s)\mathrm{d}s,$$

where

$$f_t(s,x) = \frac{\partial}{\partial t}f(t,x), \quad f_x(s,x) = \frac{\partial}{\partial x}f(t,x),$$
$$f_{xx}(s,x) = \frac{\partial^2}{\partial x^2}f(t,x).$$

Example 1.2.3 *If we take $f(t,x) = \exp(ax - \frac{1}{2}a^2 t)$, $a \in \mathbb{R}$, we have*

$$\begin{aligned}\exp(aW_t - \frac{1}{2}a^2 t) = 1 - \frac{a^2}{2}\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}s \\ + a\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}W_s \\ + \frac{a^2}{2}\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}s.\end{aligned}$$

That is,

$$\exp(aW_t - \frac{1}{2}a^2 t) = 1 + a\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}W_s.$$

Example 1.2.4 *Suppose a financial market with a single risky stock, $S_t = W_t$, and a bank account with interest rate $r = 0$. Given a strategy $\phi_t = (\phi_t^0, \phi_t^1)$ the value of our portfolio at time t, is*

$$V_t = \phi_t^0 + \phi_t^1 W_t,$$

If the strategy is self-financing we will have

$$\mathrm{d}V_t = \phi_t^1 \mathrm{d}W_t$$

Assume now that $V_t = V(t,S_t)$, then, by applying the previous stochastic calculus

$$\mathrm{d}V_t = \mathrm{d}V(t,S_t) = V_t(t,W_t)\mathrm{d}t + V_x(t,W_t)\mathrm{d}W_t + \frac{1}{2}V_{xx}(t,W_t)dt,$$

therefore

$$V_t(t,W_t) + \frac{1}{2}V_{xx}(t,W_t) = 0 \tag{1.4}$$
$$V_x(t,W_t) = \phi_t^1 \tag{1.5}$$

if we want to replicate $H = F(W_T)$, we have to find a solution of (1.4) with the boundary condition $V(T,W_T) = F(W_T)$. The equation 1.5 solves the hedging problem.

The definite integral

We are going to build a stochastic integral in the sense of L^2 convergence.

Definition 1.2.8 *$(H_t)_{0\leq t\leq T}$ is a* simple process *if it can be written*

$$H_t = \sum_{i=1}^{n} \phi_i \mathbf{1}_{(t_{i-1},t_i]}(t),$$

where $0 = t_0 < t_1 < ... < t_n = T$ and ϕ is $\left(\mathcal{F}_{t_{i-1}}\right)$-measurable and bounded.

Definition 1.2.9 *If $(H_t)_{0\leq t\leq T}$ is a simple process, we define*

$$\int_0^T H_s \mathrm{d}W_s = \sum_{i=1}^{n} \phi_i (W_{t_i} - W_{t_{i-1}})$$

Proposition 1.2.7 *If $(H_t)_{0\leq t\leq T}$ is a simple process $E(\int_0^T H_s dW_s)^2 = \int_0^T E(H_s^2)ds$ (isometry property)*

Proof.

$$\begin{aligned}
E(\int_0^T H_s \mathrm{d}W_s)^2 &= E(\sum_{i=1}^{n} \phi_i (W_{t_i} - W_{t_{i-1}}) \sum_{j=1}^{n} \phi_j (W_{t_j} - W_{t_{i-1}})) \\
&= E(\sum_{i=1}^{n} \phi_i^2 (W_{t_i} - W_{t_{i-1}})^2) \\
&+ 2 \sum_{i=1}^{n-1} \sum_{j>i} E(\phi_i (W_{t_i} - W_{t_{i-1}}) \phi_j E(W_{t_j} - W_{t_{i-1}} | \mathcal{F}_{t_{j-1}})) \\
&= \sum_{i=1}^{n} E(\phi_i^2 E(W_{t_i} - W_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}})) \\
&= \sum_{i=1}^{n} E(\phi_i^2)(t_i - t_{i-1}) = E \int_0^t H_s^2 \mathrm{d}s = \int_0^t E(H_s^2) \mathrm{d}s
\end{aligned}$$

■

Now we extend the class of simple integrands, $\mathcal{S}$ to the class $\mathcal{H}$:

$$\mathcal{H} = \{(H_t)_{0\leq t\leq T}, (\mathcal{F}_t)\text{-adaptado}, \int_0^T E(H_s^2)\mathrm{d}s < \infty\}.$$

It can be seen that the class $\mathcal{H}$ with the scalar product $\langle (H_t), (F_t) \rangle = \int_0^T E(H_s F_s)\mathrm{d}s$ is a Hilbert space. Note that, by the previous proposition, we have defined a linear map $I : \mathcal{S} \to \mathcal{M} = \{$ square integrable $\mathcal{F}_T$-measurable random variables$\}$, $I(H) = \int_0^T H_s \mathrm{d}W_s$. In $\mathcal{M}$ we can also define a scalar product producto escalar $\langle M, L \rangle := E(ML)$. We have then that I is an isometry.

Proposition 1.2.8 *The class $\mathcal{S}$ is dense in $\mathcal{H}$ (with respect to the norm $||H_t||^2 := \int_0^T E(H_s^2)ds$).*

Definition 1.2.10 *If H is a process of the class $\mathcal{H}$, the integral is defined as the L^2 limit*

$$\int_0^T H_s \mathrm{d}W_s = \lim_{n\to\infty} \int_0^T H_s^n \mathrm{d}W_s, \tag{1.6}$$

where H_s^n is a sequence of simple processes such that

$$\lim_{n\to\infty} \int_0^T E(H_s^n - H_s)^2 \mathrm{d}s = 0.$$

The existence of the limit (1.6) is due to the fact that the sequence of random variables $\int_0^T H_s^n \mathrm{d}W_s$ is a Cauchy sequence and $L^2(\Omega)$ is complete, in fact due to the isometry property

$$\begin{aligned} E(\int_0^T H_s^n \mathrm{d}W_s - \int_0^T H_s^m \mathrm{d}W_s)^2 &= \int_0^T E(H_s^n - H_s^m)^2 \mathrm{d}s \\ &\leq 2\int_0^T E(H_s^n - H_s)^2 \mathrm{d}s \\ &+ 2\int_0^T E(H_s^m - H_s)^2 \mathrm{d}s. \end{aligned}$$

Analogously it can be seen that the limit does not depend on the sequence H^n. It is easy to show that for all H in the class $\mathcal{H}$

- The isometry property is satisfied,

$$E(\int_0^T H_s dW_s)^2 = \int_0^T E(H_s^2) ds,$$

- The integral has zero expectation,

$$E(\int_0^T H_s dW_s) = 0,$$

- The integral is linear,

$$\int_0^T (aH_s + bF_s)dW_s = a\int_0^T H_s dW_s + b\int_0^T F_s dW_s$$

The indefinite integral

If H is in the class $\mathcal{H}$ then $H\mathbf{1}_{[0,t]}$ it is as well and we can define

$$\int_0^t H_s dW_s := \int_0^T H_s \mathbf{1}_{[0,t]}(s) dW_s,$$

we have then the process

$$\left\{I(H)_t := \int_0^t H_s dW_s, 0 \le t \le T\right\}$$

Proposition 1.2.9 *$I(H)$ is an $(\mathcal{F}_t)$-martingale.*

Proof. The result is obvious if H i simple: then $\int_0^t H_s dW_s$ is $\mathcal{F}_t$-measurable and with finite expectation, then it is sufficient to see that $\forall t > s$

$$E\left(\int_0^t H_u \mathrm{d}W_u \middle| \mathcal{F}_s\right) = \int_0^s H_u \mathrm{d}W_u.$$

We can asume that s and t are some of the points in the partition $0 = t_0 < t_1 < ... < t_n = T$. So it is enough to see that $(M_n) := \left(\int_0^{t_n} H_u \mathrm{d}W_u\right)$ is a $(\mathcal{G}_n)$-martingale with $G_n = \mathcal{F}_{t_n}$. But (M_n) is the martingale transform of the $(\mathcal{G}_n)$-martingale (W_{t_n}) by the process $(\mathcal{G}_n)$-predictable (ϕ_n) and consequently it is a martingale.

If H is not a simple process the integral is an L^2 limit of martingales but this preserves the martingale property. ■

Remark 1.2.7 *If can be shown, by using the Doob inequality for continuous martingales:*

$$E(\sup_{0\le t\le T} M_t^2) \le 4E(M_T^2)$$

that we have a continuous version of $I(H)$.

Remark 1.2.8 *We shall denote $\forall t > s$, $\int_s^t H_u dW_u := \int_0^t H_u dW_u - \int_0^s H_u dW_u$.*

To do a further extension of the integral the following results are convenient

Proposition 1.2.10 *Let A $\mathcal{F}_t$-measurable, then for all $H \in \mathcal{H}$*

$$\int_0^T \mathbf{1}_A H_s \mathbf{1}_{\{s>t\}} \mathrm{d}W_s = \mathbf{1}_A \int_t^T H_s \mathrm{d}W_s$$

Proof. If H^n is an approximate sequence of H then $1_A H^n 1_{\{\cdot>t\}}$ approximates $1_A H 1_{\{\cdot>t\}}$ and since the result is true for simple processes then the proposition follows. ■

Definition 1.2.11 *A* stopping time *with respect to a filtration $(\mathcal{F}_t)$ is a random variable*

$$\tau : \Omega \to [0, \infty]$$

such that for all $t \ge 0$, $\{\tau \le t\} \in \mathcal{F}_t$.

Proposition 1.2.11 *Let τ be an $(\mathcal{F}_t)$-topping time then*

$$\int_0^{\tau\wedge T} H_s \mathrm{d}W_s = \int_0^T \mathbf{1}_{\{s\le\tau\}} H_s \mathrm{d}W_s$$

Proof. If τ is of the form $\tau = \sum_{i=1}^{n} t_i 1_{A_i}$ where $0 < t_1 < t_2 < ... < t_n = T$ y A_i $\mathcal{F}_{t_i}$-measurable and disjoints, then it is straightforward:

$$\int_0^T \mathbf{1}_{\{s>\tau\}} H_s \mathrm{d}W_s = \int_0^T \sum_{i=1}^{n} \mathbf{1}_{\{s>t_i\}} \mathbf{1}_{A_i} H_s \mathrm{d}W_s = \sum_{i=1}^{n} \mathbf{1}_{A_i} \int_{t_i \wedge T}^{T} H_s \mathrm{d}W_s$$
$$= \int_{\tau \wedge T}^{T} H_s \mathrm{d}W_s,$$

Moreover $\int_0^{\tau\wedge T} H_s \mathrm{d}W_s = \int_0^T H_s \mathrm{d}W_s - \int_{\tau\wedge T}^T H_s \mathrm{d}W_s$. In general, it is enough to approximate τ by $\tau_n = \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n} 1_{\{\frac{kT}{2^n} \leq \tau < \frac{(k+1)T}{2^n}\}}$ and to see that $\int_0^T 1_{\{s\leq\tau_n\}} H_s dW_s \xrightarrow{L^2} \int_0^T 1_{\{s\leq\tau\}} H_s dW_s$:

$$E\left(\left|\int_0^T \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s - \int_0^T \mathbf{1}_{\{s\leq\tau\}} H_s \mathrm{d}W_s\right|^2\right) = E\left(\int_0^T \mathbf{1}_{\{\tau<s\leq\tau_n\}} H_s^2 \mathrm{d}s\right),$$

and the we apply the dominated convergence theorem. Finally we take a subsequence of $\int_0^T 1_{\{s\leq\tau_n\}} H_s dW_s$ converging almost surely. ■

Extension of the integral

We are going to do a further extension of the integrands, consider the class

$$\tilde{\mathcal{H}} = \{(H_t)_{0\leq t\leq T}, (\mathcal{F}_t)\text{-adapted}, \int_0^T H_s^2 \mathrm{d}s < \infty \ P\text{-c.s.}\}.$$

Given $H \in \tilde{\mathcal{H}}$ sea $\tau_n = \inf\{t \leq T, \int_0^t (H_s)^2 ds \geq n\}$ ($+\infty$ if the previous set is empty). That $\int_0^t (H_s)^2 ds$ is $\mathcal{F}_t$-measurable can be deduced from the fact that it is an a.s. limit of $\mathcal{F}_t$-measurable random variables, from here τ_n is a stopping time. Set $A_n = \{\int_0^T (H_s)^2 ds < n\}$ we can define

$$\tilde{J}(H)_t^n := \left(\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s\right) \mathbf{1}_{A_n}, \text{ para todo } n \geq 1$$

Note that this is consistently defined: if $m \geq n$ and $\omega \in A_n$ then

$$\tilde{J}(H)_t^m(\omega) = \tilde{J}(H)_t^n(\omega),$$

in fact:

$$\tilde{J}(H)_t^m(\omega) = \int_0^{t\wedge\tau_n(\omega)} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s,$$

but

$$\int_0^{t\wedge\tau_n} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s = \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s$$
$$= \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s,$$

in such a way that

$$\int_0^{t\wedge\tau_n(\omega)} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s = \tilde{J}(H)_t^n(\omega)$$

Now we can define

$$\tilde{J}(H)_t = \lim_{n\to\infty}\left(\left(\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s\right)\mathbf{1}_{A_n}\right) = \lim_{n\to\infty}\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s.$$

Note that if $H \in \mathcal{H}$

$$\begin{aligned}\tilde{J}(H)_t &= \lim_{n\to\infty}\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s = \lim_{n\to\infty}\int_0^{t\wedge\tau_n} H_s \mathrm{d}W_s \\ &= \int_0^t H_s \mathrm{d}W_s = J(H)_t,\end{aligned}$$

so it is really an *extension* of the integral.

Exercise 1.2.4 *Prove that the previous extension of the integral does not depend on the sequence of localizing stopping times of (H_s). In other words, that if we take $\tilde{\tau}_n \uparrow \infty$ and $\left(\mathbf{1}_{\{\cdot<\tilde{\tau}_n\}} H_\cdot\right)$ is in $\mathcal{H}$) then the limit is the same.*

It can be shown that the previous extension is a limit in probability of integrals fo simple processes H^n which converge to H in the sense that

$$P(\int_0^t |H_s^n - H_s|^2 ds > \varepsilon) \to 0.$$

Note that by construction the extension of the integral is a a.s. limit of another limit in quadratic norm.

We lose then the martingale property. In general we have that if (τ_m) is a localizing sequence

$$\begin{aligned}\tilde{J}(H)_{t\wedge\tau_m} &= \lim_{n\to\infty}\int_0^{t\wedge\tau_m} \mathbf{1}_{\{s\leq\tilde{\tau}_n\}} H_s \mathrm{d}W_s \\ &= \lim_{n\to\infty}\int_0^t \mathbf{1}_{\{s\leq\tilde{\tau}_n\wedge\tau_m\}} H_s \mathrm{d}W_s \\ &= \lim_{n\to\infty}\int_0^t \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s \\ &= \int_0^t \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s\end{aligned}$$

in such a way that $\tilde{J}(H)_{t\wedge\tau_m}$ is a martingale. Then it is said that $\tilde{J}(H)$ is a *local martingale*(when we stop it by τ_m it is a martingale, *in* t, and $\tau_m \uparrow \infty$).

1.2.3 Itô's Calculus

We are going to develop a differential calculus based in the previous integral. We have seen that

$$\int_0^t f'(W_s)\mathrm{d}W_s = f(W_t) - f(W_0) - \frac{1}{2}\int_0^t f^{''}(W_s)\mathrm{d}s$$

for $f \in C^2$, or in differential form

$$\mathrm{d}f(W_t) = f'(W_s)\mathrm{d}W_s + \frac{1}{2}f''(W_t)\mathrm{d}t \tag{1.7}$$

Then we want to extend this result.

Definition 1.2.12 *A process $(X_t)_{0\leq t\leq T}$ is said to be an Itô process if it can be written as*

$$X_t = X_0 + \int_0^t K_s\mathrm{d}s + \int_0^t H_s\mathrm{d}W_s$$

where

- *X_0 is $\mathcal{F}_0$-measurable.*
- *(K_t) and (H_t) are $(\mathcal{F}_t)$-adapted.*
- *$\int_0^T(|K_s| + |H_s|^2)\mathrm{d}s < \infty$ P-a.s..*

Proposition 1.2.12 *If $(M_t)_{0\leq t\leq T}$ is a continuous $(\mathcal{F}_t)$-martingale such that $M_t = \int_0^t K_s\mathrm{d}s$, where (K_s) is an $(\mathcal{F}_t)$-adapted process with $\int_0^T |K_s|\mathrm{d}s < \infty$ P-a.s., then*

$$M_t = 0, \ \textit{a.s for all } t \leq T$$

Proof. Without loss of generality we can assume that $M_t = \int_0^t |K_s|\mathrm{d}s \leq n < \infty$. Otherwise we can define the stopping time

$$\tau_n = \inf\{t, \int_0^t |K_s|\mathrm{d}s \geq n\} \wedge T,$$

and the martingale $(M_{t\wedge\tau})$ would be bounded by n. This would make $M_{t\wedge\tau_n} \equiv 0$ and we can let n go to infinity to conclude that $M_t \equiv 0$.

Let $(M_t)_{0\leq t\leq T}$ be a continuous $(\mathcal{F}_t)$-martingale bounded by C, then if we take $t_i^n = T\frac{i}{n}, 0 \leq i \leq n$, then

$$\begin{aligned}\sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2 &\leq \sup_{1\leq i\leq n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| \sum_{i=1}^n \left|M_{t_i^n} - M_{t_{i-1}^n}\right| \\ &\leq \sup_{1\leq i\leq n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| \sum_{i=1}^n \int_{t_i^n}^{t_{i+1}^n} |K_s|\mathrm{d}s \\ &\leq C \sup_{1\leq i\leq n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right|\end{aligned}$$

and $(M_t)_{0\le t\le T}$ is continuous, so

$$\lim_{n\to\infty}\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2=0,\ \text{a.s.},$$

Moreover $\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2\le C^2$, so by the dominated convergence theorem

$$\lim_{n\to\infty}E(\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2)=0.$$

On the other hand, since $(M_t)_{0\le t\le T}$ is a martingale and simultaneously

$$\begin{aligned}
E(\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2) &= E\left(\sum_{i=1}^{n}(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_i^n}M_{t_{i-1}^n})\right)\\
&= E\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_{i-1}^n}E(M_{t_i^n}|\mathcal{F}_{t_{i-1}^n})\right)\right)\\
&= E\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_{i-1}^n}^2\right)\right)\\
&= E\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2-M_{t_{i-1}^n}^2\right)\right)\\
&= E(M_T^2-M_0^2)
\end{aligned}$$

and that consequently that $M_T\equiv 0$ a.s., and so $M_t\equiv E(M_T|\mathcal{F}_t)=0$ a.s, for all $t\le T$. ∎

Corollary 1.2.1 *The expression of an Itô process is unique.*

Theorem 1.2.2 *Let $(X_t)_{0\le t\le T}$ be an Itô process and $f(t,x)\in C^{1,2}$ then:*

$$f(t,X_t)=f(0,X_0)+\int_0^t f_t(s,X_s)\mathrm{d}s+\int_0^t f_x(s,X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t f_{xx}(s,X_s)\mathrm{d}\langle X,X\rangle_s,$$

where

$$\begin{aligned}
\int_0^t f_x(s,X_s)\mathrm{d}X_s &= \int_0^t f_x(s,X_s)K_s\mathrm{d}s+\int_0^t f_x(s,X_s)H_s\mathrm{d}W_s\\
\langle X,X\rangle_s &= \int_0^t H_s^2\mathrm{d}s.
\end{aligned}$$

Example 1.2.5 *Suppose we want to find a solution $(S_t)_{0\le t\le T}$ for the equation*

$$S_t=x_0+\int_0^t S_s(\mu\mathrm{d}s+\sigma\mathrm{d}W_s)$$

or in differential form

$$\mathrm{d}S_t = S_t(\mu\mathrm{d}t + \sigma\mathrm{d}W_t), \quad S_0 = x_0.$$

By the previous theorem

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t = \mathrm{d}(\log S_t) + \frac{1}{2S_t^2}\sigma^2 S_t^2\mathrm{d}t,$$

that is

$$\mathrm{d}(\log S_t) = (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma\mathrm{d}W_t$$

in such a way that

$$S_t = S_0\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$$

Proposition 1.2.13 *(Integration by parts formula) Let X_t and Y_t two Itô processes, $X_t = X_0 + \int_0^t K_s\mathrm{d}s + \int_0^t H_s\mathrm{d}W_s$ e $Y_t = Y_0 + \int_0^t K_s'\mathrm{d}s + \int_0^t H_s'\mathrm{d}W_s$. Then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s\mathrm{d}Y_s + \int_0^t Y_s\mathrm{d}X_s + \langle X, Y\rangle_t$$

where

$$\langle X, Y\rangle_t = \int_0^t H_sH_s'\mathrm{d}s.$$

Proof. By the Itô formula

$$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2\int_0^t (X_s + Y_s)\mathrm{d}(X_s + Y_s) + \frac{1}{2}\int_0^t 2(H_s + H_s')^2\mathrm{d}s$$

and

$$X_t{}^2 = X_0^2 + 2\int_0^t X_s\mathrm{d}X_s + \frac{1}{2}\int_0^t 2H_s^2\mathrm{d}s,$$

$$Y_t{}^2 = Y_0^2 + 2\int_0^t Y_s\mathrm{d}Y_s + \frac{1}{2}\int_0^t 2H_s'^2\mathrm{d}s$$

so, by subtracting the sum of these latter expressions from the first one we obtain:

$$2X_tY_t = 2X_0Y_0 + 2\int_0^t X_s\mathrm{d}Y_s + 2\int_0^t Y_s\mathrm{d}X_s + \int_0^t 2H_sH_s'\mathrm{d}s.$$

■

Consider the differential equation

$$\mathrm{d}X_t = -cX_t\mathrm{d}t + \sigma\mathrm{d}W_t, X_0 = x$$

then if we apply the previous formula to

$$X_te^{ct}$$

we have

$$\mathrm{d}\left(X_t e^{ct}\right) = e^{ct}\mathrm{d}X_t + cX_t e^{ct}\mathrm{d}t$$

and therefore

$$e^{-ct}\mathrm{d}\left(X_t e^{ct}\right) = \sigma \mathrm{d}W_t$$

in such a way that

$$X_t = xe^{-ct} + \sigma e^{-ct}\int_0^t e^{cs}\mathrm{d}W_s.$$

A new integration by parts lead us to

$$X_t = xe^{-ct} + \sigma e^{-ct}(e^{ct}W_t - \int_0^t ce^{cs}W_s\mathrm{d}s).$$

So, it is a Gaussian process with expectation xe^{-ct} and variance

$$\begin{aligned}\mathrm{Var}(X_t) &= \sigma^2 e^{-2ct}\int_0^t e^{2cs}\mathrm{d}s \\ &= \sigma^2\frac{1-e^{-2ct}}{2c}.\end{aligned}$$

Exercise 1.2.5 *Solve the stochastic differential equation*

$$\mathrm{d}X_t = tX_t\mathrm{d}t + e^{t^2/2}\mathrm{d}B_t,\ X_0 = x_0,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion.

1.2.4 The Girsanov theorem

Lemma 1.2.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, $\mathcal{F}_T = \mathcal{F}$. Let $Z_T > 0$ such that $E(Z_T) = 1$ and $Z_t := E(Z_T|\mathcal{F}_t), 0 \leq t \leq T$. Then if we define $\tilde{P}(A) := E(\mathbf{1}_A Z_T), \forall A \in \mathcal{F}$, and Y is an $\mathcal{F}_t$-measurable such that $\tilde{E}(|Y|) < \infty$ then, for all $s \leq t$,*

$$\tilde{E}(Y|\mathcal{F}_s) = \frac{1}{Z_s}E(YZ_t|\mathcal{F}_s). \tag{1.8}$$

Proof. Take $A \in \mathcal{F}_s$ then

$$\begin{aligned}\tilde{E}(\mathbf{1}_A Y) &= E(\mathbf{1}_A Y Z_T) = E(\mathbf{1}_A E(YZ_t|\mathcal{F}_s)) \\ &= \tilde{E}(\mathbf{1}_A\frac{1}{Z_s}E(YZ_t|\mathcal{F}_s)).\end{aligned}$$

■

Theorem 1.2.3 *(Girsanov) Consider a probability space as before and* $(\theta_t)_{0\leq t\leq T}$ *an adapted process such thate* $\int_0^T \theta_t^2 \mathrm{d}t < \infty$ *a.s. where*

$$Z_t := \exp\{\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s\},$$

is assumed to be a martingale and W *is an* $(\mathcal{F}_t)$*-Brownian motion. Then under the probability* $\tilde{P}(\cdot) := E(\mathbf{1}.Z_T)$, $X_t = W_t - \int_0^t \theta_s \mathrm{d}s, 0 \leq t \leq T$, *is an* $(\mathcal{F}_t)$*-Brownian motion.*

Proof. $(X_t)_{0\leq t\leq T}$ is adapted and continuous. We can see that the increments are independent and homogeneous.

$$\begin{aligned}
&\tilde{E}(\exp\{iu(X_t - X_s)\}|\mathcal{F}_s) \\
&= \frac{1}{Z_s} E(\exp\{iu(X_t - X_s)\} Z_t|\mathcal{F}_s) \\
&= E(\exp\{\int_s^t (iu + \theta_u)\mathrm{d}W_u - \frac{1}{2}\int_s^t (2iu\theta_u + \theta_u^2)\mathrm{d}u\}|\mathcal{F}_s).
\end{aligned}$$

But, if we write

$$N_t := \exp\{iuX_t\}$$

and we apply the Itô formula to

$$Z_t N_t = \exp\{\int_0^t (iu + \theta_s)\mathrm{d}W_s - \frac{1}{2}\int_0^t (2iu\theta_s + \theta_s^2)\mathrm{d}s\}$$

we obtain

$$\begin{aligned}
&Z_t N_t \\
&= 1 + \int_0^t Z_s N_s \left((iu + \theta_s)\mathrm{d}W_s - \frac{1}{2}(2iu\theta_s + \theta_s^2)\mathrm{d}s\right) + \frac{1}{2}\int_0^t Z_s N_s (iu + \theta_s)^2 \mathrm{d}s \\
&= 1 + \int_0^t Z_s N_s (iu + \theta_s)\mathrm{d}W_s - \frac{u^2}{2}\int_0^t Z_s N_s \mathrm{d}s.
\end{aligned}$$

Then (by localizing with $\tau_n = \inf\{t \leq T, \int_0^t |(Z_s N_s(iu + \theta_s))|^2 \mathrm{d}s \geq n\}$)

$$E(Z_{t\wedge\tau_n} N_{t\wedge\tau_n}|\mathcal{F}_s) = Z_{s\wedge\tau_n} N_{s\wedge\tau_n} - \frac{u^2}{2} E\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} Z_v N_v \mathrm{d}v \middle| \mathcal{F}_s\right).$$

That is

$$\tilde{E}(N_{t\wedge\tau_n}|\mathcal{F}_s) = N_{s\wedge\tau_n} - \frac{u^2}{2}\tilde{E}\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} N_v \mathrm{d}v \middle| \mathcal{F}_s\right),$$

taking now the limit when $n \to \infty$ and by the dominated convergence and Fubini theorems, we obtain

$$\tilde{E}(\frac{N_t}{N_s}|\mathcal{F}_s) = 1 - \frac{u^2}{2}\int_s^t \tilde{E}(\frac{N_v}{N_s}|\mathcal{F}_s)\mathrm{d}v.$$

This gives an equation for $g_s(t) := \tilde{E}(\frac{N_t}{N_s}|\mathcal{F}_s)(\omega)$, such that

$$
\begin{aligned}
g_s'(t) &= -\frac{u^2}{2} g_s(t) \\
g_s(s) &= 1
\end{aligned}
$$

De manera que

$$g_s(t) = \exp\{-\frac{u^2}{2}(t-s)\}$$

that is

$$\tilde{E}(\exp\{iu(X_t - X_s)\}|\mathcal{F}_s) = \exp\{-\frac{u^2}{2}(t-s),\}$$

so the increments are independent and homogeneous with law $\mathrm{N}(0, t-s)$. ■

Exercise 1.2.6 *Consider the process* $(S_t)_{0\le t\le T}$

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right),\ 0 \le t \le T,$$

$(B_t)_{0\le t\le T}$ *a standard Brownina motion. Using Girsanov's theorem compute a probability* $\mathbb{Q}$ *under which* $\tilde{S}_t := S_t e^{-rt}$, $0 \le t \le T$ *is a martingale.*

1.2.5 The Black-Scholes model

The Samuelson model, more known as the Black-Scholes model, consist in a model of financial market with two stocks. One without risk, S^0, (or bank account) that evolves as:

$$\mathrm{d}S_t^0 = rS_t^0 \mathrm{d}t, \quad t \ge 0$$

where r is a non-negative constant, that is

$$S_t^0 = e^{rt}, \quad t \ge 0$$

and a risky stock S that evolves as

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right) \quad t \ge 0$$

where(B_t) is a Brownian motion. As we haven seen this implies that

$$S_t = S_0 \exp\{\mu t - \frac{\sigma^2}{2}t + \sigma B_t\}.$$

Then $\log(S_t)$ is a Brownian motion, no necessarily standard, and by the properties of the Brownian motion we have that S_t :

- has continuous trajectories

- the relative increments $\frac{S_t - S_u}{S_u}$ are independent of $\sigma(S_s, 0 \le s \le u)$:

$$\frac{S_t - S_u}{S_u} = \frac{S_t}{S_u} - 1$$

and

$$\frac{S_t}{S_u} = \exp\{\mu(t-u) - \frac{\sigma^2}{2}(t-u) + \sigma(B_t - B_u)\}$$

that is independent of $\sigma(B_s, 0 \le s \le u) = \sigma(S_s, 0 \le s \le u)$.

- the relative increments are homogeneous:

$$\frac{S_t - S_u}{S_u} \sim \frac{S_{t-u} - S_0}{S_0}.$$

In fact we could formulate the model in terms of these three hypothesis.

Self-financing strategies

A strategy is a process $\phi = (\phi_t)_{0 \le t \le T} = \left(\left(H_t^0, H_t\right)\right)_{0 \le t \le T}$ with values in $\mathbb{R}^2$ adapted to the natural filtration generated by the Brownian motion, (B_t), (that coincides with that generated by (S_t)), the value of the portfolio is

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t.$$

In the discrete-time setting, we said that the portfolio was self-financing if

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1}^0(S_{n+1}^0 - S_n^0) + \phi_{n+1}(S_{n+1} - S_n),$$

the corresponding version in the continuous case will be:

$$\mathrm{d}V_t = H_t^0 \mathrm{d}S_t^0 + H_t \mathrm{d}S_t.$$

To give sense to this equality we put the condition: $\int_0^T \left(|H_s^0| + H_s^2\right) \mathrm{d}s < \infty$ P c.s., then the integrals (differencials) are well defined:

$$\int_0^T H_t^0 \mathrm{d}S_t^0 = \int_0^T H_t^0 r e^{rt} \mathrm{d}t$$
$$\int_0^T H_t \mathrm{d}S_t = \int_0^T H_t S_t \mu \mathrm{d}t + \int_0^T \sigma H_t S_t \mathrm{d}B_t.$$

We have then then the following definition

Definition 1.2.13 *A self-financing strategy ϕ, is a pair of adapted processes $\left(H_t^0\right)_{0 \le t \le T}, (H_t)_{0 \le t \le T}$ that satisfy*

- $\int_0^T \left(|H_s^0| + H_s^2\right) \mathrm{d}s < \infty$ P *a.s.*
- $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_{0t} S_0 + \int_0^t H_s^0 r e^{rs} \mathrm{d}s + \int_0^t H_s \mathrm{d}S_s, \quad 0 \le t \le T.$

Denote $\tilde{S}_t = e^{-rt}S_t$, in such a way that we use the tilde as in the discrete-time setting: to indicate any discounted value.

Proposition 1.2.14 *ϕ is self-financing strategy if and only if:*

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s \mathrm{d}\tilde{S}_s$$

Proof. Suppose that ϕ is self-financing, then since $\tilde{V}_t = e^{-rt}V_t$, we will have that

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t \\
&= -re^{-rt}(H_t^0 S_t^0 + H_t S_t)\mathrm{d}t \\
&+ e^{-rt}(H_t^0 \mathrm{d}S_t^0 + H_t \mathrm{d}S_t) \\
&= -re^{-rt}(H_t^0 S_t^0 + H_t S_t)\mathrm{d}t \\
&+ e^{-rt}(H_t^0 r S_t^0 \mathrm{d}t + H_t \mathrm{d}S_t) \\
&= -re^{-rt}H_t S_t \mathrm{d}t + e^{-rt}H_t \mathrm{d}S_t \\
&= H_t(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t) \\
&= H_t \mathrm{d}\tilde{S}_t.
\end{aligned}$$

Analogously if

$$\mathrm{d}\tilde{V}_t = H_t \mathrm{d}\tilde{S}_t$$

we have that

$$\mathrm{d}V_t = H_t^0 \mathrm{d}S_t^0 + H_t \mathrm{d}S_t.$$

■

Pricing and hedging contingent claims in the Black-Scholes model

We have to find a probability under which discounted prices are martingale. We know that

$$\begin{aligned}
\mathrm{d}\tilde{S}_t &= \mathrm{d}\left(e^{-rt}S_t\right) = -re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t \\
&= e^{-rt}S_t\left(-r\mathrm{d}t + \mu\mathrm{d}t + \sigma\mathrm{d}B_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}\left(-\frac{r-\mu}{\sigma}t + B_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}W_t \qquad (1.9)
\end{aligned}$$

with

$$W_t = B_t - \frac{r-\mu}{\sigma}t.$$

Then by the Girsanov theorem with $\theta_t = \frac{r-\mu}{\sigma}$ it turns out that $(W_t)_{0\le t\le T}$ is a Brownian motion with respect to the probability P^*

$$\mathrm{d}P^* = \exp\{\frac{r-\mu}{\sigma}B_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\}\mathrm{d}P. \qquad (1.10)$$

From (1.9) we deduce that

$$\tilde{S}_t = S_0 \exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t\}$$

and that $\left(\tilde{S}_t\right)_{0\le t\le T}$ is a P^*-martingale. We also have that

$$S_t = S_0 \exp\{rt - \frac{1}{2}\sigma^2 t + \sigma W_t\}.$$

Definition 1.2.14 *A strategy ϕ is admissible if it is self-financing and its discounted value $\tilde{V}_t = H_t^0 + H_t \tilde{S}_t \ge 0, \forall t$.*

Definition 1.2.15 *We say that an option is replicable if its payoff is equal to the final value of an admissible strategy.*

Proposition 1.2.15 *In the Black-Scholes model any option with payoff (non negative) of the form $h = f(S_T)$, square integrable with respect to P^*, with $E_{P^*}(h|\mathcal{F}_t)$ a $C^{1,2}$ function of the time and of S_t, is replicable, its value is given by $C(t,S_t) = E_{P^*}(e^{-r(T-t)}h|\mathcal{F}_t)$ and the strategy that replicates h is given by (H_t^0, H_t) con*

$$H_t = \frac{\partial C(t,S_t)}{\partial S_t}$$
$$H_t^0 e^{rt} = C(t,S_t) - H_t S_t$$

Proof. First of all, by the independence of the relative increments

$$\begin{aligned} E_{P^*}(e^{-r(T-t)}f(S_T)|\mathcal{F}_t) &= E_{P^*}(e^{-r(T-t)}f(\frac{S_T}{S_t}S_t)|\mathcal{F}_t) \\ &= E_{P^*}(e^{-r(T-t)}f(\frac{S_T}{S_t}x))_{x=S_t} \\ &= C(t,S_t), \end{aligned}$$

so what we shall call price of the contingent claim at t depends only on S_t and t.

If we apply now the Itô formula to $\bar{C}(t,\tilde{S}_t) = e^{-rt}C(t,\tilde{S}_t e^{rt})$, we have

$$\begin{aligned} &\bar{C}(t,\tilde{S}_t) \\ &= C(0,S_0) + \int_0^t \frac{\partial \bar{C}(s,\tilde{S}_s)}{\partial s}\mathrm{d}s + \int_0^t \frac{\partial \bar{C}(s,\tilde{S}_s)}{\partial \tilde{S}_s}\mathrm{d}\tilde{S}_s + \frac{1}{2}\int_0^t \frac{\partial^2 \bar{C}(s,\tilde{S}_s)}{\partial \tilde{S}_s^2}\mathrm{d}\langle \tilde{S},\tilde{S}\rangle_s \end{aligned}$$

and since

$$\mathrm{d}\tilde{S}_t = \sigma \tilde{S}_t \mathrm{d}W_t$$

we obtain

$$\begin{aligned} &\bar{C}(t,\tilde{S}_t) \\ &= C(0,S_0) + \int_0^t \frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_s}\sigma \tilde{S}_s \mathrm{d}W_s + \int_0^t \left(\frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial s} + \frac{1}{2}\frac{\partial^2 \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_s^2}\sigma^2 \tilde{S}_s^2\right)\mathrm{d}s \end{aligned}$$

but $\bar{C}(t,\tilde{S}_t)$ is a square integrable martingale:

$$\bar{C}(t,\tilde{S}_t)=\tilde{C}(t,S_t)=E_{P^*}(e^{-rT}f(S_T)|\mathcal{F}_t)$$

and therefore, since the decomposition of an Itô process is unique we have:

$$\tilde{C}(t,S_t)=C(0,S_0)+\int_0^t \frac{\partial\bar{C}(t,\tilde{S}_t)}{\partial\tilde{S}_s}\mathrm{d}\tilde{S}_s$$

$$\frac{\partial\bar{C}(t,\tilde{S}_t)}{\partial s}+\frac{1}{2}\frac{\partial^2\bar{C}(t,\tilde{S}_t)}{\partial\tilde{S}_s^2}\sigma^2\tilde{S}_s^2=0.$$

Now since

$$\begin{aligned}\frac{\partial\bar{C}(t,\tilde{S}_t)}{\partial\tilde{S}_s}&=e^{-rt}\frac{\partial C(s,S_s)}{\partial S_s}\frac{\partial S_s}{\partial\tilde{S}_s}\\&=\frac{\partial C(s,S_s)}{\partial S_s}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\bar{C}(t,\tilde{S}_t)}{\partial\tilde{S}_s^2}&=\frac{\partial^2 C(s,S_s)}{\partial S_s^2}\frac{\partial S_s}{\partial\tilde{S}_s}\\&=e^{rt}\frac{\partial^2 C(s,S_s)}{\partial S_s^2},\end{aligned}$$

we can write

$$\tilde{C}(t,S_t)=C(0,S_0)+\int_0^t \frac{\partial C(s,S_s)}{\partial S_s}\mathrm{d}\tilde{S}_s \tag{1.11}$$

$$\frac{\partial C(s,S_s)}{\partial s}+rS_s\frac{\partial C(s,S_s)}{\partial S_s}+\frac{1}{2}\sigma^2S_s^2\frac{\partial^2 C(t,S_s)}{\partial S_s^2}=rC(s,S_s). \tag{1.12}$$

From (1.11) we have a self-financing strategy whose final value is $f(S_T)$ and such that $\left(H_t^0,H_t\right)$ are given by

$$H_t=\frac{\partial C(t,S_t)}{\partial S_t}$$

and

$$e^{rt}H_t^0=C(t,S_t)-\frac{\partial C(t,S_t)}{\partial S_t}S_t.$$

■

Pricing and hedging of a call option. The Black-Scholes tormula.

If we take $h=(S_T-K)_+$, we have

$$C(t,S_t)=S_t\Phi(d_+)-Ke^{-r(T-t)}\Phi(d_-)\ (\text{ Black-Scholes' formula})$$

where $\Phi(x)$ is the standard normal distribution function

$$d_{\pm} = \frac{\log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

In fact

$$\begin{aligned}
&C(t, S_t) \\
&= E_{P^*}(e^{-r(T-t)}(S_T - K)_+ | \mathcal{F}_t) \\
&= e^{-r(T-t)} E_{P^*}(S_T \mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t) - K e^{-r(T-t)} E_{P^*}(\mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t) \\
&= e^{-r(T-t)} S_t E_{P^*}(\frac{S_T}{S_t} \mathbf{1}_{\{\frac{S_T}{S_t} > \frac{K}{x}\}})_{x=S_t} - K e^{-r(T-t)} E_{P^*}(\mathbf{1}_{\{\frac{S_T}{S_t} > \frac{K}{x}\}})_{x=S_t},
\end{aligned}$$

but

$$\begin{aligned}
\frac{S_T}{S_t} &= \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\,(W_T - W_t)\} \\
&\overset{\text{Ley}}{=} \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}\}
\end{aligned}$$

then

$$\begin{aligned}
E_{P^*}(\mathbf{1}_{\{\frac{S_T}{S_t} > \frac{K}{x}\}}) &= P^*(\frac{S_T}{S_t} > \frac{K}{x}) \\
&= P^*(\log \frac{S_T}{S_t} > \log \frac{K}{x}) \\
&= P^*(\frac{W_{T-t}}{\sqrt{(T-t)}} > \frac{\log \frac{K}{x} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}) \\
&= \Phi\left(\frac{\log \frac{x}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi(d_-) \text{ (after substituting for} x \text{ by } S_t)
\end{aligned}$$

On the other hand, if we write Y to indicate a standard normal random variable

$$
\begin{aligned}
&e^{-r(T-t)}E_{P^*}(\frac{S_T}{S_t}\mathbf{1}_{\{\frac{S_T}{S_t}>\frac{K}{x}\}}) \\
&= e^{-r(T-t)}E_{P^*}(\exp\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}\}\mathbf{1}_{\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\}}) \\
&= E_{P^*}(\exp\{-\frac{1}{2}\sigma^2(T-t)+\sigma W_{T-t}\}\mathbf{1}_{\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\}}) \\
&= E_{P^*}(\exp\{-\frac{1}{2}\sigma^2(T-t)-\sigma\sqrt{(T-t)}Y\}\mathbf{1}_{\{Y<\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\}}) \\
&= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}} \exp\{-\frac{1}{2}\sigma^2(T-t)-\sigma\sqrt{(T-t)}y-\frac{1}{2}y^2\}\mathrm{d}y \\
&= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}} \exp\{-\frac{1}{2}(\sigma\sqrt{(T-t)}+y)^2\}\mathrm{d}y \\
&= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}} \exp\{-\frac{1}{2}u^2\}\mathrm{d}u \\
&= \Phi(d_+) \text{ (after substituting for } x \text{ by } S_t)
\end{aligned}
$$

From here

$$\frac{\partial C(t,S_t)}{\partial S_t} = \Phi(d_+) := \Delta.$$

In fact:

$$
\begin{aligned}
\frac{\partial C(t,S_t)}{\partial S_t} &= \Phi(d_+) + S_t\frac{\partial\Phi(d_+)}{\partial S_t} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial S_t} \\
&= \Phi(d_+) + S_t\frac{1}{\sqrt{(2\pi)}}e^{-\frac{d_+^2}{2}}\frac{\partial d_+}{\partial S_t} \\
&\quad - Ke^{-r(T-t)}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{d_-^2}{2}}\frac{\partial d_-}{\partial S_t}.
\end{aligned}
$$

But

$$\frac{\partial d_\pm}{\partial S_t} = \frac{1}{S_t\sigma\sqrt{(T-t)}},$$

therefore

$$
\begin{aligned}
\frac{\partial C(t,S_t)}{\partial S_t} &= \Phi(d_+) + \frac{1}{\sqrt{(2\pi)}}\frac{\partial d_+}{\partial S_t}\left(S_te^{-\frac{d_+^2}{2}} - Ke^{-r(T-t)}e^{-\frac{d_-^2}{2}}\right) \\
&= \Phi(d_+) + \frac{1}{\sqrt{(2\pi)}}\frac{\partial d_+}{\partial S_t}S_te^{-\frac{d_+^2}{2}}\left(1-\frac{K}{S_t}e^{-r(T-t)}e^{\frac{d_+^2}{2}-\frac{d_-^2}{2}}\right).
\end{aligned}
$$

Moreover

$$d_+ = d_- + \sigma\sqrt{(T-t)}$$

so

$$\begin{aligned} d_+^2 - d_-^2 &= (d_- + \sigma\sqrt{(T-t)})^2 - d_-^2 \\ &= 2d_-\sigma\sqrt{(T-t)} + \sigma^2(T-t) \\ &= 2\log\frac{S_t}{K} + 2r(T-t) \end{aligned}$$

and therefore

$$1 - \frac{K}{S_t} e^{-r(T-t)} e^{\frac{d_+^2}{2} - \frac{d_-^2}{2}} = 0.$$

Exercise 1.2.7 *In the Black-Scholes model compute the price and the self-financing hedging portfolios of contingent claims with payoffs:*

$$\begin{aligned} &(1)\ X = S_T^2, \\ &(2)\ X = S_T/S_{T_0},\ 0 \le T_0 \le T. \end{aligned}$$

Analysis of sensitivity. The Greeks.

One of the most important things besides pricing and hedging is the calculation of sensitivities of the prices. These sensitivities are given Greek letters and this is why they are called Greeks. Let $C(t, S_t)$ the value of a portfolio based in a risky asset (S_t) (and bonds). By practical reasons is often very important to have an idea of the sensitivity of C with respect to changes in the value of S_t (to measure the risk of our portfolio for instance) and with respect to changes in the parameters of the model (to measure a bad specification of the model). The standard notation is:

- $\Delta = \frac{\partial C}{\partial S_t}$
- $\Gamma = \frac{\partial^2 C}{\partial S_t^2}$
- $\rho = \frac{\partial C}{\partial r}$
- $\Theta = \frac{\partial C}{\partial t}$
- $\mathcal{V} = \frac{\partial C}{\partial \sigma}$

All these indexes of sensitivity are known as the Greeks. These include $\mathcal{V}$ that is pronounced Vega and that is not a Greek letter (κ was previously used). A portfolio that is not sensitive to small changes with respect to some parameter is said to be neutral: : delta neutral, gamma neutral,..

Proposition 1.2.16 *In the Black-Scholes model the portfolio that replicates a call with strike K and maturity time T has the following Greeks:*

- $\Delta = \Phi(d_+) > 0$

- $\Gamma = \frac{\phi(d_+)}{S_t\sigma\sqrt{(T-t)}} > 0$ *(where ϕ is the density of a standard normal random variable)*
- $\rho = K(T-t)e^{-r(T-t)}\Phi(d_+) > 0$
- $\Theta = -\frac{S_t\sigma}{2\sqrt{(T-t)}}\phi(d_+) - Kre^{-r(T-t)}\Phi(d_-) < 0$
- $\mathcal{V} = S_t\phi(d_+)\sqrt{(T-t)} > 0$

Exercise 1.2.8 *Prove that* $\Theta = -\frac{S_t\sigma}{2\sqrt{(T-t)}}\phi(d_+) - Kre^{-r(T-t)}\Phi(d_-)$.

Remark 1.2.9 *Note that equation (1.12), can be written*

$$\Theta + rS_s\Delta + \frac{1}{2}\sigma^2 S_s^2\Gamma = rC(s, S_s).$$

Exotic Options

Not all the options have a payoff $h = f(S_T)$. For instance we have the Asian options whose payoff is

$$h = \left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K\right)_+$$

the lookback options,

$$(\text{"lookback call"})\ h = S_T - S_*,\ \text{where} S_* = \min_{0\le t\le T} S_t$$

$$(\text{"lookback put"})\ h = S^* - S_T,\ \text{where} S^* = \max_{0\le t\le T} S_t,$$

or the barrier options

$$(\text{"down-and-out-call"})\ h = (S_T - K)_+\mathbf{1}_{\{S_*\ge K\}}$$

$$(\text{"down-and-in-call"})\ h = (S_T - K)_+\mathbf{1}_{\{S_*\le K\}}.$$

For all these options we need a more general theorem of replication in the Black-Scholes model.

Theorem 1.2.4 *In the Black-Scholes model any option with payoff $h \ge 0$, $\mathcal{F}_T$-measurable and square integrable under P^* is replicable and its value is given by*

$$C_t = E_{P^*}(e^{-r(T-t)}h|\mathcal{F}_t)$$

Proof. Under P^*

$$M_t := E_{P^*}(e^{-rT}h|\mathcal{F}_t), 0 \leq t \leq T$$

is a square integrable martingale, then by the representation theorem of Brownian martingales there exists a unique adapted process (Y_t) such that

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}W_s$$

with

$$E_{P^*}(\int_0^T Y_s^2 \mathrm{d}s) < \infty,$$

then we can define H_t by

$$H_t = \frac{Y_t}{\sigma \tilde{S}_t}$$

and we have that

$$M_t = M_0 + \int_0^t H_s \mathrm{d}\tilde{S}_s$$

that is

$$\tilde{C}_t = C_0 + \int_0^t H_s \mathrm{d}\tilde{S}_s.$$

Therefore the strategy $\left(H_t^0, H_t\right)$ with $H_t^0 = C_t - H_t S_t$ is self-financing and replicates h. To see that it is admissible it is enough to take into account that since $h \geq 0$, $C_t \geq 0$. ■

Example 1.2.6 *(Asian options) Consider an Asian option with payoff*

$$h = \left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K\right)_+,$$

by the previous theorem $C_t = E_{P^*}(e^{-r(T-t)}h|\mathcal{F}_t)$. *Define*

$$\varphi(t,x) = E_{P^*}((\frac{1}{T}\int_t^T \frac{S_u}{S_t}\mathrm{d}u - x)_+).$$

Then

$$\begin{aligned}
&C_t \\
&= e^{-r(T-t)}E_{P^*}\left(\left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K\right)_+ \middle| \mathcal{F}_t\right) \\
&= e^{-r(T-t)}E_{P^*}\left(\left(\frac{1}{T}\int_t^T S_u \mathrm{d}u - (K - \frac{1}{T}\int_0^t S_u \mathrm{d}u)\right)_+ \middle| \mathcal{F}_t\right) \\
&= e^{-r(T-t)}S_t E_{P^*}\left(\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}\mathrm{d}u - \frac{K - \frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}\right)_+ \middle| \mathcal{F}_t\right) \\
&= e^{-r(T-t)}S_t \varphi(t, Z_t)
\end{aligned}$$

where $Z_t = \frac{K-\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}$. *Is easy to see that*

$$\mathrm{d}Z_t = \left((\sigma^2 - r)Z_t - \frac{1}{T}\right)\mathrm{d}t - \sigma Z_t \mathrm{d}W_t.$$

In fact, applying the integration by parts formula and the Itô formula:

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}\left(\frac{K}{S_t}\right) - \frac{1}{TS_t}\mathrm{d}\left(\int_0^t S_u \mathrm{d}u\right) - \mathrm{d}\left(\frac{1}{S_t}\right)\frac{1}{T}\int_0^t S_u \mathrm{d}u \\
&= -\frac{K}{S_t^2}\mathrm{d}S_t + \frac{K}{S_t^3}\mathrm{d}\langle S_t\rangle - \frac{S_t}{TS_t}\mathrm{d}t + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^2}\mathrm{d}S_t - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^3}\mathrm{d}\langle S_t\rangle,
\end{aligned}$$

but since $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$, *we have that*

$$\begin{aligned}
\mathrm{d}Z_t &= \left(-\frac{K}{S_t}r + \frac{K}{S_t}\sigma^2 + r\frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}\sigma^2 - \frac{1}{T}\right)\mathrm{d}t \\
&+ \left(-\frac{K}{S_t}\sigma + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}\sigma\right)\mathrm{d}W_t \\
&= \left((\sigma^2 - r)Z_t - \frac{1}{T}\right)\mathrm{d}t - \sigma Z_t \mathrm{d}W_t.
\end{aligned}$$

Then, we know that $\tilde{C}_t = e^{-r(T-t)}\tilde{S}_t\varphi(t, Z_t), t \le T$ *is a martingale. So if we assume that* $\varphi(t, x) \in C^{1,2}$ *we will have that*

$$\begin{aligned}
\mathrm{d}\varphi &= \frac{\partial\varphi}{\partial t}\mathrm{d}t + \frac{\partial\varphi}{\partial Z_t}\mathrm{d}Z_t + \frac{1}{2}\frac{\partial^2\varphi}{\partial {Z_t}^2}\sigma^2 Z_t^2 \mathrm{d}t \\
&= \left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left((\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t \\
&- \frac{\partial\varphi}{\partial Z_t}\sigma Z_t \mathrm{d}W_t.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathrm{d}\tilde{C}_t &= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&+ e^{-r(T-t)}\mathrm{d}\langle\tilde{S}, \varphi\rangle_t \\
&= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&- e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t \mathrm{d}t \\
&= e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t \\
&+ re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t - e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t \mathrm{d}t \\
&+ e^{-r(T-t)}\tilde{S}_t\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left((\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t,
\end{aligned}$$

by identifying the martingale parts

$$\mathrm{d}\tilde{C}_t = e^{-r(T-t)}\left(\varphi - Z_t \frac{\partial \varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t$$

$$r\varphi + \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial Z_t}\left(rZ_t + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial Z_t^2}\sigma^2 Z_t^2 = 0.$$

Therefore the hedging strategy is given by $\left(H_t^0, H_t\right)$ *with* $H_t^0 = C_t - H_t S_t$ *and*

$$H_t = e^{-r(T-t)}\left(\varphi - Z_t \frac{\partial \varphi}{\partial Z_t}\right),$$

where φ *is the solution of the partial differential equation*

$$r\varphi + \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x}\left(rx + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}\sigma^2 x^2 = 0 \tag{1.13}$$

with the boundary condition $\varphi(T,x) = x_-$ *(negative part of x). These equation can be solved numerically.*

Exercise 1.2.9 *Demostrar que el precio de una optionasiatica con strike flotante (payoff=*$\left(\frac{1}{T}\int_0^T S_u \mathrm{du} - S_T\right)_+$*) viene dado en el instante inicial por*

$$C = e^{-rT} S_0 \varphi(0,0)$$

whereφ *es solución de la ecuación (1.13) con la condición de contorno* $\varphi(T,x) = (1+x)_-$

Lemma 1.2.2 *Consider stepwise functions*

$$f(t) = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{(t_{i-1},t_i]}(t)$$

with $\lambda_i \in \mathbb{R}$ *and* $0 \leq t_0 < t_1 ... < t_n \leq T$*. Denote by* $\mathcal{J}$ *that set of functions. Set* $\mathcal{E}_T^f = \exp\{\int_0^T f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^T f^2(s)\mathrm{d}s\}, f \in \mathcal{J}$ *. If* $Y \in L^2(\mathcal{F}_T, P)$ *is orthogonal to* $\mathcal{E}_T^f, f \in \mathcal{J}$ *then* $Y = 0$.

Proof. Consider $Y \geq 0 \in L^2(\mathcal{F}_T, P)$ orthogonal to $\mathcal{E}_T^f$. Let $\mathcal{G}_n := \sigma(B_{t_1}, B_{t_2}, ..., B_{t_n})$, we have

$$E(\exp\{\sum_{i=1}^{n} \lambda_i (B_{t_i} - B_{t_{i-1}})\}Y) = 0,$$

and

$$E(\exp\{\sum_{i=1}^{n} \lambda_i (B_{t_i} - B_{t_{i-1}})\}E(Y|\mathcal{G}_n)) = 0.$$

Let X be the map

$$\begin{aligned}&X:\Omega\rightarrow\mathbb{R}^n\\&\omega\longmapsto X(\omega)=(B_{t_1}(\omega),B_{t_2}(\omega)-B_{t_1}(\omega),...,B_{t_n}(\omega)-B_{t_{n-1}}(\omega))\end{aligned}$$

then

$$\int_{\mathbb{R}^n}\exp\{\sum_{i=1}^n\lambda_i x_i\}E(Y|\mathcal{G}_n)(x_1,x_2,...,x_n)\mathrm{d}P^X(x_1,x_2,...,x_n)=0,$$

in such a way that the Laplace transform of $E(Y|\mathcal{G}_n)(x_1,x_2,...,x_n)\mathrm{d}P^X$ is zero and therefore $E(Y|\mathcal{G}_n)(x_1,x_2,...,x_n)$ is identically null P^X a.s.. From here $E(Y|\mathcal{G}_n)=0$ P a.s., and finally since this is true for any $\mathcal{G}_n$ of the previous type it turns out that Y is zero P a.s.. Finally for a general Y we can decompose $Y=Y_+-Y_-$ and we would arrive to the conclusion that $Y_+=Y_-$ P a.s. by the uniqueness of the Laplace transform of a measure. ■

Proposition 1.2.17 *For all random variable $F\in L^2(\mathcal{F}_T,P)$ there exists and adapted process $(Y_t)_{0\leq t\leq T}$, with $E(\int_0^T Y_t^2\mathrm{d}t)<\infty$, such that*

$$F=E(F)+\int_0^T Y_t\mathrm{d}B_t$$

Proof. Suppose that $F-E(F)$ is orthogonal to $\int_0^T Y_t\mathrm{d}B_t$ for all $(Y_t)_{0\leq t\leq T}$, with $E(\int_0^T Y_t^2\mathrm{d}t)<\infty$, then if we prove that $F-E(F)=0$ P a.s. then we have finished, since the Hilbert space of centered random variables of $L^2(\mathcal{F}_T,P)$ will coincide with the Hilbert space of random variables $\int_0^T Y_t\mathrm{d}B_t$ with $E(\int_0^T Y_t^2\mathrm{d}t)<\infty$. Write $Z=F-E(F)$, we have

$$E((F-E(F))\int_0^T Y_t\mathrm{d}B_t)=0.$$

Take $Y_t=\mathcal{E}_t^f f(t)$, with the $\mathcal{E}_t^f$ define previously, then

$$E((F-E(F))\int_0^T\mathcal{E}_t^f f(t)\mathrm{d}B_t)=0$$

and also that

$$E((F-E(F))(1+\int_0^T\mathcal{E}_t^f f(t)\mathrm{d}B_t))=0$$

but, by the Itô formula

$$\mathcal{E}_T^f=1+\int_0^T\mathcal{E}_t^f f(t)\mathrm{d}B_t.$$

So

$$E((F-E(F))\mathcal{E}_T^f)=0$$

and by the previous lemma $F-E(F)=0$ P a.s.. ■

Theorem 1.2.5 *Any square integrable martingale* $(M_t)_{0\leq t\leq T}$ *can be written as*

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}B_s, 0 \leq t \leq T$$

whereY_s *is an adapted process with* $E(\int_0^T Y_t^2 \mathrm{d}t) < \infty$.

Proof. We can write

$$M_t = E(M_T|\mathcal{F}_t)$$

and by the previous proposition

$$M_T = E(M_T) + \int_0^T Y_s \mathrm{d}B_s$$

then it is enough to take conditional expectations. ∎

1.2.6 Multidimensional Black-Scholes model with continuous dividends

The model of the financial market consists in $(d+1)$ stocks S_t^0, $S_t^1, ...,$ S_t^d in such a way that

$$\mathrm{d}S_t^0 = S_t^0 r(t)\mathrm{d}t, S_0^0 = 1,$$

and

$$\mathrm{d}S_t^i = S_t^i(\mu^i(t)\mathrm{d}t + \sum_{j=1}^d \sigma^{ij}(t)\mathrm{d}W_t^j), i = 1, ..., d$$

where $W = (W^1, ..., W^d)$ is a d-dimensional Brownian motion. By simplicity we assume that μ, σ and r are deterministic and cadlag. We shall consider the natural filtration associated with W.

An investment strategy will be an adapted process $\phi = ((\phi_t^0, \phi_t^1, ..., \phi_t^d))_{0\leq t\leq T}$ in R^{d+1}. The value of the portfolio at time t is given by the scalar product

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^d \phi_t^i S_t^i,$$

and its discounted value is

$$\tilde{V}_t(\phi) = e^{-\int_0^t r_s ds} V_t(\phi) = \phi_t \cdot \tilde{S}_t.$$

We assume that the stocks can produce dividends in a continuous and deterministic way: $((\delta_t^1, ..., \delta_t^d))_{0\leq t\leq T}$. Then if the strategy is self-financing

$$\mathrm{d}V_t(\phi) = \sum_{i=0}^d \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^d \phi_t^i S_t^i \delta_t^i \mathrm{d}t.$$

Now we look for a probability under which the discounted values of the self-financing portfolios are martingales. We know that

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-\int_0^t r_s ds} V_t(\phi)\right) = -r_t e^{-\int_0^t r_s ds} V_t \mathrm{d}t + e^{-\int_0^t r_s ds} \mathrm{d}V_t \\
&= -r_t e^{-\int_0^t r_s ds} V_t \mathrm{d}t + e^{-\int_0^t r_s ds} \left(\sum_{i=0}^d \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^d \phi_t^i S_t^i \delta_t^i \mathrm{d}t \right) \\
&= e^{-\int_0^t r_s ds} r_t (\phi_t^0 S_t^0 - V_t)\mathrm{d}t + e^{-\int_0^t r_s ds} \sum_{i=1}^d \left(\phi_t^i \mathrm{d}S_t^i + \phi_t^i S_t^i \delta_t^i \mathrm{d}t \right) \\
&= e^{-\int_0^t r_s ds} \sum_{i=1}^d \phi_t^i S_t^i (\delta_t^i - r_t)\mathrm{d}t + e^{-\int_0^t r_s ds} \sum_{i=1}^d \phi_t^i \mathrm{d}S_t^i \\
&= e^{-\int_0^t r_s ds} \left(\sum_{i=1}^d \phi_t^i S_t^i (\delta_t^i + \mu_t^i - r_t)\mathrm{d}t + \sum_{i=1}^d \phi_t^i S_t^i \sum_{j=1}^d \sigma^{ij}(t) \mathrm{d}W_t^j \right) \\
&= e^{-\int_0^t r_s ds} \sum_{i=1}^d \phi_t^i S_t^i \sum_{j=1}^d \sigma^{ij}(t) \left(\mathrm{d}W_t^j + \sum_{k=1}^d \left(\sigma_t^{-1}\right)^{jk}(t)(\delta_t^k + \mu_t^k - r_t)\mathrm{d}t \right) \\
&= e^{-\int_0^t r_s ds} \sum_{i=1}^d \phi_t^i S_t^i \sum_{j=1}^d \sigma^{ij}(t) \mathrm{d}\tilde{W}_t^j
\end{aligned}$$

with

$$\mathrm{d}\tilde{W}_t^j = \mathrm{d}W_t^j + \sum_{k=1}^d \left(\sigma^{-1}\right)^{jk}(t)(\delta_t^k + \mu_t^k - r_t)\mathrm{d}t, j = 1, ..., d$$

Then by the Girsanov theorem with $\theta_j(t) = \left(\sigma^{-1}\right)^{jk}(t)(r_t - \delta_t^k - \mu_t^k)$ it turns out that $\left(\tilde{W}_t\right)_{0 \le t \le T}$ is a d-dimensional Brownian motion with respect to the probability P^*:

$$\mathrm{d}P^* = \Pi_{j=1}^n \exp\{-\int_0^T \theta_j(t)\mathrm{d}W_t^j - \frac{1}{2}\int_0^T \theta_j^2(t)\mathrm{d}t\}\mathrm{d}P.$$

Then

$$E_{P^*}(\tilde{V}_T|\mathcal{F}_t) = \tilde{V}_t,$$

and any replicable payoff X will have a price at t given by

$$V_t = e^{\int_0^t r_s ds} E_{P^*}(\tilde{X}|\mathcal{F}_t).$$

On the other hand if $\tilde{X}$ is square integrable the representation theorem of Brownian martingales allows us to write

$$E_{P^*}(\tilde{X}|\mathcal{F}_t) = E_{P^*}(\tilde{X}) + \sum_{j=1}^d \int_0^t h_s^j \mathrm{d}\tilde{W}_s^j,$$

in such a way that we can take

$$\phi_t^i = \frac{1}{\tilde{S}_t^i} \sum_{k=1}^{d} \left(\sigma_t^{-1}\right)^{ik} h_t^k, i = 1, ..., d.$$

Remark 1.2.10 *We have assume that* $\left(\sigma_t^{ij}\right)$ *is invertible and from here we conclude that the model is free of arbitrage and complete. For the lack of arbitrage it is sufficient to have* $\theta(t)$ *such that* $\sum_{k=1}^{d} \sigma^{jk}(t)\theta_k(t) = \delta_t^j + \mu_t^j - r_t$. *But for completeness we need that* $\left(\sigma_t^{ij}\right)$ *is invertible. In this way we can have viable models where the dimension of* W *is greater than the number of stocks but then they are no complete.*

Remark 1.2.11 *Note that a portfolio with a constant number of assets is NOT a self-financing portfolio, except for the trivial case where you have only riskless assets. This is due to the fact that risky assets generate dividends and then your bank account change if you mantain the number of risky assets in your portfolio.*

Price of a call option

First note that under P^*

$$\mathrm{d}S_t^i = S_t^i(\left(r_t - \delta_t^i\right) \mathrm{d}t + \sum_{j=1}^{d} \sigma_t^{ij} \mathrm{d}\tilde{W}_t^j), i = 1, ..., d,$$

so $\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s}\right)$ are martingales under P^*:

$$\begin{aligned} \mathrm{d}\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s}\right) &= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \left(-S_t^i \left(r_t - \delta_t^i\right) \mathrm{d}t + \mathrm{d}S_t^i\right) \\ &= \sum_{j=1}^{d} \sigma_t^{ij} S_t^i \mathrm{d}\tilde{W}_t^j. \end{aligned}$$

Then

$$C_t := E_{P^*}\left(\left.\frac{(S_T^i - K)_+}{\exp\{\int_t^T r_s \mathrm{d}s\}}\right| \mathcal{F}_t\right) = \exp\{-\int_t^T \delta_s^i \mathrm{d}s\} E_{P^*}\left(\frac{(S_T^i - K)_+}{\exp\{\int_t^T (r_s - \delta_s^i) \mathrm{d}s\}} | \mathcal{F}_t\right),$$

under P^*, and conditional to $\mathcal{F}_t$,

$$\log S_T^i - \log S_t^i \sim N(\int_t^T (r_s - \delta_s^i) \mathrm{d}s - \frac{1}{2}\int_t^T \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \mathrm{d}s, \int_t^T \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \mathrm{d}s).$$

Therefore

$$C_t = \exp\{-\int_t^T \delta_s^i \mathrm{d}s\} \left(S_t^i \Phi(d_+) - K \exp\{-\int_t^T (r_s - \delta_s^i) \mathrm{d}s\} \Phi(d_-)\right),$$

with

$$d_{\pm} = \frac{\log \frac{S_t^i}{K} + \int_t^T \left(r_s - \delta_s^i \pm \frac{1}{2} \sum_{j=1}^d \left(\sigma_s^{ij}\right)^2 \right) \mathrm{d}s}{\sqrt{\int_t^T \sum_{j=1}^d \left(\sigma_s^{ij}\right)^2 \mathrm{d}s}}.$$

If we take $d = 1$ a constant interest rate r and constant dividend rate δ, we have the following formula for a call option, with strike K:

$$C_t = S_t e^{-\delta(T-t)} \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-),$$

with

$$d_{\pm} = \frac{\log \frac{S_t}{K} + (r - \delta \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{(T-t)}}.$$

If we take $d = 1$ a constant interest rate r and constant dividend rate δ, we have the following formula for a call option, with strike K:

$$C_t = S_t e^{-\delta(T-t)} \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-),$$

with

$$d_{\pm} = \frac{\log \frac{S_t}{K} + (r - \delta \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{(T-t)}}.$$

1.2.7 Currency options

A foreign currency can be thought as a kind of risky stock whose value at t, say X_t, changes in a random way at that generates some interests (or dividends) at the foreign rate, say r_f. In this way, if we assume a Black-Scholes for X and with domestic interest rates r_d, the price of a call option with strike K can be obtained by using the previous formula with $\delta = r_f$ y $r = r_d$.

Remark 1.2.12 *The previous arguments can be extended to the cases where μ, r and δ are adapted processes, cadlag and such that*

$$\Pi_{j=1}^n \exp\{-\int_0^t \theta_j(s) \mathrm{d}W_s^j - \frac{1}{2} \int_0^t \theta_j^2(s) \mathrm{d}s\}, 0 \leq t \leq T,$$

is a martingale. Also to the cases where σ is adapted and invertible for all ω and t, but in these cases we will not have formulas of Black-Scholes type since the discounted values of the stocks will not be log-normal distributed.

1.2.8 Stochastic volatility

Suppose that under P^*

$$\mathrm{d}S_t = S_t(r_t \mathrm{d}t + \sigma(W_t^2, t) \mathrm{d}W_t^1)$$

where W_t^1 and W_t^2 are two independent Brownian motions. Then the price of a call option with strike K is given by

$$\begin{aligned} C_t &= E(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ | \mathcal{F}_t) \\ &= E(E(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ | \sigma(W_s^2, s), t \le s \le T, \mathcal{F}_t)|\mathcal{F}_t) \\ &= E(S_t \Phi(d_+) - K e^{-\int_t^T r_s \mathrm{d}s}\Phi(d_-)|\mathcal{F}_t), \end{aligned}$$

with

$$d_\pm = \frac{\log \frac{S_t}{K} + \int_t^T (r_s \pm \frac{1}{2}\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T \sigma^2(W_s^2, s)\mathrm{d}s}},$$

If we assume a covariance $\int_0^t \rho_s \mathrm{d}s$ between W_t^1 and W_t^2 we obtain

$$E(S_t \xi_t \Phi(d_+) - K e^{-\int_t^T r_s \mathrm{d}s}\Phi(d_-)|\mathcal{F}_t),$$

with

$$d_\pm = \frac{\log \frac{S_t \xi_t}{K} + \int_t^T (r_s \pm \frac{1}{2}(1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}},$$

and

$$\xi_t = \exp\{\int_t^T \rho_s \sigma(W_s^2, s)\mathrm{d}W_s^2 - \frac{1}{2}\int_t^T \rho_s^2 \sigma^2(W_s^2, s)\mathrm{d}s\}.$$

In fact, first note that a process Z such that

$$Z_t := W_t^1 - \int_0^t \rho_s \mathrm{d}W_s^2,$$

is independent of W^2:

$$E(Z_t W_t^2) = \int_0^t \rho_s \mathrm{d}s - \int_0^t \rho_s \mathrm{d}s = 0.$$

So, we can write

$$\mathrm{d}W_t^1 = \sqrt{1-\rho_t^2}\mathrm{d}\hat{W}_t + \rho_t \mathrm{d}W_t^2,$$

with $\mathrm{d}\hat{W}_t = \frac{1}{\sqrt{1-\rho_t^2}}\mathrm{d}Z_t$. Then $\hat{W}$ is a Brownian motion independent of W^2. Therefore we have

$$\mathrm{d}S_t = S_t\left(r\mathrm{d}t + \sigma(W_t^2, t)\left(\sqrt{1-\rho_t^2}\mathrm{d}\hat{W}_t + \rho_t \mathrm{d}W_t^2\right)\right)$$

and by the Itô formula:

$$\begin{aligned} S_T = S_t \exp\{&\int_t^T r_s \mathrm{d}s + \int_t^T \rho_s \sigma(W_s^2, s)\mathrm{d}W_s^2 - \frac{1}{2}\int_t^T \rho_s \sigma^2(W_s^2, s)\mathrm{d}s\} \\ &\times \exp\{\int_t^T \sqrt{1-\rho_s^2}\sigma(W_s^2, s)\mathrm{d}\hat{W}_s - \frac{1}{2}\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s)\mathrm{d}s\}. \end{aligned}$$

1.2.9 Fourier methods for pricing

Define the Fourier transform of f by

$$(\mathbf{F}f)(v) = \int_{\mathbb{R}} e^{ixv} f(x) dx.$$

If f is integrable then it exists. Its inverse, if f is integrable, is given by, a.e., by

$$\left(\mathbf{F}^{-1}f\right)(v) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixv} f(x) dx.$$

Suppose that the model is, under P^*, of the form

$$S_t = e^{rt+X_t},$$

where (X_t) is a process with independent increments and homogeneous, $X_0 = 0$. c.s., and with density $f_{X_t}(x)$. The price at time zero of a call with strike e^k is given by

$$C(k) = e^{-rT} E((e^{rT+X_T} - e^k)_+).$$

Then if we consider the function

$$z_T(k) = e^{-rT} E((e^{rT+X_T} - e^k)_+) - (1 - e^{k-rT})_+,$$

it turns out that

$$\varsigma_T(v) := (\mathbf{F}z_T)(v) = e^{ivrT} \frac{\varphi_{X_T}(v-i) - 1}{iv(iv+1)},$$

where φ_{X_T} is the characteristic function of X_T. $C(k)$ can be obtained now by inverting $\varsigma_T(v)$. In fact En efecto

$$z_T(k) = e^{-rT} \int_{\mathbb{R}} f_{X_T}(x)(e^{rT+x} - e^k)(\mathbf{1}_{\{rT+x>k\}} - \mathbf{1}_{\{rT>k\}}) \mathrm{d}x.$$

Then if we apply the Fubini theorem

$$
\begin{aligned}
\varsigma_T(v) &= \int_{\mathbb{R}} e^{ikv} z_T(k)\mathrm{d}k \\
&= e^{-rT}\int_{\mathbb{R}} e^{ikv}\left(\int_{\mathbb{R}} f_{X_T}(x)(e^{rT+x}-e^k)(\mathbf{1}_{\{rT+x>k\}}-\mathbf{1}_{\{rT>k\}})\mathrm{d}x\right)\mathrm{d}k \\
&= e^{-rT}\int_{\mathbb{R}} f_{X_T}(x)\left(\int_{rT}^{rT+x} e^{ikv}(e^{rT+x}-e^k)\mathrm{d}k\right)\mathrm{d}x \\
&= e^{-rT}\int_{\mathbb{R}} f_{X_T}(x)\left(e^{rT+x}\left[\frac{e^{ikv}}{iv}\right]_{rT}^{rT+x} - \left[\frac{e^{k(iv+1)}}{iv+1}\right]_{rT}^{rT+x})\right)\mathrm{d}x \\
&= e^{irTv}\int_{\mathbb{R}} f_{X_T}(x)\frac{e^{x(iv+1)}-e^x}{iv}\mathrm{d}x - e^{irTv}\int_{\mathbb{R}} f_{X_T}(x)\frac{e^{x(iv+1)}-1}{iv+1}\mathrm{d}x \\
&= \frac{e^{irTv}}{iv}(\varphi_{X_T}(v-i)-1) - \frac{e^{irTv}}{iv+1}(\varphi_{X_T}(v-i)-1) \\
&= \frac{e^{irTv}}{iv(iv+1)}(\varphi_{X_T}(v-i)-1).
\end{aligned}
$$

The next step is to invert $\varsigma_T(v)$, since it is assumed that we know φ_{X_T}, and then we recover $z_T(k)$.

To do this last step we can use numerical methods. If we want to calculate the inverse Fourier transform of $f(x)$ we can do the approximation

$$
\int_{\mathbb{R}} e^{-iux} f(x)\mathrm{d}x \approx \int_{-A/2}^{A/2} e^{-iux} f(x)\mathrm{d}x \approx \frac{A}{N}\sum_{k=0}^{N-1} w_k f(x_k) e^{-iux_k},
$$

where $x_k = -A/2 + k\Delta$, with $\Delta = A/(N-1)$. w_k depends of the kind of approximation. For instance the trapezoidal approximation $w_0 = w_{N-1} = 1/2$ and the rest of weights 1. If now we take $u = u_n = \frac{2\pi n}{N\Delta}$ we have that

$$
F^{-1}(f)(u_n) \approx \frac{A}{N} e^{iu_n A/2}\sum_{k=0}^{N-1} w_k f(x_k) e^{-2\pi i n k/N}.
$$

Then, there exists an algorithm *fast Fourier transform* (FFT) to calculate very fast

$$
\sum_{k=0}^{N-1} g_k e^{-2\pi i n k/N}, n = 0, 1, ..., N-1,
$$

that requires $O(N\log N)$ calculations. Note that the step in the net of points u_n is given by $d = \frac{2\pi}{N\Delta}$. So $d\Delta = \frac{2\pi}{N}$. Then if we want d and Δ small we have to raise N in a major way. Another limitation is that to use the FFT algorithm the net of points has to be uniform and a power of two ($N = 2^k$).

Chapter 2

Interest rates models

Interest rates models are used mainly for valuing and hedging bonds and options on bonds. To remark that there is not a reference model as the Black-Scholes on stocks.

2.1 Basic facts

2.1.1 The yield curve

In the models we studied we assumed a constant interest rate. In practice the interest rate depends on the emission data of the loan and the final or maturity time.

Someone borrows one euro at time t, till maturity T, he will have to pay an amount $F(t,T)$ at time T, this is equivalent to a mean rate of continuous interest $R(t,T)$ given by the equality:

$$F(t,T) = e^{(T-t)R(t,T)}.$$

If we assume that interest rates are known: $(R(t,T))_{0\leq t\leq T}$, and there is not arbitrage then

$$F(t,s) = F(t,u)F(u,s), \forall t \leq u \leq s,$$

and from here together with the condition $F(t,t) = 1$, it follows, if $F(t,s)$ is differentiable as a function of s, that there exist a function $r(t)$ such that

$$F(t,T) = \exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

In fact, let $s \geq t$

$$\begin{aligned} F(t,s+h) - F(t,s) &= F(t,s)F(s,s+h) - F(t,s) \\ &= F(t,s)(F(s,s+h) - 1), \end{aligned}$$

$$\frac{F(t,s+h)-F(t,s)}{F(t,s)h}=\frac{F(s,s+h)-F(s,s)}{h},$$

taking $h\to 0$ we have

$$\frac{\partial_2 F(t,s)/\partial s}{F(t,s)}=\partial_2 F(s,s)/\partial s := r(s)$$

and from here

$$F(t,T)=\exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

Note that

$$R(t,T)=\frac{1}{T-t}\int_t^T r(s)\mathrm{d}s.$$

The function $r(s)$ is interpreted as an instantaneous interest rate, and it is also called short rate.

But look the other way round. Suppose that I want a contract to guarantee one euro at time T. We have the so called *bonds*. Which is the price of a bond at time t?. To receive $F(t,T)$ at time T we have to pay (put in the bank account) one euro, then, for the bond, we have to pay $1/F(t,T)$.

In practice we do not know the prices of the bonds in different times, these prices are changing randomly, but intuitively it seems that there must exist a relation among all these prices for different initial and maturity times. The interest rate models try to explain these prices.

The main object of our study is what is called the *zero coupon bond*

Definition 2.1.1 *A zero coupon bond with maturity T is a contract that guarantees one euro at time T. Its price at t shall be denote by $P(t,T)$.*

The bonds with coupons are those that are giving certain amounts (coupons) till the maturity of the bond.

Definition 2.1.2 *The yield curve of a zero coupon bond is the graph corresponding to the map*

$$T\longmapsto R(t,T)$$

We saw above that if we can anticipate the future or we would like to build a bond market with deterministic prices for the different trading and maturity times, the lack of arbitrage lead us to

$$P(t,T)=e^{-\int_t^T r(s)\mathrm{d}s}.$$

y

$$R(t,T)=\frac{1}{T-t}\int_t^T r(s)\mathrm{d}s$$

2.1.2 Yield curve for a random future

For a fixed t, $P(t,T)$ is a function of T whose graph gives us the prices of the bonds at t or the *term structure* at t. It is expected a smooth function. If we fix T, $p(t,T)$ will be a stochastic process. In this context, our bond market will be a market with infinitely many assets: for each T we have an asset and we ask ourselves questions like:

- which models are sensible to valuate bonds?
- which relation must the prices of the bond have to avoid arbitrage opportunities?
- can we obtain the prices of the bonds if we have a model for short rates?
- given a model of bond market how can we calculate prices of derivatives, such as call options of bonds?

2.1.3 Interest rates

Consider the following example. Suppose that we are at time t and we fix another future times S and T, $t < S < T$. The purpose is to build at time t a contract that investing at time S one euro we get a **deterministic** interest rate in the period $[S,T]$, in such a way that we obtain a deterministic amount at T. This can done in the following way:

1. At time t we sell a bond with maturity S. This gives us $P(t,S)$ euros.
2. At time t we buy $P(t,S)/P(t,T)$ bonds with maturity T.

Note that this implies the following:

1. The cost of the operation at t is zero.
2. At time S we have to pay one euro.
3. At time T we receive $P(t,S)/P(t,T)$ euros.

The amount we receive $P(t,S)/P(t,T)$ can be quoted by simple or continuously compounded rates:

- The simple forward interest rate (LIBOR), $L = L(t;S,T)$, which is the solution of the equation:

$$1 + (T-S)L = \frac{P(t,S)}{P(t,T)}$$

that is the simple interest rate guaranteed for the period $[S,T]$ *at time* t.

- The continuously compounded forward interest rate $R = R(t; S, T)$, solution of the equation:
$$e^{R(T-S)} = \frac{P(t,S)}{P(t,T)}.$$
analogously to the previous case, is continuously compounded interest guaranteed at time t, for the period $[S,T]$. The quotation using simple interest rates is the usual at financial markets whereas continuously compounded rates are used in theoretical frameworks.

So, in the bond market we can define different interest rates. That is the prices of the bonds can be quoted in different ways.

Definition 2.1.3 *1. The simple forward rate for the interval $[S,T]$ contracted at t, (LIBOR ("London Interbank Offer Rate") is defined as*

$$L(t; S, T) = -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}$$

2. The simple spot rate for $[t,T]$, spot LIBOR, is defined as

$$L(t,T) = -\frac{P(t,T) - 1}{(T-t)P(t,T)},$$

it is the previous one with $S = t$.

3. The continuously compounded forward rate contracted at t for $[S,T]$ as

$$R(t; S, T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}$$

4. The continuously compounded spot rate for $[t,T]$ as

$$R(t,T) = -\frac{\log P(t,T)}{T-t}$$

5. The instantaneous forward rate with maturity T contracted at t as

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T} = \lim_{T \to S} R(t; S, T)$$

6. The instantaneous (spot) short rate at t

$$r(t) = f(t,t) = \lim_{T \to t} f(t,T)$$

Note that the instantaneous forward rate with maturity T contracted at t can be seen as the deterministic rate contracted a t for the infinitesimal period $[T, T + \mathrm{d}T]$.

Fixed t, any of the rates defined previously, from 1 to 5, alow us to recover the prices of the bonds. Then, modelling these rates is equivalent to modelling the bond prices.

2.1.4 Bonds with coupons, swaps, caps and floors

Fixed coupons bonds

The simplest of the bonds with coupons is the bond with fixed coupons. It is a bond that at some times in between gives predetermined profits (coupons) to the owner of the bond. Its formal description is:

- Let $T_0, T_1, ..., T_n$, fixed times. T_0 is the emission time of the bond, whereas $T_1, ..., T_n$ are the payment times.
- At time T_i the owner receives the amount c_i.
- At time T_n there is an extra payment: K.

It is obvious that this bond can be replicated with a portfolio with c_i zero-coupon bonds with maturities T_i, $i = 1, .., n-1$ and K zero-coupon bonds with maturity T_n. So, the price at time $t < T_1$ will be given by

$$p(t) = KP(t, T_n) + \sum_{i=1}^{n} c_i P(t, T_i).$$

Usually the coupons are expressed in terms of certain rates r_i instead of quantities, in such a way that for instance

$$c_i = r_i(T_i - T_{i-1})K.$$

For a standard coupon the intervals of time are equal:

$$T_i = T_0 + i\delta,$$

y $r_i = r$, de manera que

$$p(t) = K\left(P(t, T_n) + r\delta \sum_{i=1}^{n} P(t, T_i)\right).$$

Floating rate coupon

Quite often the coupons are not fixed in advance, but rather they are updated for every coupon period. On example is to take $r_i = L(T_{i-1}, T_i)$ where L is the spot LIBOR. Since

$$L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1$$

we have (taking $K = 1$)

$$c_i = L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1.$$

It is easy to see that we can replicate this amount selling a bond (without coupons) with maturity T_i and buying one with maturity T_{i-1} :

- With the bond sold we will have at T_i a payoff -1.
- With the bond bought, we will have 1 at T_{i-1} and we can buy $\frac{1}{P(T_{i-1},T_i)}$ bonds with maturity T_i giving a payoff $\frac{1}{P(T_{i-1},T_i)}$.
- The total cost is $P(t, T_{i-1}) - P(t, T_i)$.

The for any time $t < T_0$ the price of this bond with random coupons is

$$p(t) = P(t, T_n) + \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) = P(t, T_0)!.$$

Thsi means that a unit of money at T_0, evolves as a coupon bond with floating rates given by the simple Libor rates.

Interest rate Swaps

There are many types of rate swaps but all of the are basically exchanges of payments with fixed rates with random payments. We shall consider the so called *forwards swaps settled in arrears*. Denote the principal by K and the swap rate (fixed rate) by R . Suppose equally spaced dates T_i, at time T_i, $i \geq 1$ we receive

$$K\delta L(T_{i-1}, T_i)$$

by paying $K\delta R$, so the cash flow at T_i is $K\delta[L(T_{i-1}, T_i) - R]$,. The value at $t \leq T_0$ off tis cash flow is

$$\begin{aligned} &K(P(t, T_{i-1}) - P(t, T_i)) - K\delta R P(t, T_i) \\ &= KP(t, T_{i-1}) - K(1 + R\delta)P(t, T_i), \end{aligned}$$

so in total

$$\begin{aligned} p(t) &= \sum_{i=1}^{n} (KP(t, T_{i-1}) - K(1 + R\delta)P(t, T_i)) \\ &= KP(t, T_0) - KP(t, T_n) - KR\delta \sum_{i=1}^{n} P(t, T_i) \\ &= KP(t, T_0) - K \sum_{i=1}^{n} d_i P(t, T_i), \end{aligned}$$

with $d_i = R\delta, i = 1, .., n-1$ and $d_n = 1 + R\delta$.

R is usually taken in such a way that the value of the contract is zero when it is issued. If $t < T_0$,

$$R = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

Caps and Floors

A *cap* is a contract that protects you from paying more than a fixed rate (the *cap rate*) R even though the loan has floating rate. We can also define a *floor* that is a contract that guarantees that the rate is always above the so called *floor rate* R even for an investment with random rate.

Technically a cap is a sum of *captlets*, they consist on these basic contracts.

- The interval $[0, T]$ is divided by equidistant points: $0 = T_0, T_1, ..., T_n = T$, with distance δ. Typically 1/4 of the year or half year.
- The cap works on a principal, say K, and the cap rate is R.
- The floating rate is for instance the LIBOR $L(T_{i-1}, T_i)$.
- The caplet i is defined as a contract with payoff en T_i given by

$$K\delta(L(T_{i-1}, T_i) - R)_+.$$

Proposition 2.1.1 *The value of a cap with principal K and* cap rate *R is that of one portfolio with $K(1+R\delta)$ put options with maturities T_{i-1}, $i = 1, ..., n$ on bonds with maturities T_i and with strike $\frac{1}{1+R\delta}$.*

Proof.

$$\begin{aligned}
K\delta(L(T_{i-1}, T_i) - R)_+ &= K\left(\frac{1}{P(T_{i-1}, T_i)} - 1 - \delta R\right)_+ \\
&= \frac{K(1+R\delta)}{P(T_{i-1}, T_i)}\left(\frac{1}{(1+R\delta)} - P(T_{i-1}, T_i)\right)_+,
\end{aligned}$$

but a payoff $\frac{1}{P(T_{i-1}, T_i)}$ in T_i is equivalent to 1 at T_{i-1}. In other words, with the cash amount $K(1+R\delta)(\frac{1}{(1+R\delta)} - P(T_{i-1}, T_i))_+$ at T_{i-1} I can buy $\frac{K(1+R\delta)}{P(T_{i-1}, T_i)}(\frac{1}{(1+R\delta)} - P(T_{i-1}, T_i))_+$ bonds with maturity T_i and I get this amount. ■

Note that

$$\text{Cap(t)} - \text{Floor(t)} = \text{Swap(t)}.$$

Swaptions

I a contract s that gives the right to enter in a swap at the maturity time of the *swaption*. A payer swaption gives the right to enter in a swap as payer of the fixed rate. A *receiver swaption* gives the right to enter as the receiver of the fixed rates.

A payer swaption has similarities with the cap contract. In the cap the owner has the right to receive a random rate and to pay a constant rate and he will exercise in each period where the random rate is greater than the fixed one. Similarly the owner of payer swaption has the right to receive a floating rate and to pay a constant rate, however in the cap you chose if paying or not at each period, in the case of a swaption te decision is taken once for ever at the maturity time of the swaption. The value of the "swap", with principal 1, at the maturity time of the swaption, say T, is

$$P(T,T_0) - P(T,T_n) - R\delta \sum_{i=1}^{n} P(T,T_i),$$

so the payoff of a swaption is

$$\begin{aligned}&\left(P(T,T_0) - P(T,T_n) - R\delta \sum_{i=1}^{n} P(T,T_i)\right)_+ \\ &= (S(T) - Z(T))_+ ,\end{aligned}$$

where

$$S(T) = P(T,T_0) - P(T,T_n)$$

that is the value of the payments with floating rate and

$$Z(T) = R\delta \sum_{i=1}^{n} P(T,T_i)$$

that is the value of the payments with fixed rate.

It is also interesting the decomposition of the payer swaption payoff as

$$\left(P(T,T_0) - (P(T,T_n) + R\delta \sum_{i=1}^{n} P(T,T_i))\right)_+$$

where $P(T,T_0)$ is the value of a coupon bond (at T) with floating payments and $P(T,T_n) + R\delta \sum_{i=1}^{n} P(T,T_i)$ of a coupond bond with fixed payments. Then a swaption can be seen as an option to exchange one coupon by another. If $T = T_0$ a swaption becomes a put with strike 1 on a bond with fixed coupons.

2.2 A general framework for short rates

We are going to define the process *bank account* or *riskless* asset. We shall create a random scenario for the instantaneous rates $r(s)$. More concretely we consider a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T})$, and we assume that $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by a Brownian motion $(W_s)_{0\le t\le T}$ and that $\mathcal{F}_T = \mathcal{F}$. In this context we introduce the *riskless* asset:

$$S_t^0 = \exp\{\int_0^t r(s)\mathrm{d}s\}$$

where $(r(t))_{0\le t\le T}$ is an adapted process with $\int_0^t |r(s)|\mathrm{d}s < \infty$. In our market we shall assume the existence of risky assets: the bonds! (without coupons) with maturity less or equal than the horizon T. For each time $u \leq T$ we define an adapted process $(P(t,u))_{0\leq t\leq u}$ satisfying $P(t,t) = 1$.

We make the following hypothesis:

(H) There exist a probability P^* equivalent to P such that for all $0 \leq u \leq T$, $(\tilde{P}(t,u))_{0\leq t\leq u}$ defined by

$$\tilde{P}(t,u) = e^{-\int_0^t r(s)\mathrm{d}s} P(t,u)$$

is a martingale.

This hypothesis has the following interesting consequences:

Proposition 2.2.1

$$P(t,u) = E_{P^*}\left(e^{-\int_t^u r(s)\mathrm{d}s}|\mathcal{F}_t\right)$$

Proof.

$$\begin{aligned}\tilde{P}(t,u) &= E_{P^*}(\tilde{P}(u,u)|\mathcal{F}_t) = E_{P^*}(e^{-\int_0^u r(s)\mathrm{d}s}P(u,u)|\mathcal{F}_t)\\ &= E_{P^*}(e^{-\int_0^u r(s)\mathrm{d}s}|\mathcal{F}_t),\end{aligned}$$

so, by eliminating the discount factor

$$P(t,u) = E_{P^*}(e^{-\int_t^u r(s)\mathrm{d}s}|\mathcal{F}_t)$$

■

If we write, as usually, $Z_T = \frac{\mathrm{d}P^*}{\mathrm{d}P}$, we know that $Z_t := E(\frac{\mathrm{d}P^*}{\mathrm{d}P}|\mathcal{F}_t)$ is a martingale strictly positive, then since the filtration is that the generated by the Brwonian motion, we have the following representation:

Proposition 2.2.2 *There exists an adapted process $(q(t))_{0\leq t\leq T}$ such that, for all $0 \leq t \leq T$,*

$$Z_t = \exp\{\int_0^t q(s)\mathrm{d}W_s - \frac{1}{2}\int_0^t q^2(s)\mathrm{d}s\}, \quad c.s.$$

Proof. Since Z_t is a Brownian martingale, a localization argument (since we do not know if it is square integrable) allows us to extend the Theorem (1.2.5) and to conclude that there is a process (H_t) satisfying $\int_0^T H_t^2\mathrm{d}t < \infty$, a.s., such that

$$Z_t = 1 + \int_0^t H_s\mathrm{d}W_s,$$

now since $Z_t > 0$, P a.s., by applyin theItô formula, we have

$$\log Z_t = \int_0^t \frac{H_s}{Z_s}\mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{H_s^2}{Z_s^2}\mathrm{d}s$$

so $q(s) = \frac{H_s}{Z_s}$, *c.s.* ■

Corollary 2.2.1 *The price at time t of a bond (without coupons) with maturity $u \leq T$ is given by*

$$P(t,u) = E(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s} | \mathcal{F}_t)$$

Proof.

$$\begin{aligned} E_{P^*}(e^{-\int_t^u r(s)\mathrm{d}s}|\mathcal{F}_t) &= \frac{E(e^{-\int_t^u r(s)\mathrm{d}s} Z_u|\mathcal{F}_t)}{Z_t} \\ &= E(e^{-\int_t^u r(s)\mathrm{d}s}\frac{Z_u}{Z_t}|\mathcal{F}_t) \\ &= E(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s}|\mathcal{F}_t). \end{aligned}$$

■

The following proposition gives an economic interpretation of the process q.

Proposition 2.2.3 *For each maturity u, there exists an adapted process $(\sigma_t^u)_{0\leq t\leq u}$ such that, for all $0 \leq t \leq u$,*

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

Proof. Since $\left(\tilde{P}(t,u)\right)$ is a martingale under P^* it turns out that $\left(\tilde{P}(t,u)Z_t\right)$ is a martingale under P, it is strictly positive as well and by reasoning as before we

$$\tilde{P}(t,u)Z_t = P(0,u)e^{\int_0^t \theta_s^u \mathrm{d}W_s - \frac{1}{2}\int_0^t (\theta_s^u)^2\mathrm{d}s}$$

for a certain adapted process $(\theta_s^u)_{0\leq t\leq u}$, in such a way that

$$\begin{aligned} P(t,u) = P(0,u)\exp\{&\int_0^t r(s)\mathrm{d}s + \int_0^t (\theta_s^u - q(s))\mathrm{d}W_s \\ &- \frac{1}{2}\int_0^t ((\theta_s^u)^2 - q^2(s))\mathrm{d}s\}, \end{aligned}$$

consequently, by applying the Itô formula,

$$\begin{aligned} \frac{\mathrm{d}P(t,u)}{P(t,u)} &= r(t)\mathrm{d}t + (\theta_t^u - q(t))\mathrm{d}W_t \\ &- \frac{1}{2}((\theta_t^u)^2 - q^2(t))\mathrm{d}t \\ &+ \frac{1}{2}(\theta_t^u - q(t))^2\mathrm{d}t \\ &= (r(t) + q^2(t) - \theta_t^u q(t))\mathrm{d}t \\ &+ (\theta_t^u - q(t))\mathrm{d}W_t, \end{aligned}$$

and the result follows by taking $\sigma_t^u = \theta_t^u - q(t)$. ■

Remark 2.2.1 *If we compare the formula*

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

with

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r(t)\mathrm{d}t$$

we find that the bonds are assets with greater risk the riskless *asset S^0. Note also that, under P^**

$$\tilde{W}_t := W_t - \int_0^t q(s)\mathrm{d}s$$

is a standard $(\mathcal{F}_t)$- Brownian (by the Girsanov (1.2.3 theorem)) and we can write

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = r(t)\mathrm{d}t + \sigma_t^u \mathrm{d}\tilde{W}_t$$

justifying the name of risk neutral probability that we use for P^*.

2.3 Options on bonds

Suppose a European contingent claim with maturity T and payoff

$$(P(T,T^*) - K)_+$$

where $T^* > T$ and $P(T,T^*)$ is the price of a bond with maturity T^*. Te purpose is to valuate and hedge this call option of the bond with maturity T^*. It seems sensible to try to hedge this derivative with the riskless stock

$$S_t^0 = e^{\int_0^t r(s)\mathrm{d}s}$$

and the risky one

$$P(t,T^*) = P(0,T^*)\exp\{\int_0^t (r(s) - \frac{1}{2}\left(\sigma_s^{T^*}\right)^2)\mathrm{d}s + \int_0^t \sigma_s^{T^*}\mathrm{d}\tilde{W}_s,$$

in such a way that a strategy will be a pair of adapted processes $\left(\phi_t^0, \phi_t^1\right)_{0\leq t\leq T^*}$ that represent the amount od assets without risk and the bonds with maturity T^* respectively. The value of the self-financing portfolio at time t is given by

$$V_t = \phi_t^0 S_t^0 + \phi_t^1 P(t,T^*)$$

and the self-financing condition implies that

$$\begin{aligned}
\mathrm{d}V_t &= \phi_t^0 \mathrm{d}S_t^0 + \phi_t^1 \mathrm{d}P(t,T^*) \\
&= \phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s}\mathrm{d}t + \phi_t^1 P(t,T^*)(r(t)\mathrm{d}t + \sigma_t^{T^*}\mathrm{d}\tilde{W}_t) \\
&= (\phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s} + \phi_t^1 r(t) P(t,T^*))\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t,T^*)\mathrm{d}\tilde{W}_t \\
&= r(t)V_t\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t,T^*)\mathrm{d}\tilde{W}_t,
\end{aligned}$$

we shall impose the conditions $\int_0^T |r(t)V_t|\mathrm{d}t < \infty$ y $\int_0^T |\phi_t^1 \sigma_t^{T^*} P(t,T)|^2 \mathrm{d}t < \infty$, to get well defined objects.

Definition 2.3.1 *An strategy $\phi = (\phi_t^0, \phi_t^1)_{0 \le t \le T}$ is admissible if it is self-financing and its discounted value, $\tilde{V}_t$, is non negative.*

Proposition 2.3.1 *Let $T < T^*$. Suppose that $\sup_{0 \le t \le T} r(t) < \infty$ a.s. and that $\sigma_t^{T^*} \neq 0$ a.s. for all $0 \le t \le T$. Let h be a random variable $\mathcal{F}_T$-measurable such that $\tilde{h} = e^{-\int_0^T r(s)\mathrm{d}s} h$ is square integrable under P^*. Then there exists an admissible strategy such that at time T its value is h and at time $t \le T$ it is given by*

$$V_t = E_{P^*}(e^{-\int_t^T r(s)\mathrm{d}s} h | \mathcal{F}_t).$$

Proof. $\tilde{h}$ is a variable $\mathcal{F}_T$-measurable, with $\mathcal{F}_T = \sigma(W_t, 0 \le t \le T)$, it is square integrable, as well, with respect to P^*, so

$$M_t := E_{P^*}(\tilde{h} | \mathcal{F}_t)$$

is a, square integrable, P^*-martingala. Then $(M_t Z_t)$ is a P-martingala, no necessarily square integrable. In fact, we know that

$$E_{P^*}(\tilde{h}|\mathcal{F}_t) = \frac{E(\tilde{h} Z_T | \mathcal{F}_t)}{Z_t}$$

in such a way that

$$M_t Z_t = E(\tilde{h} Z_T | \mathcal{F}_t)$$

and $\Big(E(\tilde{h} Z_T | \mathcal{F}_t)\Big)$ is clearly a P-martingale. In that way we have, by a small extension of the Theorem (1.2.5),

$$M_t Z_t = E(M_t Z_t) + \int_0^t J_s \mathrm{d}W_s,$$

with (J_s) adapted and such that $\int_0^T J_s^2 \mathrm{d}s < \infty$ a.s., so

$$Z_t \mathrm{d}M_t + M_t \mathrm{d}Z_t + \mathrm{d}\langle M, Z\rangle_t = J_s \mathrm{d}W_s,$$

that is

$$\begin{aligned}
\mathrm{d}M_t &= -M_t \frac{\mathrm{d}Z_t}{Z_t} - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= -M_t q(t)\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}W_t - (\frac{J_t}{Z_t} - M_t q(t))q(t)\mathrm{d}t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}\tilde{W}_t = H_t \mathrm{d}\tilde{W}_t
\end{aligned}$$

with $H_t := \frac{J_t}{Z_t} - M_t q(t), 0 \le t \le T$. Therefore if we take

$$\phi_t^1 = \frac{H_t}{\sigma_t^{T^*} \tilde{P}(t,T^*)}, \phi_t^0 = E_{P^*}(\tilde{h}|\mathcal{F}_t) - \frac{H_t}{\sigma_t^{T^*}}$$

we will have a self-financing portfolio with final value $e^{\int_0^T r(s)\mathrm{d}s} M_T = h$. In fact

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}(e^{-\int_0^t r(s)\mathrm{d}s} V_t) = -e^{-\int_0^t r(s)\mathrm{d}s} r(t) V_t \mathrm{d}t + e^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}V_t \\
&= e^{-\int_0^t r(s)\mathrm{d}s}(-r(t)V_t\mathrm{d}t + r(t)V_t\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t,T^*)\mathrm{d}\tilde{W}_t) \\
&= \phi_t^1 \sigma_t^{T^*} \tilde{P}(t,T^*)\mathrm{d}\tilde{W}_t = H_t \mathrm{d}\tilde{W}_t = \mathrm{d}M_t
\end{aligned}$$

It is obvious that $\tilde{V}_t \ge 0$. The condition $\sup_{0\le t\le T} r(t) < \infty$ a.s. guarantees that $\int_0^T |r(t)V_t|\mathrm{d}t < \infty$ a.s.. ■

2.4 Short rate models

Consider an evolution of the form

$$\mathrm{d}r(t) = \mu(t, r(t))\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W_t \tag{2.1}$$

and suppose that

$$P(t,T) = F(t, r(t); T) \tag{2.2}$$

where F is a smooth function in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. Obviously the boundary condition $F(T, r(T); T) = 1$, should be fulfilled for all value of $r(T)$. Considere two bonds with different maturities T_1 and $T_2 > T_1$. Assume there exists a self-financing portfolio (ϕ_t^0, ϕ_t^1), based on the bank account and such that the bond matures at T_2 and that, at time $T_3 < T_1$, replicates the bond with maturity T_1, that is

$$P(T_3, T_1) = \phi_{T_3}^0 e^{\int_0^{T_3} r(s)\mathrm{d}s} + \phi_{T_3}^1 P(T_3, T_2)$$

then, if there is not arbitrage, we will have the equality

$$\mathrm{d}P(t, T_1) = r(t)\phi_t^0 e^{\int_0^t r(s)\mathrm{d}s}\mathrm{d}t + \phi_t^1 \mathrm{d}P(t, T_2)$$

for all $t \le T_3$, and applying the Itô formula to (2.2) we have

$$\begin{aligned}
&\frac{\partial F^{(1)}}{\partial t}\mathrm{d}t + \frac{\partial F^{(1)}}{\partial r}\mathrm{d}r(t) + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2\mathrm{d}t \\
&= r(t)\phi_t^0 S_t^0 \mathrm{d}t + \phi_t^1 \frac{\partial F^{(2)}}{\partial t}\mathrm{d}t + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\mathrm{d}r(t) + \phi_t^1 \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2\mathrm{d}t
\end{aligned}$$

So, by equating the $\mathrm{d}W_t$ and $\mathrm{d}t$ terms,

$$\begin{aligned}
&\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2 \\
&= r\phi_t^0 S_t^0 + \phi_t^1 \frac{\partial F^{(2)}}{\partial t} + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\mu + \phi_t^1 \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2
\end{aligned} \tag{2.3}$$

$$\sigma \frac{\partial F^{(1)}}{\partial r} = \phi_t^1 \frac{\partial F^{(2)}}{\partial r} \sigma,$$

hence

$$\phi_t^1 = \frac{\frac{\partial F^{(1)}}{\partial r}}{\frac{\partial F^{(2)}}{\partial r}}$$

and

$$r\phi_t^0 S_t^0 = r(F^{(1)} - \frac{\frac{\partial F^{(1)}}{\partial r}}{\frac{\partial F^{(2)}}{\partial r}} F^{(2)}).$$

Then, by substituting in (2.3) we have

$$\frac{1}{\frac{\partial F^{(1)}}{\partial r}} \left(\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2 - rF^{(1)} \right)$$
$$= \frac{1}{\frac{\partial F^{(2)}}{\partial r}} \left(\frac{\partial F^{(2)}}{\partial t} + \frac{\partial F^{(2)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2 - rF^{(2)} \right).$$

Since this is true for all, $T_1, T_2 < T$, it turns out that there exists a $\lambda(t,r)$ such that

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = \lambda\sigma\frac{\partial F}{\partial r} \text{ (structure equation)} \qquad (2.4)$$

As we see there is an indetermination in λ and this has to do with the fact that the dynamics of $r(t)$ under P does **not** determine the prices of the bonds.

We have the following proposition

Proposition 2.4.1 *Let P^* be equivalent to P such that*

$$\frac{\mathrm{d}P^*}{\mathrm{d}P} = \exp\{-\int_0^T \lambda(s, r(s))\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda^2(s, r(s))\mathrm{d}s\},$$

assume that

$$F(t, r(t); T) = E_{P^*}(e^{-\int_t^T r(s)ds}|\mathcal{F}_t)$$

is $C^{1,2}$, then it is a solution of (2.4) with the boundary condition $F(T, r(T); T) = 1$. Also, under P^*

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma d\tilde{W}_t$$

with $\tilde{W}$ $(\mathcal{F}_t)$ being a P^*-Brownian motion.

Proof. Let P^* be equivalent to P such that

$$\frac{\mathrm{d}P^*}{\mathrm{d}P} = \exp\{-\int_0^T \lambda(s, r)\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda^2(s, r)\mathrm{d}s\}$$

(a sufficient condition is the Novikov condition $E(\exp\{\frac{1}{2}\int_0^T \lambda^2(s, r(s))\mathrm{d}s\}) < \infty$) then we know, by the Girsanov theorem, that

$$\tilde{W}_{\cdot} = W_{\cdot} + \int_0^{\cdot} \lambda(s, r(s))\mathrm{d}s$$

is an $(\mathcal{F}_t)$-Brownian motion with respect to P^*. If we apply the Itô formula to $e^{-\int_0^t r(s)\mathrm{d}s}F(t, r(t); T)$ we have:

$$
\begin{aligned}
&e^{-\int_0^t r(s)\mathrm{d}s}F(t, r(t); T)\\
&= F(0, r(0); T) + \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF\right)\mathrm{d}s\\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}W_s\\
&= F(0, r(0); T) + \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r}\right)\mathrm{d}s\\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}\tilde{W}_s.
\end{aligned}
$$

Then, since $e^{-\int_0^t r(s)\mathrm{d}s}F(t, r(t); T) = E_{P^*}((e^{-\int_0^T r(u)\mathrm{d}u} | \mathcal{F}_t)$ it turns out that $\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r} = 0$. The boundary condition $F(T, r(T); T) = 1$ is obviously satisfied. ■

In this situation several models for $r(t)$, under the risk neutral probability, has been proposed:

1. Vasicek
$$dr(t) = (b - ar(t))dt + \sigma dW_t.$$

2. Cox-Ingersoll-Ross (CIR)
$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW_t$$

3. Dothan
$$dr(t) = ar(t)dt + \sigma r(t)dW_t$$

4. Black-Derman-Toy
$$dr(t) = \Theta(t)r(t)dt + \sigma(t)r(t)dW_t$$

5. Ho-Lee
$$dr(t) = \Theta(t)dt + \sigma dW_t$$

6. Hull-White (Vasicek generalizado)
$$dr(t) = (\Theta(t) - a(t)r(t))dt + \sigma(t)dW_t$$

7. Hull-White (CIR generalizado)
$$dr(t) = (\Theta(t) - a(t)r(t))dt + \sigma(t)\sqrt{r(t)}dW_t$$

2.4.1 Inversion of the yield curve

In the previous models we have several unknown parameters, that we shall denote by α. These parameters cannot be estimated from the observed values of $r(s)$, since that evolve not P^* but under the real probability P. Where we can note the effect of P^* is in the real prices of the bonds, because if the model is correct

$$P(t,T) = E_{P^*}(e^{-\int_t^T r(s)\mathrm{d}s}|\mathcal{F}_t) = F(t,r(t);T,\alpha),$$

this latter equality if the model is Markovian under P^*. Then, if, for instance, the evolution of r under P^* is given by

$$dr(t) = \mu(t,r(t);\alpha)dt + \sigma(t,r(t);\alpha)dW_t$$

we can try to solve the partial differential equation

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = 0, \tag{2.5}$$

$$F(T,r(T);T,\alpha) = 1 \tag{2.6}$$

and then try to adjust the value of α for fitting $P(t,T) = F(t,r(t);T,\alpha)$ to the observed values of the bonds. Evidently some models will be more tractable than others.

2.4.2 Affine term structures

Definition 2.4.1 *If the term structure $\{P(t,T);\ 0 \le t \le T\}$ has the form*

$$P(t,T) = F(t,r(t);T)$$

where F is given by

$$F(t,r(t);T) = e^{A(t,T)-B(t,T)r}$$

and where $A(t,T)$ and $B(t,T)$ are deterministic, then we say that the model has an affine term structure (Affine Term Structure: ATS).

The structure equation (2.5) lead us to

$$\frac{\partial A}{\partial t} - \{1 + \frac{\partial B}{\partial t}\}r - \mu B + \frac{1}{2}\sigma^2 B^2 = 0$$

and the boundary condition (2.6) to

$$A(T,T) = 0$$
$$B(T,T) = 0.$$

Then, if $\mu(t,r(t))$ and $\sigma^2(t,r(t))$ are also affine, that is

$$\mu(t,r(t)) = \alpha(t)r + \beta(t)$$
$$\sigma(t,r(t)) = \sqrt{(\gamma(t)r + \delta(t))}$$

we have

$$\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 - \{1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2\}r = 0$$

and since this is satisfied for all values of $r(t)(\omega)$ we conclude

$$\begin{aligned}\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 &= 0\\ 1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 &= 0.\end{aligned}$$

Exercise 2.4.1 *Consider all the above mentioned models except for the Dothan and Black-Derman-Toy models, and show that they are ATS.*

2.4.3 The Vasicek model

We shall apply the previous technique to the Vasicek model

$$\mathrm{d}r(t) = (b - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, b, \sigma > 0$$

Note that

$$\begin{aligned}\mathrm{d}r(t) + ar(t)\mathrm{d}t &= b\mathrm{d}t + \sigma\mathrm{d}W_t\\ &= e^{-at}\mathrm{d}\left(e^{at}r(t)\right).\end{aligned}$$

Hence

$$\mathrm{d}\left(e^{at}r(t)\right) = e^{at}b\mathrm{d}t + e^{at}\sigma\mathrm{d}W_t,$$

and finally

$$r(t) = \frac{b}{a} + e^{-at}\left(r(0) - \frac{b}{a}\right) + \sigma\int_0^t e^{-a(t-s)}\mathrm{d}W_s.$$

Then, we have that r is a Gaussian process and when $t \to \infty$, the distribution of $r(t)$ tends to a limit distribution $\mathrm{N}(b/a, \sigma^2/(2a))$. This process is named the Ornstein-Uhlenbeck process and its main feature is its mean reverting property: if the process $r(t)$ is greater than $\frac{b}{a}$, then the drift is negative and the process tends to go down. If the process $r(t)$ is less than $\frac{b}{a}$ then it tends to go up. So, in the end, it finished oscillating around the mean value $\frac{b}{a}$ with a constant variance. A drawback of this model is that it can give negative values for $r(t)$, producing arbitrage opportunities. This model is an ATS model with $\alpha(t) = -a, \beta(t) = b, \gamma(t) = 0$ y $\delta(t) = \sigma^2$, so

$$\begin{aligned}\frac{\partial A}{\partial t} - bB + \frac{1}{2}\sigma^2B^2 = 0, &\quad A(T,T) = 0\\ 1 + \frac{\partial B}{\partial t} - aB = 0, &\quad B(T,T) = 0 \qquad (2.7)\end{aligned}$$

It is easy to see that

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

then , from (2.7), we have

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2 \mathrm{d}s - b\int_t^T B\mathrm{d}s$$

and substituting for B we obtain

$$A(t,T) = \frac{B(t,T) - (T-t)}{a^2}(ab - \frac{1}{2}\sigma^2) - \frac{\sigma^2}{4a}B^2(t,T).$$

If we consider the continuous forward interest rate for the period $[t,T]$: $R(t,T)$, since

$$P(t,T) = \exp\{-(T-t)R(t,T)\}$$

and since

$$P(t,T) = \exp\{A(t,T) - B(t,T)r(t)\},$$

it turns out that

$$R(t,T) = -\frac{A(t,T) - B(t,T)r(t)}{T-t}.$$

So, in this model

$$\lim_{T\to\infty} R(t,T) = \frac{b}{a} - \frac{\sigma^2}{2a^2}$$

and this is consider as another imperfection of the model by praticcioners since it does not depend on $r(t)$.

2.4.4 The Ho-Lee model

In the Ho-Lee model

$$dr(t) = \Theta(t)dt + \sigma dW_t$$

So, $\alpha(t) = \gamma(t) = 0$, $\beta(t) = \Theta(t)$ and $\delta(t) = \sigma^2$. Then, we have the equations

$$\begin{aligned} \frac{\partial A}{\partial t} - \Theta(t)B + \frac{\sigma^2}{2}B^2 = 0, &\quad A(T,T) = 0 \\ 1 + \frac{\partial B}{\partial t} = 0, &\quad B(T,T) = 0, \end{aligned}$$

therefore

$$\begin{aligned} B(t,T) &= T - t \\ A(t,T) &= \int_t^T \Theta(s)(s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}. \end{aligned}$$

Note that, contrarily to the previous model, we do not have an explicit expression in terms of the parameters. Now, we have an infinite-dimension parameter

$\Theta(s)$. One way of estimating it is to try to fit the initially observed term structure $\{\hat{P}(0,T), T \geq 0\}$ to the theoretical values. That is

$$P(0,T) \approx \hat{P}(0,T), T \geq 0.$$

This gives

$$-\frac{\partial^2 \log P(0,T)}{\partial T^2} \approx -\frac{\partial^2 \log \hat{P}(0,T)}{\partial T^2} = \frac{\partial \hat{f}(0,T)}{\partial T}$$

and therefore

$$\Theta(T) = \frac{\partial \hat{f}(0,T)}{\partial T} + \sigma^2 T$$

2.4.5 The CIR model

In this model model

$$dr(t) = a(b - r(t))\mathrm{d}t + \sigma\sqrt{r}(t)\mathrm{d}W_t$$

where $a, b, \sigma > 0$. As in the Vasicek model there is a reversion to the mean, here given by b, but the volatility factor $\sqrt{r}(t)$ keeps the process above zero: when the process is close to zero there is only contribution of a positive drift.

Proposition 2.4.2 *Let W_1, W_2 be two independent Brownian motions and let $X_i, i = 1, 2$ be two Ornstein-Uhlenbeck process, solutions of*

$$\mathrm{d}X_i(t) = -\frac{a}{2}X_i(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_i(t), i = 1, 2.$$

Then the process

$$r(t) := X_1^2(t) + X_2^2(t),$$

satisfies

$$\mathrm{d}r(t) = (\frac{\sigma^2}{2} - ar(t))\mathrm{d}t + \sigma\sqrt{r(t)})\mathrm{d}W(t)$$

where W is a standard Brownian motion.

Proof. By the Itô formula for the bidimensional case

$$\begin{aligned}
\mathrm{d}r(t) &= 2\sum_{i=1,2} X_i(t)\mathrm{d}X_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= -ar(t)\mathrm{d}t + \sigma\sum_{i=1,2} X_i(t)\mathrm{d}W_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= (\frac{\sigma^2}{2} - ar(t))\mathrm{d}t + \sigma\sqrt{r(t)}\sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t).
\end{aligned}$$

Write

$$\mathrm{d}W(t) := \sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t),$$

then W is an Itô process with quadratic variation t:

$$\begin{aligned}[W,W]_t &= \sum_{i=1,2}\int_0^t \frac{X_i^2(s)}{r(s)}\mathrm{d}s \\ &= t.\end{aligned}$$

And by he Itô formula

$$e^{i\lambda W_t} = e^{i\lambda W_u} + i\lambda\int_u^t e^{i\lambda W_s}\mathrm{d}W_s - \frac{\lambda^2}{2}\int_u^t e^{i\lambda W_s}.\mathrm{d}s$$

Consequently

$$E(e^{i\lambda(W_t-W_u)}|\mathcal{F}_u) = 1 - \frac{\lambda^2}{2}\int_u^t E(e^{i\lambda(W_t-W_u)}|\mathcal{F}_u)\mathrm{d}s,$$

and

$$E(e^{i\lambda(W_t-W_u)}|\mathcal{F}_u) = e^{-\frac{1}{2}\lambda^2(t-u)}.$$

Hence W has continuous trajectories, with independent and homogeneous increments (and $\mathrm{N}(0,t)$). In other words, W is a Brownian motion. ∎

Remark 2.4.1 *From the previous calculations we deduce that if $ab > \frac{\sigma^2}{2}$, the values of $r(t)$ hold strictly positive.*

Bond prices for the CIR model

We have to solve

$$\begin{aligned}\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 &= 0, \\ 1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 &= 0.\end{aligned}$$

con $\beta = ab, \delta = 0, \alpha = -a$ y $\gamma = \sigma^2$. That is

$$\begin{aligned}\frac{\partial A}{\partial t} - abB &= 0, \\ 1 + \frac{\partial B}{\partial t} - aB - \frac{1}{2}\sigma^2 B^2 &= 0,\end{aligned}$$

with the boundary condition $B(T,T) = A(T,T) = 0$. It is easy to see that, by taking derivatives, we have

$$B(t,T) = \frac{2(e^{c(T-t)}-1)}{d(t)}$$

with $c = \sqrt{a^2+2\sigma^2}$ and $d(t) = (c+a)(e^{c(T-t)}-1) + 2c$. By integrating

$$A(t,T) = \frac{2ab}{\sigma^2}\left(\frac{(a+c)(T-t)}{2} + \log\frac{2c}{d(t)}\right).$$

2.4.6 The Hull-White model

In the calibration step we try to adjust the real bond prices to the the theoretical ones. If we use the notation $\{\hat{P}(0,T), T \geq 0\}$ for the observed prices, we shall find that

$$P(0,T;\alpha) = \hat{P}(0,T), \quad T \geq 0.$$

but this is not possible if our set of parameters, α, is finite dimensional. We have seen that in the Ho-Lee model this was possible due to the fact that the involved parameter $\Theta(t)$ was infinite dimensional. The Hull-White model combines this fact with the mean reverting property we have in the Vasicek model. By this reason it is quite popular. The dynamics we consider is

$$\mathrm{d}r(t) = (\Theta(t) - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, \sigma > 0.$$

Then, we have

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

and

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2 \mathrm{d}s - \int_t^T \Theta(s) B \mathrm{d}s$$

then we have a theoretical forward rates given by

$$\begin{aligned}
f(0,T) &= -\partial_T \log P(0,T) = \partial_T\left(B(0,T)r(0) - A(0,T)\right) \\
&= \partial_T\left(B(0,T)\right) r(0) - \sigma^2 \int_0^T B(s,T)\partial_T B(s,T)\mathrm{d}s + \int_0^T \Theta(s)\partial_T B(s,T)\mathrm{d}s \\
&= e^{-aT} r(0) - \sigma^2 \int_0^T \frac{1}{a}(1 - e^{-a(T-s)})e^{-a(T-s)}\mathrm{d}s + \int_0^T \Theta(s) e^{-a(T-s)}\mathrm{d}s \\
&= e^{-aT} r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2 + \int_0^T \Theta(s) e^{-a(T-s)}\mathrm{d}s.
\end{aligned}$$

We have to solve $f(0,T) = \hat{f}(0,T)$. By differentiating with respect to T and we call $g(T) := e^{-aT} r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2$, we have

$$\begin{aligned}
\partial_T f(0,T) &= \partial_T g(T) + \Theta(T) - a\int_0^T \Theta(s) e^{-a(T-s)}\mathrm{d}s \\
&= \partial_T g(T) + \Theta(T) - a(f(0,T) - g(T)),
\end{aligned}$$

so

$$\Theta(T) = \partial_T f(0,T) - \partial_T g(T) + a(f(0,T) - g(T)).$$

We can then to capture $\hat{f}(0,T)$ doing

$$\Theta(T) = \partial_T \hat{f}(0,T) - \partial_T g(T) + a(\hat{f}(0,T) - g(T)).$$

Exercise 2.4.2 *Let $(W_1, W_2, ..., W_n)$ n be independent standard Brownian motions and let $X_i, i = 1, ..., n,$ be Ornstein-Uhlenbeck processes solving*

$$dX_i(t) = -aX_i(t)dt + \sigma dW_i(t), i = 1, ..., n.$$

Consider the process

$$r(t) := X_1^2(t) + ... + X_n^2(t).$$

Show that

$$dr(t) = (n\sigma^2 - 2ar(t))dt + 2\sigma\sqrt{r(t)})dW(t)$$

where W is a standard Brownian motion.

2.5 Forward rate models

As we have seen one drawback of the short rate models is their difficulty in capturing the term structure observed at initial time. An alternative is to model the forward rates $f(t,T)$ and to use the relation $r(t) = f(t,t)$, this is the so-called éste es el enfoque de Heath-Jarrow-Morton (HJM) approach. We have that

$$P(t,T) = \exp\{-\int_t^T f(t,s)\mathrm{d}s\},$$

so $f(t,s)$ represent the instantaneous rates (at s) *anticipated* by the market at t. Suppose that under a risk neutral probability P^*

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t \quad ,T \geq 0 \tag{2.8}$$

with

$$f(0,T) = \hat{f}(0,T).$$

We shall try to deduce the evolution of $P(t,T)$ from that of $f(t,T)$. If we write $X_t = -\int_t^T f(t,s)\mathrm{d}s$, we have $P(t,T) = e^{X_t}$ and from the equation (2.8) we obtain

$$\begin{aligned}
\mathrm{d}X_t &= f(t,t)\mathrm{d}t - \int_t^T \mathrm{d}f(t,s)\mathrm{d}s = \\
&= f(t,t)\mathrm{d}t - \int_t^T \alpha(t,s)\mathrm{d}t\mathrm{d}s - \int_t^T \sigma(t,s)\mathrm{d}W_t\mathrm{d}s \\
&= (f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s)\mathrm{d}t - (\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}W_t,
\end{aligned}$$

where we have applied a *stochastic*Fubini theorem. Then

$$\begin{aligned}\frac{\mathrm{d}P(t,T)}{P(t,T)} &= \mathrm{d}X_t + \frac{1}{2}\mathrm{d}\langle X\rangle_t \\ &= (f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s)\mathrm{d}t - (\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}W_t \\ &+ \frac{1}{2}(\int_t^T \sigma(t,s)\mathrm{d}s)^2\mathrm{d}t \\ &= (f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}(\int_t^T \sigma(t,s)\mathrm{d}s)^2)\mathrm{d}t \\ &- (\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}W_t.\end{aligned}$$

And if we compare with that obtained in (2.2.1) and we have into account that $f(t,t) = r(t)$ it turns out that

$$-\int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}(\int_t^T \sigma(t,s)\mathrm{d}s)^2 = 0,$$

therefore

$$\alpha(t,T) = (\int_t^T \sigma(t,s)ds)\sigma(t,T)$$

and we can write the evolution equation (2.8) as

$$\mathrm{d}f(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t.$$

Note that all depends on $\sigma(t,s)$, that is on certain volatility. We have *eliminated* the drift $\alpha(t,T)$, as in certain way happened for the call prices in the Black-Scholes model.

Then the algorithm to use the HJM approach is

1. Specify the volatilities $\sigma(t,s)$
2. Integrate $\mathrm{d}f(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t$ with the initial condition $f(0,T) = \hat{f}(0,T)$.
3. Calculate the prices of the bonds from the formula $P(t,T) = \exp\{-\int_t^T f(t,s)\mathrm{d}s\}$.
4. To use the previous results to calculate contingent claim prices.

Example 2.5.1 *Suppose that $\sigma(t,T)$ is constant that we denote σ. Then*

$$\mathrm{d}f(t,T) = \sigma^2(T-t)\mathrm{d}t + \sigma\mathrm{d}W_t,$$

so

$$f(t,T) = \hat{f}(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma W_t.$$

In particular

$$r(t) = f(t,t) = \hat{f}(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t$$

and therefore

$$\mathrm{d}r(t) = (\frac{\partial \hat{f}(0,T)}{\partial T}|_{T=t} + \sigma^2 t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

but this is the Ho-Lee adjusted to the initial structure of the forward rates.

Example 2.5.2 *A usual assumption consist of assuming that the forward rates with greater maturity time has a lower fluctuation than that with a lower maturity time. To capture this feature we can take, for instance,* $\sigma(t,T) = \sigma e^{-b(T-t)}$, $b > 0$. *We have then*

$$\int_t^T \sigma(t,s)ds = \int_t^T e^{-b(s-t)}ds = -\frac{\sigma}{b}\left(e^{-b(T-t)} - 1\right),$$

and

$$\mathrm{d}f(t,T) = -\frac{\sigma^2}{b}e^{-b(T-t)}(e^{-b(T-t)} - 1)\mathrm{d}t + \sigma e^{-b(T-t)}\mathrm{d}W_t.$$

Therefore

$$f(t,T) = f(0,T) + \frac{\sigma^2 e^{-2bT}}{2b^2}\left(1 - e^{2bt}\right) - \frac{\sigma^2 e^{-bT}}{b^2}(1 - e^{bt}) + \sigma e^{-bT}\int_0^t e^{bs}\mathrm{d}W_s.$$

In particular

$$r(t) = f(0,t) + \frac{\sigma^2}{2b^2}\left(e^{-2bt} - 1\right) - \frac{\sigma^2}{b^2}(e^{-bt} - 1) + \sigma e^{-bt}\int_0^t e^{bs}\mathrm{d}W_s,$$

that corresponds to the Hull-White model considered above.

Remark 2.5.1 *A sufficient condition to guarantee the equality* $\int_0^T \sigma(t,s)\mathrm{d}W_t\mathrm{d}s = \int_0^T \sigma(t,s)\mathrm{d}s)\mathrm{d}W_t$ *es* $\int_0^T E(\sigma^2(t,s))\mathrm{d}s\mathrm{d}t < \infty$, *see Lamberton and Lapeyre (1996) page 138.*

2.5.1 The Musiela equation

Define

$$r(t,x) := f(t,t+x)$$

and assume a model HJM under the neutral probability, in such a way that

$$df(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)ds)dt + \sigma(t,T)d\tilde{W}_t,$$

We have the following proposition,

Proposition 2.5.1

$$dr(t,x) = \{\frac{\partial}{\partial x} r(t,x) + \sigma_0(t,x)(\int_0^x \sigma_0(t,s)ds\}dt + \sigma_0(t,x)d\tilde{W}_t$$

where

$$\sigma_0(t,x) := \sigma(t,t+x)$$

Proof.

$$\begin{aligned} dr(t,x) &= df(t,T)_{|T=t+x} + \frac{\partial}{\partial T} f(t,T)_{|T=t+x} dt \\ &= \sigma(t,t+x)(\int_t^{t+x} \sigma(t,s)ds)dt + \sigma(t,t+x)d\tilde{W}_t \\ &+ \frac{\partial}{\partial x} r(t,x)dt \end{aligned}$$

■

Note that the Musiela equation is a stochastic partial differential equation.

2.6 Change of numeraire. The forward measure

We are going to study a procedure that is useful when we want to calculate prices of options in a bond market. It has to do with the use of the so-called *forward measure*. Let P^* the neutral probability. By definition P^* is a probability such that

$$\left(\tilde{P}(t,T)\right)_{0 \leq t \leq T}$$

are martingales, for all values of T. Fix a maturity time T and consider the values of bonds with another maturity time $\tilde{T} > T$ in terms of the bond with maturity T:

$$U_{T,\tilde{T}}(t) := \frac{P(t,\tilde{T})}{P(t,T)}.$$

That is instead of taking as reference (*numeraire*) the value of a unit of money in the bank account, we take the value of a bond with maturity T. Let P^T ne a probability with respect to which $\left(U_{T,\tilde{T}}(t)\right)_{0 \leq t \leq T}$ are martingales for all $\tilde{T} > T$. We call P^T the *forward measure*. Define a probability at $\mathcal{F}_T$, P^T such that

$$\frac{\mathrm{d}P^T}{\mathrm{d}P^*} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)}.$$

We can see that it is a forward measure.

Proposition 2.6.1 *If $(V_t)_{0 \leq t \leq T}$ is the value of a self-financing portfolio then its discounted value using as reference (*numeraire*) the bond value $P(t,T)$, is a P^T-martingale. That is*

$$\frac{V_t}{P(t,T)}, \quad 0 \leq t \leq T,$$

is a P^T-martingale.

Proof. Define

$$Z_t := E_{P^*}(\frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)}|\mathcal{F}_t),$$

then

$$Z_t = \frac{\tilde{P}(t,T)}{P(0,T)}.$$

By the Bayes (1.8) rule

$$\begin{aligned} E_{P^T}(\frac{V_T}{P(T,T)}|\mathcal{F}_t) = E_{P^T}(V_T|\mathcal{F}_t) &= \frac{E_{P^*}(V_T Z_T|\mathcal{F}_t)}{Z_t} \\ &= \frac{E_{P^*}(\tilde{V}_T|\mathcal{F}_t)}{P(0,T)Z_t} = \frac{\tilde{V}_t}{\tilde{P}(t,T)} \\ &= \frac{V_t}{P(t,T)}. \end{aligned}$$

■

Corollary 2.6.1 *The price of a replicable T-payoff Y is given by*

$$P(t,T)E_{P^T}(Y|\mathcal{F}_t).$$

Proof. Let $(V_t)_{0\leq t\leq T}$ the self-financing portfolio that replicates Y, then $V_T = Y$ and therefore

$$E_{P^T}(Y|\mathcal{F}_t) = \frac{V_t}{P(t,T)}.$$

■

Proposition 2.6.2 *Suppose that*

$$\frac{\partial}{\partial T}E_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}|\mathcal{F}_t) = E_{P^*}(\frac{\partial}{\partial T}\left(e^{-\int_t^T r_s \mathrm{d}s}\right)|\mathcal{F}_t),$$

then

$$E_{P^T}(r_T|\mathcal{F}_t) = f(t,T).$$

Proof.

$$\begin{aligned} f(t,T) &= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{1}{P(t,T)}\frac{\partial}{\partial T}E_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}|\mathcal{F}_t) \\ &= -\frac{1}{P(t,T)}E_{P^*}(\frac{\partial}{\partial T}\left(e^{-\int_t^T r_s \mathrm{d}s}\right)|\mathcal{F}_t) = \frac{1}{P(t,T)}E_{P^*}(r_T e^{-\int_t^T r_s \mathrm{d}s}|\mathcal{F}_t) \\ &= E_{P^T}(r_T|\mathcal{F}_t). \end{aligned}$$

■

Let $(S_t)_{0\leq t\leq T}$ an asset strictly positive and denote by $P^{(S)}$ the probability (in $\mathcal{F}_T$) that makes

$$\left(\frac{V_t}{S_t}\right)_{0\leq t\leq T}$$

a martingale, where $(V_t)_{0\leq t\leq T}$ is a self-financing portfolio. We have a general formula general for an option price.

Proposition 2.6.3 *The price of a replicable T-payoff Y is given by*

$$S_t E_{P^{(S)}}(\frac{Y}{S_T}|\mathcal{F}_t).$$

Proof. Let $(V_t)_{0\leq t\leq T}$ be the self-financing portfolio that replicates Y, then $V_T = Y$ and therefore

$$E_{P^{(S)}}(\frac{V_T}{S_T}|\mathcal{F}_t) = \frac{V_t}{S_t}.$$

■

Proposition 2.6.4 *Let $(S_t)_{0\leq t\leq T}$ be an asset strictly positive, then the price of a call option with maturity T of the asset S and strike K is given by*

$$\Pi(t;S) = S_t P^{(S)}(S_T \geq K|\mathcal{F}_t) - KP(t,T)P^T(S_T \geq K|\mathcal{F}_t).$$

Proof.

$$\begin{aligned}
\Pi(t;S) &= E_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+|\mathcal{F}_t)\\
&= E_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)\mathbf{1}_{\{S_T\geq K\}}|\mathcal{F}_t)\\
&= E_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}S_T\mathbf{1}_{\{S_T\geq K\}}|\mathcal{F}_t) - KE_{P^*}(e^{-\int_t^T r_s \mathrm{d}s}\mathbf{1}_{\{S_T\geq K\}}|\mathcal{F}_t)\\
&= S_t P^{(S)}(S_T \geq K|\mathcal{F}_t) - KP(t,T)P^T(S_T \geq K|\mathcal{F}_t),
\end{aligned}$$

with

$$\frac{\mathrm{d}P^{(S)}}{\mathrm{d}P^*} = \frac{e^{-\int_0^T r_s \mathrm{d}s}S_T}{S_0}.$$

■

Suppose that S is another bond with maturity $\bar{T} > T$, then the option (with maturity T) on this bond has a price given by

$$\begin{aligned}
\Pi(t;S) &= P(t,\bar{T})P^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t)) - P(t,T)P^T(P(T,\bar{T}) \geq K|\mathcal{F}_t))\\
&= P(t,\bar{T})P^{\bar{T}}(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^T(\frac{P(T,\bar{T})}{P(T,T)} \geq K|\mathcal{F}_t).
\end{aligned}$$

Define,

$$U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}.$$

In the context of affine structures

$$U(t,T,\bar{T}) = \frac{P(t,T)}{P(t,\bar{T})} = \exp\{-A(t,\bar{T}) + A(t,T) + (B(t,\bar{T}) - B(t,T))r_t\}$$

and with respect to P^*

$$\mathrm{d}U(t) = U(t)(...\mathrm{d}t + (B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t).$$

Then under $P^{\bar{T}}$ and P^T we have

$$\mathrm{d}U(t) = U(t)(B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t^{\bar{T}},$$
$$\mathrm{d}U^{-1}(t) = -U^{-1}(t)(B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t^{T}.$$

in such a way that

$$U(T) = \frac{P(t,T)}{P(t,\bar{T})}\exp\{-\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{\bar{T}} - \frac{1}{2}\int_t^T \sigma^2_{\bar{T},T}(s)\mathrm{d}s\},$$
$$U^{-1}(T) = \frac{P(t,\bar{T})}{P(t,T)}\exp\{\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{T} - \frac{1}{2}\int_t^T \sigma^2_{\bar{T},T}(s)\mathrm{d}s\}.$$

with

$$\sigma_{\bar{T},T}(t) = -(B(t,\bar{T}) - B(t,T))\sigma_t.$$

Therefore, if σ_t is **deterministic** the law of $\log U(T)$ conditional to $\mathcal{F}_t$ is Gaussiana with respect to P^T and $P^{\bar{T}}$, with variance

$$\Sigma^2_{t,T,\bar{T}} := \int_t^T \sigma^2_{\bar{T},T}(s)\mathrm{d}s,$$

$$\text{Ley}\left(\frac{\log U(T) - \log\frac{P(t,T)}{P(t,\bar{T})} + \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}|\mathcal{F}_t\right) \sim N(0,1) \text{ bajo } P^{\bar{T}}$$
$$\text{Ley}\left(\frac{\log U^{-1}(T) - \log\frac{P(t,\bar{T})}{P(t,T)} + \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}|\mathcal{F}_t\right) \sim N(0,1) \text{ bajo } P^{T}$$

Note finally that

$$\Pi(t;S) = P(t,\bar{T})P^{\bar{T}}(\frac{P(T,T)}{P(T,\bar{T})} \le \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^T(\frac{P(T,\bar{T})}{P(T,T)} \ge K|\mathcal{F}_t) \tag{2.9}$$
$$= P(t,\bar{T})P^{\bar{T}}(U(T) \le \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^T(U^{-1}(T) \ge K|\mathcal{F}_t)$$
$$= P(t,\bar{T})P^{\bar{T}}(\log U(T) \le -\log K|\mathcal{F}_t) - KP(t,T)P^T(\log U^{-1}(T) \ge \log K|\mathcal{F}_t)$$
$$= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-),$$

with

$$d\pm = \frac{\log\frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}.$$

Example 2.6.1 *In the Ho-Lee model*

$$\sigma_{\bar{T},T} = -\sigma(\bar{T} - T),$$
$$\Sigma_{t,T,\bar{T}} = \sigma(\bar{T} - T)\sqrt{T - t}.$$

Example 2.6.2 *For the Vasicek model*

$$\sigma_{\bar{T},T} = \frac{\sigma}{a} e^{at}(e^{-a\bar{T}} - e^{-aT}),$$
$$\Sigma^2_{t,T,\bar{T}} = \frac{\sigma^2}{2a^3}(1 - e^{-2(T-t)})(1 - e^{-(\bar{T}-T)})^2.$$

and the same for the Hull-White model!.

2.7 Market models

2.7.1 A market model for Swaptions

Consider a *payer swpation* with maturity $T < T_0$, *tenor structure* $T_1, T_2, ..., T_n$, and *swap rate* R. Its payoff is

$$(S(T) - Z(T))_+$$

con

$$S(T) = P(T, T_0) - P(T, T_n)$$

that is the value of the floating payments and

$$Z(T) = R\delta \sum_{i=1}^{n} P(T, T_i)$$

the value of payments with fixed rate. We can take $Z(t)$ as *numeraire* and the price will be

$$Z(t)E_{P^{(Z)}}(\frac{(S(T) - Z(T))_+}{Z(T)}|\mathcal{F}_t)) = Z(t)E_{P^{(Z)}}((\frac{S(T)}{Z(T)} - 1)_+|\mathcal{F}_t)).$$

Then, if we assume that under P, or P^* we have an evolution

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right) = \frac{S(t)}{Z(t)}\left(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\right),$$

with σ constant, it turns out that, under $P^{(Z)}$

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right) = \frac{S(t)}{Z(t)}\sigma \mathrm{d}W^Z_t,$$

so

$$\frac{S(T)}{Z(T)} = \frac{S(t)}{Z(t)} \exp\left\{\int_t^T \sigma \mathrm{d}W^Z_s - \frac{1}{2}\int_t^T \sigma^2 \mathrm{d}s\right\},$$

and we obtain the Black-Scholes formula of a call with strike 1 and $r = 0$, multiplied by $Z(t)$:

$$Z(t)\left(\frac{S(t)}{Z(t)}\Phi(d_+) - \Phi(d_-)\right) = S(t)\Phi(d_+) - Z(t)\Phi(d_-),$$

with

$$\Phi(d_\pm) = \frac{\log\frac{S(t)}{Z(t)} \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}.$$

This formula is known as the Margrabe formula. Remember that the *forward swap rate* was given by

$$R(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta\sum_{i=1}^n P(t,T_i)},$$

so

$$\frac{S(t)}{Z(t)} = \frac{P(t,T_0) - P(t,T_n)}{R\delta\sum_{i=1}^n P(t,T_i)} = \frac{R(t)}{R}.$$

Therefore the volatility σ corresponds to the volatility of e $R(t)$. The previous formula can be written more explicitly as

$$\text{Swaption}_t = \left(P(t,T_0) - P(t,T_n)\right)\Phi(d_+) - \left(R\delta\sum_{i=1}^n P(t,T_i)\right)\Phi(d_-),$$

where

$$\Phi(d_\pm) = \frac{\log\left(P(t,T_0) - P(t,T_n)\right) - \log\left(R\delta\sum_{i=1}^n P(t,T_i)\right) \pm \sigma^2(T-t)}{\sigma\sqrt{(T-t)}}.$$

2.7.2 A LIBOR market model

First of all note that

$$L(t;T_{i-1},T_i) = -\frac{P(t,T_i) - P(t,T_{i-1})}{\delta P(t,T_i)},$$

so

$$U(t,T_{i-1},T_i) = \frac{P(t,T_{i-1})}{P(t,T_i)} = 1 + \delta L(t;T_{i-1},T_i)$$

and therefore

$$\mathrm{d}U(t,T_{i-1},T_i) = \delta\mathrm{d}L(t;T_{i-1},T_i),$$

then, respect to P^{T_i}, and if the structure is affine,

$$\begin{aligned}\mathrm{d}L(t;T_{i-1},T_i) &= \frac{1}{\delta}U(t,T_{i-1},T_i)(B(t,T_i) - B(t,T_{i-1}))\sigma_t\mathrm{d}W_t^{T_i} \\ &= \frac{1}{\delta}(1 + \delta L(t;T_{i-1},T_i))(B(t,T_i) - B(t,T_{i-1}))\sigma_t\mathrm{d}W_t^{T_i}.\end{aligned}$$

Consequently the structure of LIBOR is established . Another way is to fix a model for the LIBORs, but then we have to check the consistency and if the whole model is free of arbitrage. One way is that the whole mode implies a model for forward rates free of arbitrage. It can be seen, by a backward induction, that it is possible to build a LIBOR model such that

$$\mathrm{d}L(t;T_{i-1},T_i) = L(t;T_{i-1},T_i)\lambda(t,T_{i-1},T_i)\mathrm{d}W_t^{T_i}, i = 1,...,n$$

with initial conditions

$$L(0;T_{i-1},T_i) = -\frac{P(0,T_i) - P(0,T_{i-1})}{\delta P(0,T_i)}, i = 1,...,n.$$

In particular, if we take $\lambda(t,T_{i-1},T_i)$ deterministic we have that $L(t;T_{i-1},T_i)$ is lognormal (LLM). This model is very popular.

Let $P(t,T_n)$ fix as numeraire, then

$$U(t,T_i,T_n) = \frac{P(t,T_i)}{P(t,T_n)},$$

are P^{T_n}-martingales for $i = 0,...,n-1$ and since

$$\mathrm{d}U(t,T_i,T_n) = \delta \mathrm{d}L(t;T_i,T_n),$$

in turns out that

$$\mathrm{d}L(t;T_i,T_n) = L(t;T_i,T_n)\lambda_i^n(t)\mathrm{d}W^{T_n}.$$

We have arbitrariness choosing $\lambda_i^n(t)$. Fix $\lambda_{n-1}^n(t) = \lambda(t,T_{n-1},T_n)$ and consider now the market when t moves between 0 and T_{n-1}, take $P(t,T_{n-1})$ as reference, we have that

$$U(t,T_i,T_{n-1}) = \frac{P(t,T_i)}{P(t,T_{n-1})}, i = 0,...,n-2,$$

are $P^{T_{n-1}}$-martingales, but

$$\begin{aligned} U(t,T_i,T_{n-1}) = \frac{P(t,T_i)}{P(t,T_{n-1})} &= \frac{\frac{P(t,T_i)}{P(t,T_n)}}{\frac{P(t,T_{n-1})}{P(t,T_n)}} \\ &= \frac{U(t,T_i,T_n)}{U(t,T_{n-1},T_n)}, \end{aligned}$$

therefore we can calculate the dynamics in terms of W^{T_n}. For simplicity in the notation write

$$\mathrm{d}U(t,T_i,T_n) = \alpha \mathrm{d}W_t^{T_n}, \quad \mathrm{d}U(t,T_{n-1},T_n) = \beta \mathrm{d}W_t^{T_n}$$

$$
\begin{aligned}
\mathrm{d}U(t,T_i,T_{n-1}) &= \frac{1}{U(t,T_{n-1},T_n)}\mathrm{d}U(t,T_i,T_n) + U(t,T_i,T_n)\mathrm{d}\frac{1}{U(t,T_{n-1},T_n)} \\
&+ \mathrm{d}\langle U(\cdot,T_i,T_n), \frac{1}{U(\cdot,T_{n-1},T_n)}\rangle_t \\
&= \frac{\alpha}{U(t,T_{n-1},T_n)}\mathrm{d}W_t^{T_n} - \frac{U(t,T_i,T_n)\beta}{U(t,T_{n-1},T_n)^2}\mathrm{d}W_t^{T_n} \\
&+ \frac{U(t,T_i,T_n)\beta^2}{U(t,T_{n-1},T_n)^3}\mathrm{d}t \\
&- \frac{\alpha\beta}{U(t,T_{n-1},T_n)^2}\mathrm{d}t \\
&= \frac{\alpha U(t,T_{n-1},T_n) - \beta U(t,T_i,T_n)}{U(t,T_{n-1},T_n)}\left(\mathrm{d}W_t^{T_n} - \frac{\beta}{U(t,T_{n-1},T_n)}\mathrm{d}t\right) \\
&= \gamma_i^n(t)\left(\mathrm{d}W_t^{T_n} - \frac{\delta L(t;T_{n-1},T_n)\lambda(t;T_{n-1},T_n)}{1+\delta L(t;T_{n-1},T_n)}\mathrm{d}t\right),
\end{aligned}
$$

for certain process, γ_i^n, then , we can find a forward measure $P^{T_{n-1}}$ respect to which $U(t,T_i,T_{n-1})$, $i=1,...,n-2$ are martingales, and we will have

$$
\mathrm{d}L(t;T_i,T_{n-1}) = L(t;T_{n-2},T_{n-1})\lambda_i^{n-1}(t)\mathrm{d}W^{T_{n-1}}.
$$

Now fix $\lambda_{n-2}^{n-1}(t) := \lambda(t,T_{n-2},T_{n-1})$ and so on. Finally we can fix the evolution of all LIBOR and bonds in such a way that the market model is free of arbitrage.

2.7.3 A market model for caps

Proposition 2.7.1 *In an LLM model the price of a cap ("in arrears") with swap rate K and tenor-structure $T_i = T_0 + i\delta$, $i=1,...,n$ is given by*

$$
\Pi(t) = \sum_{i=1}^{n} \delta P(t,T_i)(L(t;T_{i-1},T_i)\Phi(d_{i+}) - K\Phi(d_{i-})),
$$

where

$$
d_{i\pm} = \frac{\log\frac{L(t;T_{i-1},T_i)}{K} \pm \frac{1}{2}\upsilon_i^2(t)}{\upsilon_i(t)},
$$

with

$$
\upsilon_i^2(t) = \int_t^{T_{i-1}} \lambda^2(s,T_{i-1},T_i)\mathrm{d}s.
$$

Proof.

$$
\Pi(t) = \sum_{i=1}^{n} K\delta P(t,T_i)E_{P^{T_i}}\left((L(T_{i-1},T_i)-K)_+|\mathcal{F}_t\right),
$$

and under P^{T_i},

$$\begin{aligned}\log L(T_{i-1},T_i) &= \log L(T_{i-1},T_{i-1},T_i)\\ &= \log L(t,T_{i-1},T_i) + \int_t^{T_{i-1}} \lambda(s,T_{i-1},T_i)\mathrm{d}W_s^{T_i}\\ &- \frac{1}{2}\int_t^{T_{i-1}} \lambda^2(s,T_{i-1},T_i)\mathrm{d}s.\end{aligned}$$

■

Remark 2.7.1 *If $\lambda^2(s,T_{i-1},T_i) = \sigma_i^2$, $i = 1,..,n$ for certain constants, then we have the so-called Black formula for caps. This market model is incompatible with a model for swaps with constant volatility for the forward swap rate.*

2.8 Miscelanea

2.8.1 Forwards and Futures

Definition 2.8.1 *Let X be a payoff at T. A forward contract on X with delivering time T is a contract established at $t < T$ that specifies a forward price $f(t;T)$* ***that will be paid at*** *T for receiving X. The price $f(t;T)$ is fixed in such a way that the contract price at t is zero.*

Proposition 2.8.1

$$\begin{aligned}f(t;T) &= \frac{1}{P(t,T)}E_{P^*}(X\exp\{-\int_t^T r_s ds\}|\mathcal{F}_t)\\ &= E_{P^T}(X|\mathcal{F}_t).\end{aligned}$$

Definition 2.8.2 *Let X a payoff at T. A contract of futures on X and delivering time T is a financial asset with the following properties*

- There exist a *future price* $F(t;T)$ on X at each time t.
- At T the owner of the contract pays $F(T;T)$ and receives X.
- For any arbitrary interval $(s,t]$ the owner receives $F(t;T) - F(s;T)$.
- At each time the price of the contract is zero.

Proposition 2.8.2
$$F(t;T) = E_{P^*}(X|\mathcal{F}_t).$$

Proof. Let V_t the value of a self-financing portfolio formed by a bank account and a contract of futures

$$\begin{aligned}V_t &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 \cdot 0\\ &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s}\end{aligned}$$

but

$$\begin{aligned}\mathrm{d}V_t &= r_t\phi_t^0 e^{\int_0^t r_s \mathrm{d}s}\mathrm{d}t + \phi_t^1 \mathrm{d}F(t;T)\\ &= r_t V_t \mathrm{d}t + \phi_t^1 \mathrm{d}F(t;T),\end{aligned}$$

so

$$\mathrm{d}\tilde{V}_t = e^{\int_0^t r_s \mathrm{d}s}\phi_t^1 \mathrm{d}F(t;T),$$

with $F(T;T) = X$ and since $\tilde{V}$ is a martingale with respect to P^* it turns out that $F(\cdot;T)$ is also a martingale and therefore

$$F(t;T) = E_{P^*}(F(T;T)|\mathcal{F}_t) = E_{P^*}(X|\mathcal{F}_t)$$

■

Corollary 2.8.1 *Future prices and forward prices coincide if and only if interest rates are deterministic.*

2.8.2 Stock options

Suppose that bonds have a volatility $\sigma_B(t,T)$, d-dimensional, **deterministic** and cadlag, that is, that under the risk neutral probability P^*

$$\mathrm{d}P(t,T) = P(t,T)(...\mathrm{d}t + \sigma_B(t,T)\cdot dW_t)$$

and that there is a stock S such that under P^*

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sigma_S(t)\cdot dW_t),$$

where $\|\sigma_S(t) - \sigma_B(t,T)\| > 0$, $\sigma_S(t)$ **determinista** and cadlag. Then the price of a call option with strike K is given by

$$C_t = S_t\Phi(d_+) - KP(t,T)\Phi(d_-), \tag{2.10}$$

with

$$d_\pm = \frac{\log\frac{S_t}{KP(t,T)} \pm \frac{1}{2}\Sigma_t^2}{\Sigma_t},$$

where

$$\Sigma_t^2 = \int_t^T \|\sigma_S(u) - \sigma_B(u,T)\|^2\, du.$$

In fact, by the general formula we have seen above

$$\Pi(t;S) = S_t P^{(S)}(S_T \geq K|\mathcal{F}_t) - KP(t,T)P^T(S_T \geq K|\mathcal{F}_t),$$

under P^*

$$F_S(t) := \frac{P(t,T)}{S_t} = \frac{P(0,T)}{S_0}\exp\{\int_0^t ..\mathrm{d}u + \int_0^t (\sigma_S(u) - \sigma_B(u,T))\cdot dW_u\},$$

and under $P^{(S)}$

$$\mathrm{d}F_S(t) = F_S||\sigma_S(u) - \sigma_B(u,T))||\mathrm{d}W_u^{(S)},$$

where $W^{(S)}$ is a $P^{(S)}$-Brownian motion. Analogously under P^T

$$F_B(t) := \frac{S_t}{P(t,T)}$$

$$\mathrm{d}F_B(t) = -F_B||\sigma_S(u) - \sigma_B(u,T))||\mathrm{d}W_u^T,$$

with W^T Brownian motion under P^T. And doing similar calculations to that in (2.9) we obtain (2.10).

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