Time series analysis

Lecture 1. Stochastic process and its main characteristics

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Stochastic Processes

Introduction

Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t,\xi)$ is assigned.

The collection of such waveforms form a stochastic process. The set of $\{\xi_k\}$ and the time index t can be continuous or discrete (countably infinite or finite) as well.

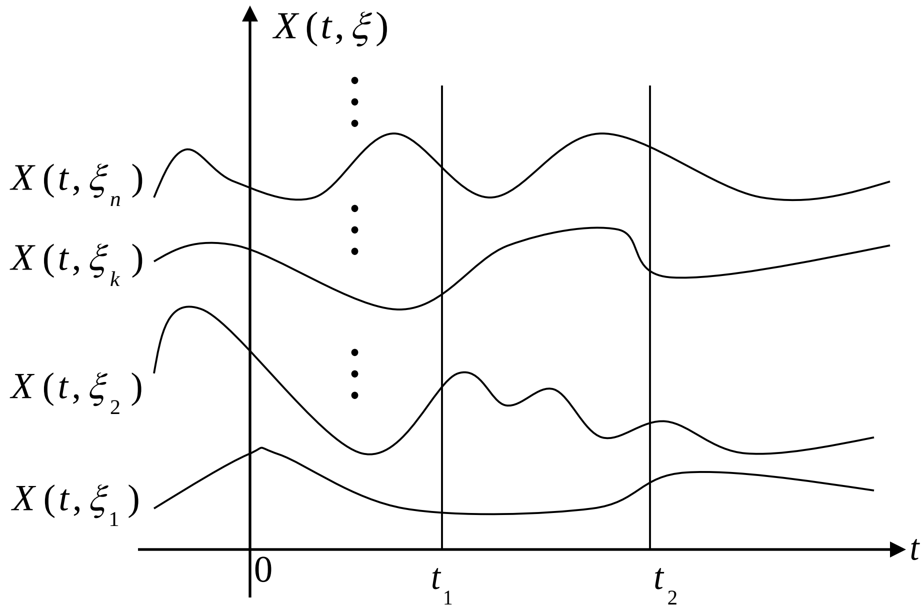

Fig. 1

For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t,\xi)$ is a specific time function. For fixed t,

$$X_1 = X(t_1,\xi_i)$$

is a random variable. The ensemble of all such realizations $X(t,\xi)$ over time represents the stochastic process $X(t)$. (see Fig 1). For example

$$X(t) = a\cos(\omega_0 t + \varphi),$$

where φ is a uniformly distributed random variable in $(0, 2\pi)$, represents a stochastic process. Stochastic processes are everywhere: Brownian motion, stock market fluctuations, various queuing systems all represent stochastic phenomena.

If $X(t)$ is a stochastic process, then for fixed t, $X(t)$ represents a random variable. Its distribution function is given by

$$F_X(x,t) = P\{X(t) \le x\} \qquad (1\text{-}1)$$

Notice that $F_X(x,t)$ depends on t, since for a different t, we obtain a different random variable. Further

$$f_X(x,t) \triangleq \frac{dF_X(x,t)}{dx} \qquad (1\text{-}2)$$

represents the first-order probability density function of the process $X(t)$.

For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively. Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\} \tag{1-3}$$

and

$$f_X(x_1, x_2, t_1, t_2) \triangleq \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \, \partial x_2} \tag{1-4}$$

represents the second-order density function of the process $X(t)$. Similarly $f_X(x_1, x_2, \cdots x_n, \; t_1, t_2 \cdots, t_n)$ represents the n^{th} order density function of the process $X(t)$. Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_X(x_1, x_2, \cdots x_n, \; t_1, t_2 \cdots, t_n)$ for all t_i, $i = 1, 2, \cdots, n$ and for all n. (an almost impossible task in reality).

Mean of a Stochastic Process:

$$\mu(t) \triangleq E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx \qquad \text{(1-5)}$$

represents the mean value of a process $X(t)$. In general, the mean of a process can depend on the time index t.

Autocorrelation function of a process $X(t)$ is defined as

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\} = \int\int x_1 x_2^* f_X(x_1,x_2,t_1,t_2)dx_1 dx_2 \quad \text{(1-6)}$$

and it represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ generated from the process $X(t)$.

Properties:

1. $R_{XX}(t_1,t_2) = R_{XX}^*(t_2,t_1) = [E\{X(t_2)X^*(t_1)\}]^*$ (1-7)

2. $R_{XX}(t,t) = E\{|X(t)|^2\} > 0.$ (Average instantaneous power)

3. $R_{XX}(t_1, t_2)$ represents a nonnegative definite function, i.e., for *any* set of constants $\{a_i\}_{i=1}^n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i, t_j) \geq 0. \qquad (1\text{-}8)$$

Eq. (14-8) follows by noticing that $E\{|Y|^2\} \geq 0$ for $Y = \sum_{i=1}^{n} a_i X(t_i).$ The function

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2) \qquad (1\text{-}9)$$

represents the **autocovariance** function of the process $X(t)$.

Example 1.1

Let

$$z = \int_{-T}^{T} X(t)dt.$$

Then

$$E[|z|^2] = \int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1, t_2)dt_1 dt_2 \qquad (1\text{-}10)$$

Example 1.2

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi). \tag{1-11}$$

This gives

$$\begin{aligned}\mu_X(t) &= E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\} \\ &= a\cos\omega_0 t\, E\{\cos\varphi\} - a\sin\omega_0 t\, E\{\sin\varphi\} = 0,\end{aligned} \tag{1-12}$$

since $E\{\cos\varphi\} = \frac{1}{2\pi}\int_0^{2\pi}\cos\varphi\, d\varphi = 0 = E\{\sin\varphi\}.$

Similarly

$$\begin{aligned}R_{XX}(t_1,t_2) &= a^2 E\{\cos(\omega_0 t_1 + \varphi)\cos(\omega_0 t_2 + \varphi)\} \\ &= \frac{a^2}{2} E\{\cos\omega_0(t_1 - t_2) + \cos(\omega_0(t_1 + t_2) + 2\varphi)\} \\ &= \frac{a^2}{2}\cos\omega_0(t_1 - t_2).\end{aligned} \tag{1-13}$$

Stationary Stochastic Processes

Stationary processes exhibit statistical properties that are invariant to shift in the time index. Thus, for example, second-order stationarity implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1+c), X(t_2+c)\}$ are the same for *any* c. Similarly first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i+c)$ are the same for any c.

In strict terms, the statistical properties are governed by the joint probability density function. Hence a process is nth-order **Strict-Sense Stationary (S.S.S)** if

$$f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n,\ t_1 + c, t_2 + c \cdots, t_n + c) \tag{14-14}$$

for *any* c, where the left side represents the joint density function of the random variables $X_1 = X(t_1),\ X_2 = X(t_2),\ \cdots,\ X_n = X(t_n)$ and the right side corresponds to the joint density function of the random variables $X_1' = X(t_1 + c),\ X_2' = X(t_2 + c),\ \cdots,\ X_n' = X(t_n + c).$

A process $X(t)$ is said to be **strict-sense stationary** if (14-14) is true for all $t_i,\ \ i = 1, 2, \cdots, n,\ \ n = 1, 2, \cdots$ and *any* c.

For a **first-order strict sense stationary process**, from (1-14) we have

$$f_X(x,t) \equiv f_X(x,t+c) \tag{1-15}$$

for any c. In particular $c = -t$ gives

$$f_X(x,t) = f_X(x) \tag{1-16}$$

i.e., the first-order density of $X(t)$ is independent of t. In that case

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x)dx = \mu, \;\; a \;\, constant. \tag{1-17}$$

Similarly, for a **second-order strict-sense stationary process** we have from (14-14)

$$f_X(x_1,x_2,\; t_1,t_2) \equiv f_X(x_1,x_2,\; t_1+c,t_2+c)$$

for any c. For $c = -t_2$ we get

$$f_X(x_1,x_2,\; t_1,t_2) \equiv f_X(x_1,x_2,\; t_1-t_2) \tag{1-18}$$

i.e., the second order density function of a strict sense stationary process depends only on the difference of the time indices $t_1 - t_2 = \tau$. In that case the autocorrelation function is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &\triangleq E\{X(t_1)X^*(t_2)\} \\ &= \int\int x_1 x_2^* f_X(x_1, x_2, \tau = t_1 - t_2) dx_1 dx_2 \\ &= R_{XX}(t_1 - t_2) \triangleq R_{XX}(\tau) = R_{XX}^*(-\tau), \end{aligned} \qquad (1\text{-}19)$$

i.e., the autocorrelation function of a second order strict-sense stationary process depends only on the difference of the time indices $\tau = t_1 - t_2$.

Notice that (1-17) and (1-19) are consequences of the stochastic process being first and second-order strict sense stationary.

On the other hand, the basic conditions for the first and second order stationarity – Eqs. (1-16) and (1-18) – are usually difficult to verify. In that case, we often resort to a looser definition of stationarity, known as **Wide-Sense Stationarity (W.S.S),** by making use of

(1-17) and (1-19) as the necessary conditions. Thus, a process $X(t)$ is said to be **Wide-Sense Stationary** if

(i) $E\{X(t)\} = \mu$ (1-20)

and

(ii) $E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1 - t_2),$ (1-21)

i.e., for wide-sense stationary processes, the mean is a constant and the autocorrelation function depends only on the difference between the time indices. Notice that (1-20)-(1-21) does not say anything about the nature of the probability density functions, and instead deal with the average behavior of the process. Since (1-20)-(1-21) follow from (1-16) and (1-18), strict-sense stationarity always implies wide-sense stationarity. However, the converse is *not true* in general, the only exception being the Gaussian process.

This follows, since if $X(t)$ is a Gaussian process, then by definition $X_1 = X(t_1), X_2 = X(t_2), \cdots, X_n = X(t_n)$ are jointly Gaussian random variables for any $t_1, t_2 \cdots, t_n$ whose joint characteristic function is given by

$$\phi_{\underline{X}}(\omega_1,\omega_2,\cdots,\omega_n)=e^{\,j\sum_{k=1}^{n}\mu(t_k)\omega_k-\sum_{l,k}^{n}\sum C_{xx}(t_i,t_k)\omega_i\omega_k/2} \qquad (1\text{-}22)$$

where $C_{XX}(t_i,t_k)$ is as defined on (14-9). If $X(t)$ is wide-sense stationary, then using (14-20)-(14-21) in (14-22) we get

$$\phi_{\underline{X}}(\omega_1,\omega_2,\cdots,\omega_n)=e^{\,j\sum_{k=1}^{n}\mu\omega_k-\frac{1}{2}\sum_{1=1}^{n}\sum_{k=1}^{n}C_{xx}(t_i-t_k)\omega_i\omega_k} \qquad (1\text{-}23)$$

and hence if the set of time indices are shifted by a constant c to generate a new set of jointly Gaussian random variables $X_1'=X(t_1+c)$, $X_2'=X(t_2+c),\cdots,X_n'=X(t_n+c)$ then their joint characteristic function is identical to (14-23). Thus the set of random variables $\{X_i\}_{i=1}^n$ and $\{X_i'\}_{i=1}^n$ have the same joint probability distribution for all n and all c, establishing the strict sense stationarity of Gaussian processes from its wide-sense stationarity.

To summarize if $X(t)$ is a Gaussian process, then

wide-sense stationarity (w.s.s) $\Rightarrow$ strict-sense stationarity (s.s.s).

Notice that since the joint p.d.f of Gaussian random variables depends only on their second order statistics, which is also the basis for wide sense stationarity, we obtain strict sense stationarity as well. From (1-12)-(1-13), (refer to Example 1.2), the process $X(t) = a\cos(\omega_0 t + \varphi)$, in (14-11) is wide-sense stationary, but not strict-sense stationary.

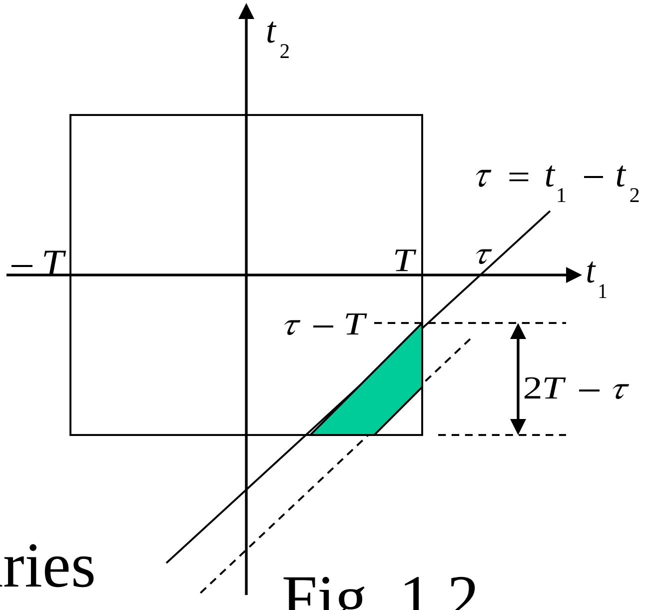

Fig. 1.2

Similarly if $X(t)$ is a zero mean wide sense stationary process in Example 1.1, then σ_z^2 in (1-10) reduces to

$$\sigma_z^2 = E\{|z|^2\} = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)dt_1 dt_2.$$

As t_1, t_2 varies from $-T$ to $+T$, $\tau = t_1 - t_2$ varies from $-2T$ to $+2T$. Moreover $R_{XX}(\tau)$ is a constant over the shaded region in Fig 1.2, whose area is given by $(\tau > 0)$

$$\frac{1}{2}(2T-\tau)^2 - \frac{1}{2}(2T-\tau-d\tau)^2 = (2T-\tau)d\tau$$

and hence the above integral reduces to

$$\sigma_z^2 = \int_{-2t}^{2T} R_{XX}(\tau)(2T - |\tau|)d\tau = \frac{1}{2T}\int_{-2t}^{2T} R_{XX}(\tau)(1 - \tfrac{|\tau|}{2T})d\tau. \qquad (1\text{-}24)$$

Systems with Stochastic Inputs

A deterministic system[1] transforms each input waveform $X(t,\xi_i)$ into an output waveform $Y(t,\xi_i)=T[X(t,\xi_i)]$ by operating only on the time variable t. Thus a set of realizations at the input corresponding to a process $X(t)$ generates a new set of realizations $\{Y(t,\xi)\}$ at the output associated with a new process $Y(t)$.

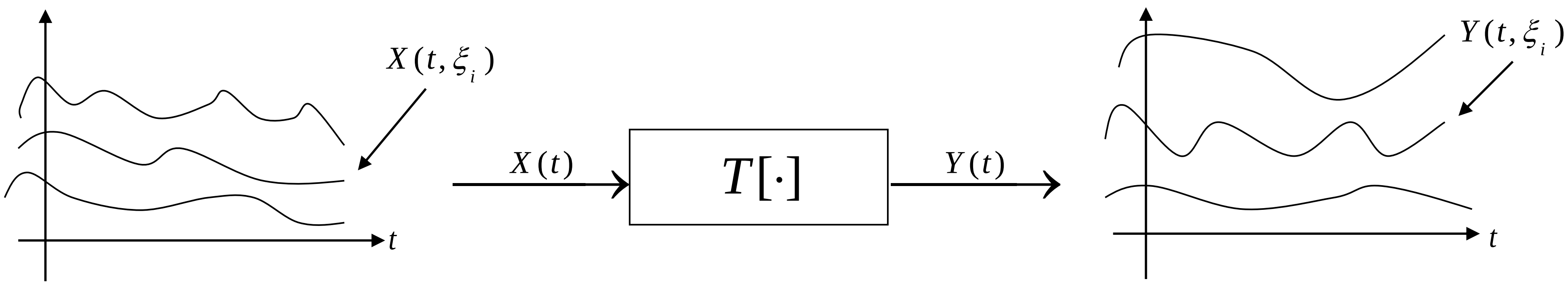

Fig. 1.3

Our goal is to study the output process statistics in terms of the input process statistics and the system function.

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[1]A stochastic system on the other hand operates on both the variables t and ξ.

Deterministic Systems

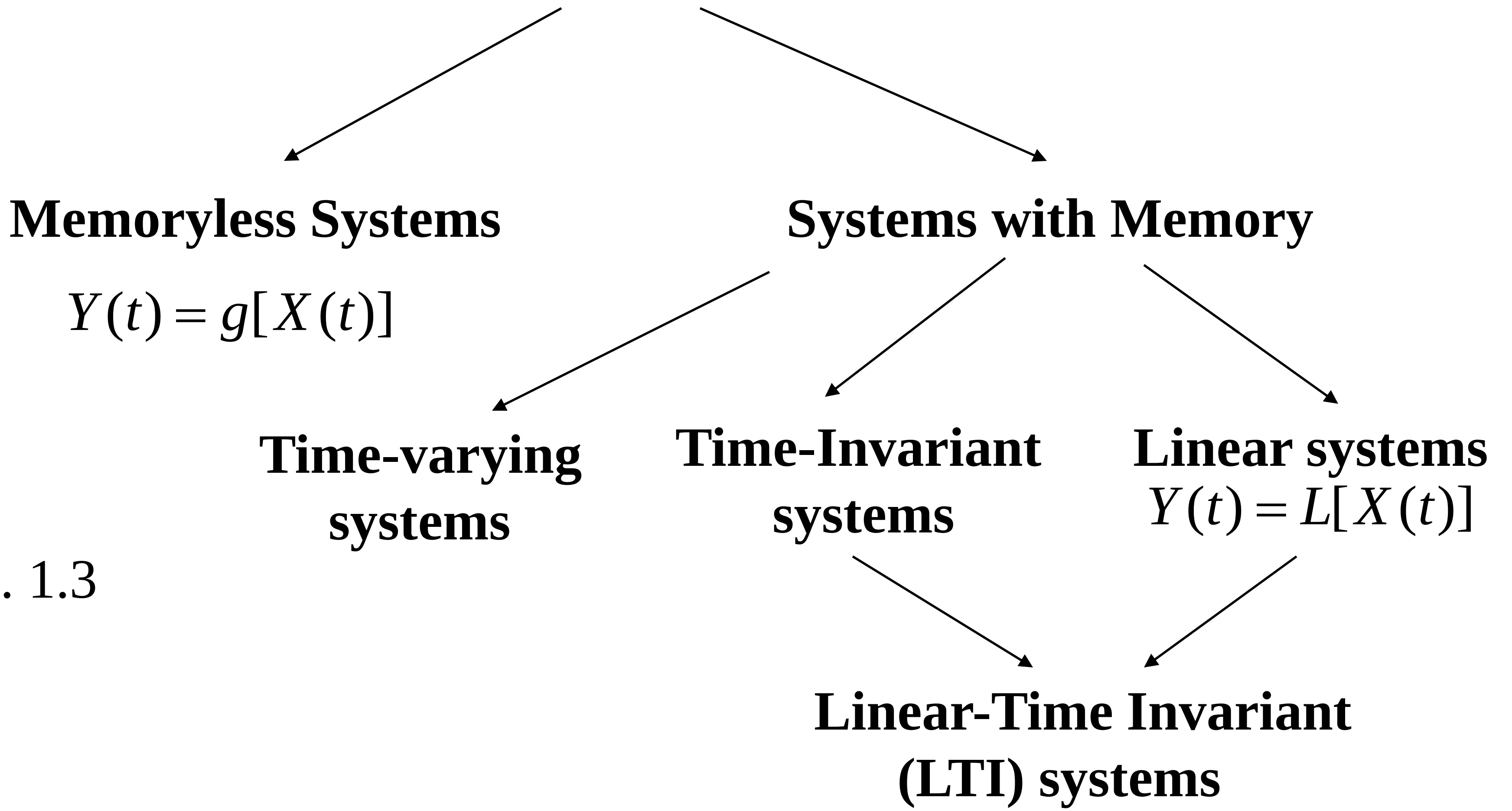
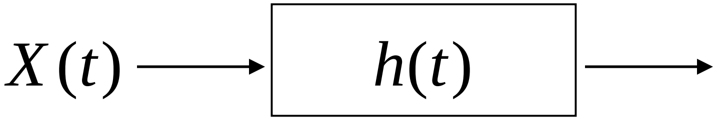


Fig. 1.3



LTI system

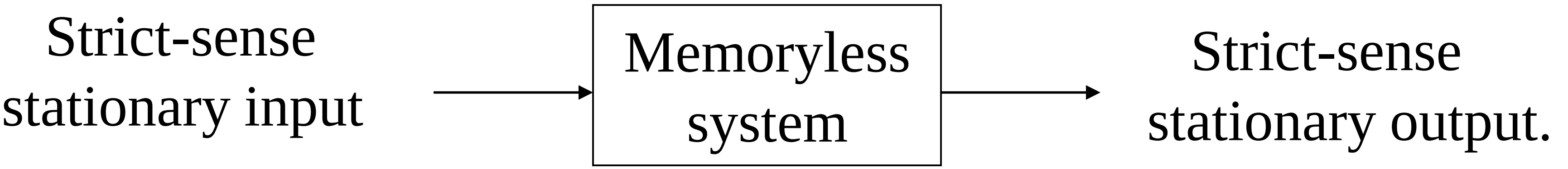
$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)\,d\tau.$$

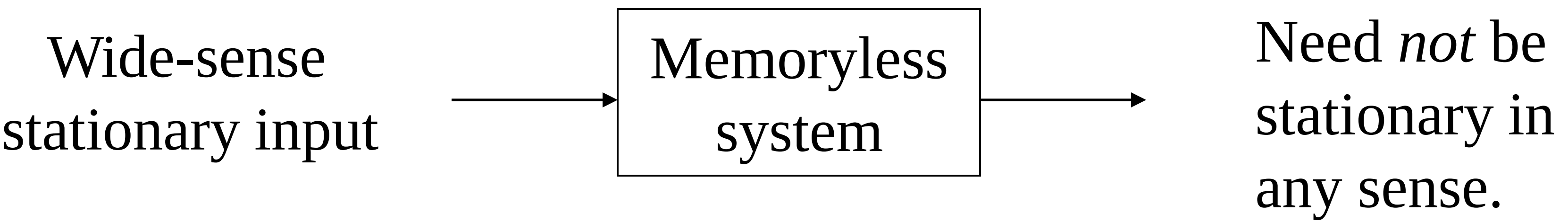
Memoryless Systems:

The output $Y(t)$ in this case depends only on the present value of the input $X(t)$. i.e.,

$$Y(t) = g\{X(t)\} \tag{1-25}$$



(see (9-76), Text for a proof.)



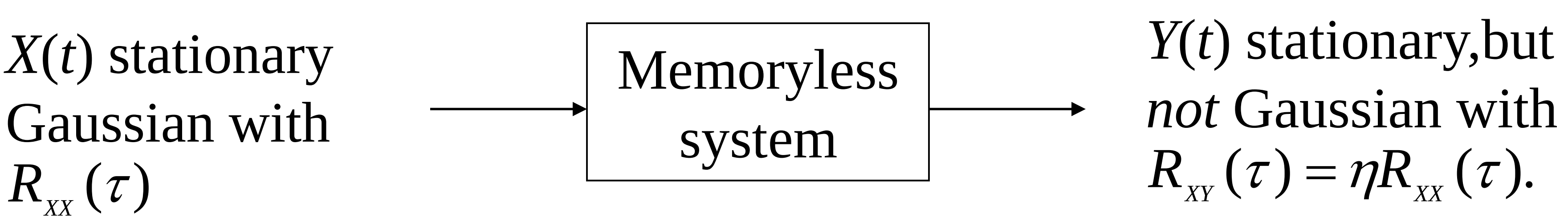


Fig. 1.4

Theorem: If $X(t)$ is a zero mean stationary Gaussian process, and $Y(t) = g[X(t)]$, where $g(\cdot)$ represents a nonlinear memoryless device, then

$$R_{XY}(\tau) = \eta R_{XX}(\tau), \qquad \eta = E\{g'(X)\}. \tag{1-26}$$

Proof:

$$R_{XY}(\tau) = E\{X(t)Y(t-\tau)\} = E[X(t)g\{X(t-\tau)\}]$$
$$= \int\int x_1 g(x_2) f_{X_1X_2}(x_1, x_2) dx_1 dx_2 \tag{1-27}$$

where $X_1 = X(t), \quad X_2 = X(t-\tau)$ are jointly Gaussian random variables, and hence

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{|A|}} e^{-\underline{x}^* A^{-1} \underline{x}/2}$$

$$\underline{X} = (X_1, X_2)^T, \qquad \underline{x} = (x_1, x_2)^T$$

$$A = E\{\underline{X}\,\underline{X}^*\} = \begin{pmatrix} R_{XX}(0) & R_{XX}(\tau) \\ R_{XX}(\tau) & R_{XX}(0) \end{pmatrix} \triangleq LL^*$$

where L is an upper triangular factor matrix with positive diagonal entries. i.e.,

$$L = \begin{pmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{pmatrix}.$$

Consider the transformation

$$\underline{Z} \triangleq L^{-1}\underline{X} = (Z_1,\ Z_2)^T, \qquad \underline{z} \triangleq L^{-1}\underline{x} = (z_1,\ z_2)^T$$

so that

$$E\{\underline{Z}\,\underline{Z}^*\} = L^{-1}E\{\underline{X}\,X^*\}L^{*-1} = L^{-1}AL^{*-1} = I$$

and hence Z_1, Z_2 are zero mean independent Gaussian random variables. Also

$$\underline{x} = L\underline{z} \;\Rightarrow\; x_1 = l_{11}z_1 + l_{12}z_2, \quad x_2 = l_{22}z_2$$

and hence

$$\underline{x}^* A^{-1}\underline{x} = \underline{z}^* L^* A^{-1} L\underline{z} = \underline{z}^*\,\underline{z} = z_1^2 + z_2^2.$$

The Jacobaian of the transformation is given by

$$|J| = |L^{-1}| = |A|^{-1/2}.$$

Hence substituting these into (14-27), we obtain

$$R_{XY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(l_{11}z_1 + l_{12}z_2)g(l_{22}z_2)\tfrac{1}{|J|}\cdot\tfrac{1}{2\pi|A|^{1/2}}e^{-z_1^2/2}e^{-z_2^2/2}$$

$$= l_{11}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z_1 g(l_{22}z_2) f_{z_1}(z_1) f_{z_2}(z_2) dz_1 dz_2$$

$$+ l_{12}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z_2 g(l_{22}z_2) f_{z_1}(z_1) f_{z_2}(z_2) dz_1 dz_2$$

$$= l_{11}\underbrace{\int_{-\infty}^{+\infty} z_1 f_{z_1}(z_1) dz_1}_{0}\int_{-\infty}^{+\infty} g(l_{22}z_2) f_{z_2}(z_2) dz_2$$

$$+ l_{12}\int_{-\infty}^{+\infty} z_2 g(l_{22}z_2)\underbrace{f_{z_2}(z_2)}_{\frac{1}{\sqrt{2\pi}}e^{-z_2^2/2}} dz_2$$

$$= \frac{l_{12}}{l_{22}^2}\int_{-\infty}^{+\infty} u g(u)\frac{1}{\sqrt{2\pi}}e^{-u^2/2l_{22}^2} du,$$

where $u = l_{22}z_2$. This gives

$$R_{XY}(\tau) = l_{12} l_{22} \int_{-\infty}^{+\infty} g(u) \underbrace{\frac{u}{l_{22}^2} \overbrace{\frac{1}{\sqrt{2\pi\, l_{22}^2}} e^{-u^2/2l_{22}^2}}^{f_u(u)}}_{-\frac{df_u(u)}{du} = -f_u'(u)} du$$

$$= -R_{XX}(\tau) \int_{-\infty}^{+\infty} g(u) f_u'(u) du,$$

since $A = LL^*$ gives $l_{12} l_{22} = R_{XX}(\tau)$. Hence

$$R_{XY}(\tau) = R_{XX}(\tau)\{-\underbrace{g(u) f_u(u) \Big|_{-\infty}^{+\infty}}_{0} + \int_{-\infty}^{+\infty} g'(u) f_u(u) du\}$$

$$= R_{XX}(\tau) E\{g'(X)\} = \eta R_{XX}(\tau),$$

the desired result, where $\eta = E[g'(X)].$ Thus if the input to a memoryless device is stationary Gaussian, the cross correlation function between the input and the output is proportional to the input autocorrelation function.

Linear Systems: $L[\cdot]$ represents a linear system if

$$L\{a_1 X(t_1) + a_2 X(t_2)\} = a_1 L\{X(t_1)\} + a_2 L\{X(t_2)\}. \tag{1-28}$$

Let

$$Y(t) = L\{X(t)\} \tag{1-29}$$

represent the output of a linear system.

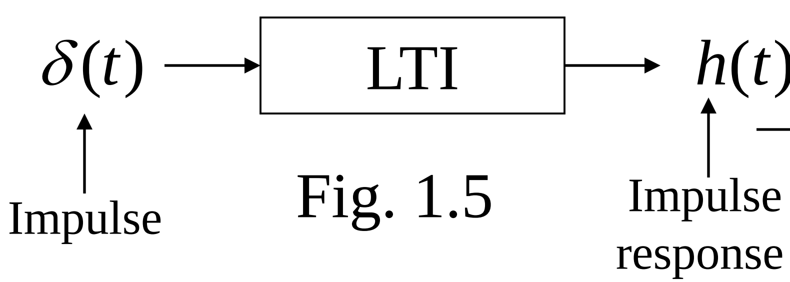
Time-Invariant System: $L[\cdot]$ represents a time-invariant system if

$$Y(t) = L\{X(t)\} \Rightarrow L\{X(t-t_0)\} = Y(t-t_0) \tag{1-30}$$

i.e., shift in the input results in the same shift in the output also.

If $L[\cdot]$ satisfies both (14-28) and (14-30), then it corresponds to a linear time-invariant (LTI) system.

LTI systems can be uniquely represented in terms of their output to a delta function



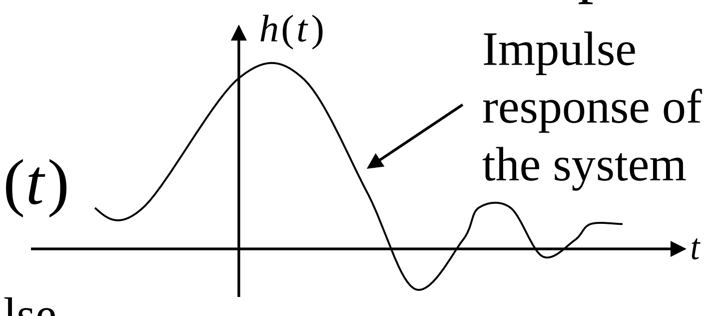


Fig. 1.5

then

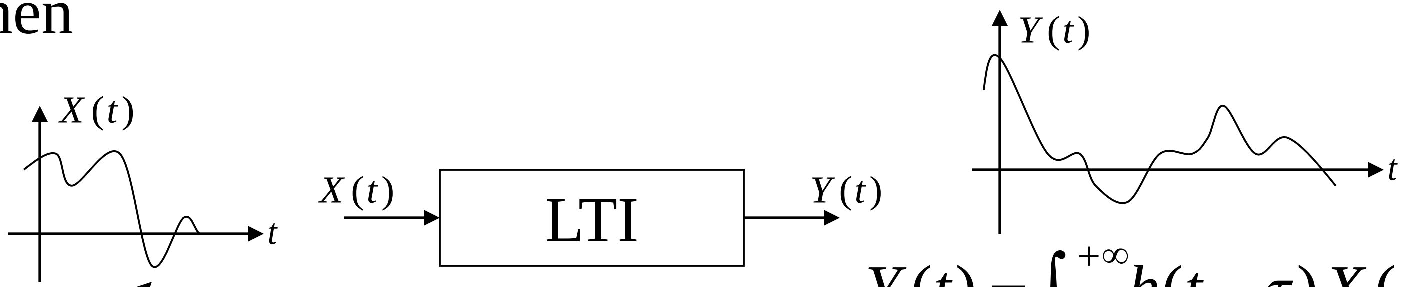


Fig. 1.6

$$Y(t)=\int_{-\infty}^{+\infty}h(t-\tau)X(\tau)d\tau$$
$$=\int_{-\infty}^{+\infty}h(\tau)X(t-\tau)d\tau \quad (1\text{-}31)$$

Eq. (1-31) follows by expressing $X(t)$ as

$$X(t)=\int_{-\infty}^{+\infty}X(\tau)\delta(t-\tau)d\tau \qquad (1\text{-}32)$$

and applying (1-28) and (1-30) to $Y(t)=L\{X(t)\}$. Thus

$$Y(t)=L\{X(t)\}=L\{\int_{-\infty}^{+\infty}X(\tau)\delta(t-\tau)d\tau\}$$

$$=\int_{-\infty}^{+\infty}L\{X(\tau)\delta(t-\tau)d\tau\}$$

By Linearity

$$=\int_{-\infty}^{+\infty}X(\tau)L\{\delta(t-\tau)\}d\tau$$

By Time-invariance

$$=\int_{-\infty}^{+\infty}X(\tau)h(t-\tau)d\tau=\int_{-\infty}^{+\infty}h(\tau)X(t-\tau)d\tau. \quad (1\text{-}33)$$

Output Statistics: Using (1-33), the mean of the output process is given by

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{+\infty} E\{X(\tau)h(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} \mu_X(\tau)h(t-\tau)d\tau = \mu_X(t) * h(t). \qquad (1\text{-}34)$$

Similarly the cross-correlation function between the input and output processes is given by

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y^*(t_2)\}$$

$$= E\{X(t_1)\int_{-\infty}^{+\infty} X^*(t_2-\alpha)h^*(\alpha)d\alpha\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1)X^*(t_2-\alpha)\}h^*(\alpha)d\alpha$$

$$= \int_{-\infty}^{+\infty} R_{XX}(t_1,t_2-\alpha)h^*(\alpha)d\alpha$$

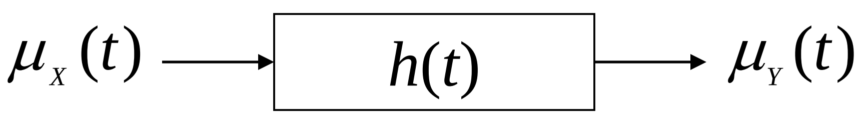
$$= R_{XX}(t_1,t_2) * h^*(t_2). \qquad (1\text{-}35)$$

Finally the output autocorrelation function is given by

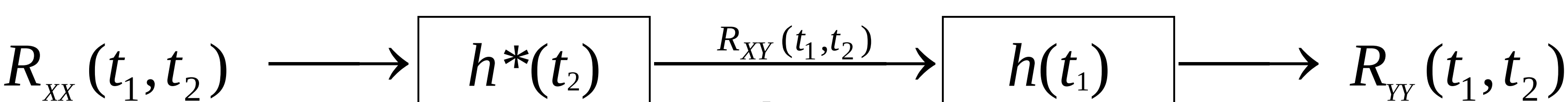
$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E\{Y(t_1)Y^*(t_2)\} \\
&= E\{\int_{-\infty}^{+\infty} X(t_1-\beta)h(\beta)d\beta\, Y^*(t_2)\} \\
&= \int_{-\infty}^{+\infty} E\{X(t_1-\beta)Y^*(t_2)\}h(\beta)d\beta \\
&= \int_{-\infty}^{+\infty} R_{XY}(t_1-\beta,t_2)h(\beta)d\beta \\
&= R_{XY}(t_1,t_2) * h(t_1),
\end{aligned} \tag{1-36}
$$

or

$$
R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h^*(t_2) * h(t_1). \tag{1-37}
$$



(a)



(b)

Fig. 1.7

In particular if $X(t)$ is wide-sense stationary, then we have $\mu_X(t) = \mu_X$ so that from (14-34)

$$\mu_Y(t) = \mu_X \int_{-\infty}^{+\infty} h(\tau) d\tau = \mu_X c, \quad a \ constant. \tag{1-38}$$

Also $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$ so that (14-35) reduces to

$$\begin{aligned} R_{XY}(t_1, t_2) &= \int_{-\infty}^{+\infty} R_{XX}(t_1 - t_2 + \alpha) h^*(\alpha) d\alpha \\ &= R_{XX}(\tau) * h^*(-\tau) \stackrel{\Delta}{=} R_{XY}(\tau), \quad \tau = t_1 - t_2. \end{aligned} \tag{1-39}$$

Thus $X(t)$ and $Y(t)$ are jointly w.s.s. Further, from (1-36), the output autocorrelation simplifies to

$$\begin{aligned} R_{YY}(t_1, t_2) &= \int_{-\infty}^{+\infty} R_{XY}(t_1 - \beta - t_2) h(\beta) d\beta, \quad \tau = t_1 - t_2 \\ &= R_{XY}(\tau) * h(\tau) = R_{YY}(\tau). \end{aligned} \tag{1-40}$$

From (14-37), we obtain

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau). \tag{1-41}$$

From (1-38)-(1-40), the output process is also wide-sense stationary. This gives rise to the following representation

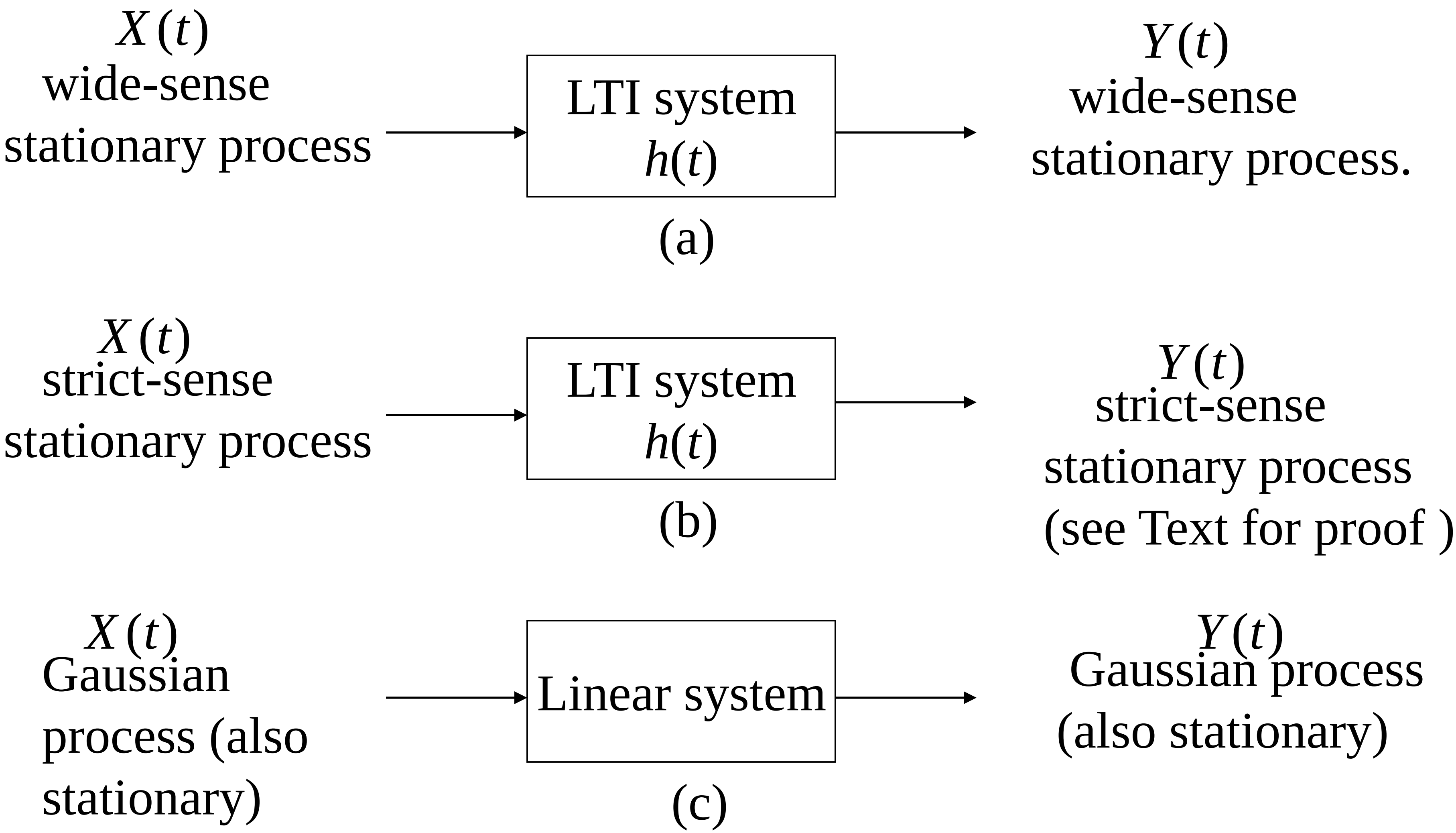


Fig. 1.8

White Noise Process:

$W(t)$ is said to be a white noise process if

$$R_{WW}(t_1, t_2) = q(t_1)\delta(t_1 - t_2), \tag{1-42}$$

i.e., $E[W(t_1)\, W^*(t_2)] = 0$ unless $t_1 = t_2$.

$W(t)$ is said to be wide-sense stationary (w.s.s) white noise if $E[W(t)]$ = constant, and

$$R_{WW}(t_1, t_2) = q\delta(t_1 - t_2) = q\delta(\tau). \tag{1-43}$$

If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables (why?).

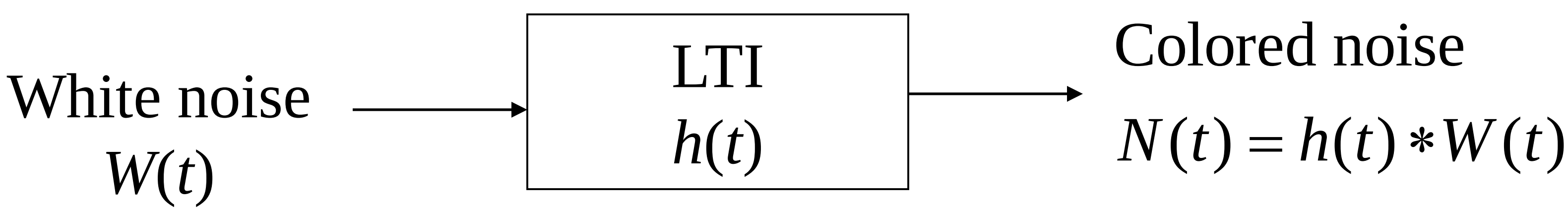


Fig. 1.9

For w.s.s. white noise input $W(t)$, we have

$$E[N(t)] = \mu_W \int_{-\infty}^{+\infty} h(\tau)\,d\tau, \quad a\ constant \tag{1-44}$$

and

$$\begin{aligned} R_{nn}(\tau) &= q\delta(\tau) * h^*(-\tau) * h(\tau) \\ &= qh^*(-\tau) * h(\tau) = q\rho(\tau) \end{aligned} \tag{1-45}$$

where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{+\infty} h(\alpha) h^*(\alpha+\tau)\,d\alpha. \tag{1-46}$$

Thus the output of a white noise process through an LTI system represents a (colored) noise process.

Note: White noise need not be Gaussian.

“White” and “Gaussian” are two different concepts!

Upcrossings and Downcrossings of a stationary Gaussian process:

Consider a zero mean stationary Gaussian process $X(t)$ with autocorrelation function $R_{XX}(\tau)$. An upcrossing over the mean value occurs whenever the realization $X(t)$ passes through zero with positive slope. Let $\rho \Delta t$ represent the probability of such an upcrossing in the interval $(t, t+\Delta t)$. We wish to determine ρ.

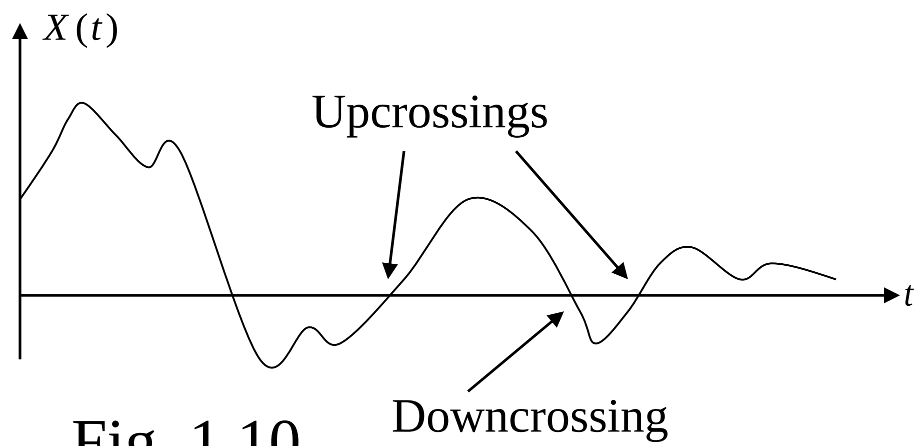


Fig. 1.10

Since $X(t)$ is a stationary Gaussian process, its derivative process $X'(t)$ is also zero mean stationary Gaussian with autocorrelation function $R_{X'X'}(\tau) = -R''_{XX}(\tau)$ (see (9-101)-(9-106), Text). Further $X(t)$ and $X'(t)$ are jointly Gaussian stationary processes, and since (see (9-106), Text)

$$R_{XX'}(\tau) = -\frac{dR_{XX}(\tau)}{d\tau},$$

we have

$$R_{XX'}(-\tau) = -\frac{dR_{XX}(-\tau)}{d(-\tau)} = \frac{dR_{XX}(\tau)}{d\tau} = -R_{XX'}(\tau) \qquad (1\text{-}47)$$

which for $\tau = 0$ gives

$$R_{XX'}(0) = 0 \quad \Rightarrow \quad E[X(t)X'(t)] = 0 \qquad (1\text{-}48)$$

i.e., the jointly Gaussian zero mean random variables

$$X_1 = X(t) \quad \text{and} \quad X_2 = X'(t) \qquad (1\text{-}49)$$

are uncorrelated and hence *independent* with variances

$$\sigma_1^2 = R_{XX}(0) \;\; \text{and} \;\; \sigma_2^2 = R_{X'X'}(0) = -R''_{XX}(0) > 0 \qquad (1\text{-}50)$$

respectively. Thus

$$f_{X_1X_2}(x_1, x_2) = f_X(x_1)\, f_X(x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\left(\frac{x_1^2}{2\sigma_1^2} + \frac{x_1^2}{2\sigma_2^2}\right)}. \qquad (1\text{-}51)$$

To determine ρ, the probability of upcrossing rate, we argue as follows: In an interval $(t, t+\Delta t),$ the realization moves from $X(t) = X_1$ to $X(t+\Delta t) = X(t) + X'(t)\Delta t = X_1 + X_2\Delta t,$ and hence the realization intersects with the zero level somewhere in that interval if

$$X_1 < 0, \;\; X_2 > 0, \quad \text{and} \quad X(t+\Delta t) = X_1 + X_2\Delta t > 0 \qquad \text{(1-52)}$$

i.e., $X_1 > -X_2\Delta t.$

Hence the probability of upcrossing in $(t, t+\Delta t)$ is given by

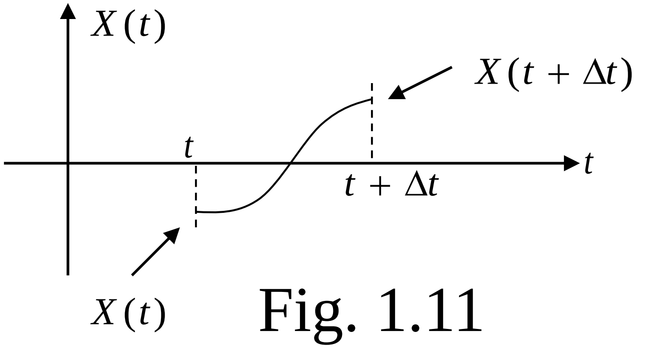


Fig. 1.11

$$\rho\Delta t = \int_{x_2=0}^{\infty}\int_{x_1=-x_2\Delta t}^{0} f_{X_1X_2}(x_1, x_2)dx_1dx_2$$

$$= \int_0^{\infty} f_{X_2}(x_2)dx_2 \int_{-x_2\Delta t}^{\infty} f_{X_1}(x_1)dx_1. \qquad \text{(1-53)}$$

Differentiating both sides of (14-53) with respect to $\Delta t,$ we get

$$\rho = \int_0^{\infty} f_{X_2}(x_2)x_2 f_{X_1}(-x_2\Delta t)dx_2 \qquad \text{(1-54)}$$

and letting $\Delta t \to 0,$ Eq. (14-54) reduce to

$$\rho = \int_0^\infty x_2 f_X(x_2) f_X(0) dx_2 = \frac{1}{\sqrt{2\pi R_{XX}(0)}} \int_0^\infty x_2 f_X(x_2) dx_2$$

$$= \frac{1}{\sqrt{2\pi R_{XX}(0)}} \frac{1}{2} (\sigma_2 \sqrt{2/\pi}) = \frac{1}{2\pi} \sqrt{\frac{-R''_{XX}(0)}{R_{XX}(0)}} \qquad (1\text{-}55)$$

[where we have made use of (5-78), Text]. There is an equal probability for downcrossings, and hence the total probability for crossing the zero line in an interval $(t, t+\Delta t)$ equals $\rho_0 \Delta t,$ where

$$\rho_0 = \frac{1}{\pi} \sqrt{-R''_{XX}(0)/R_{XX}(0)} \;>\; 0. \qquad (1\text{-}56)$$

It follows that in a long interval T, there will be approximately $\rho_0 T$ crossings of the mean value. If $-\;R''_{XX}(0)$ is large, then the autocorrelation function $R_{XX}(\tau)$ decays more rapidly as τ moves away from zero, implying a large random variation around the origin (mean value) for $X(t)$, and the likelihood of zero crossings should increase with increase in $-\;R''_{XX}(0),$ agreeing with (1-56).

Discrete Time Stochastic Processes:

A discrete time stochastic process $X_n = X(nT)$ is a sequence of random variables. The mean, autocorrelation and auto-covariance functions of a discrete-time process are gives by

$$\mu_n = E\{X(nT)\} \tag{1-57}$$

$$R(n_1, n_2) = E\{X(n_1 T) X^*(n_2 T)\} \tag{1-58}$$

and

$$C(n_1, n_2) = R(n_1, n_2) - \mu_{n_1} \mu_{n_2}^* \tag{1-59}$$

respectively. As before strict sense stationarity and wide-sense stationarity definitions apply here also.
For example, $X(nT)$ is wide sense stationary if

$$E\{X(nT)\} = \mu, \quad a\ constant \tag{1-60}$$

and

$$E[X\{(k+n)T\} X^*\{(k)T\}] = R(n) = r_n \triangleq r_{-n}^* \tag{1-61}$$

i.e., $R(n_1, n_2) = R(n_1 - n_2) = R^*(n_2 - n_1)$. The positive-definite property of the autocorrelation sequence in (1-8) can be expressed in terms of certain Hermitian-Toeplitz matrices as follows:

Theorem: A sequence $\{r_n\}_{-\infty}^{+\infty}$ forms an autocorrelation sequence of a wide sense stationary stochastic process if and only if every Hermitian-Toeplitz matrix T_n given by

$$T_n = \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_n \\ r_1^* & r_0 & r_1 & \cdots & r_{n-1} \\ & & \vdots & & \\ r_n^* & r_{n-1}^* & \cdots & r_1^* & r_0 \end{pmatrix} = T_n^* \qquad (1\text{-}62)$$

is non-negative (positive) definite for $n = 0, 1, 2, \cdots, \infty.$

Proof: Let $\underline{a} = [a_0, a_1, \cdots, a_n]^T$ represent an arbitrary constant vector. Then from (1-62),

$$\underline{a}^* T_n \underline{a} = \sum_{i=0}^{n} \sum_{k=0}^{n} a_i a_k^* r_{k-i} \qquad (1\text{-}63)$$

since the Toeplitz character gives $(T_n)_{i,k} = r_{k-i}$. Using (1-61), Eq. (1-63) reduces to

$$\underline{a}^* T_n \underline{a} = \sum_{i=0}^{n}\sum_{k=0}^{n} a_i a_k^* E\{X(kT)X^*(iT)\} = E\left\{\left|\sum_{k=0}^{n} a_k^* X(kT)\right|^2\right\} \geq 0. \quad (1\text{-}64)$$

From (1-64), if $X(nT)$ is a wide sense stationary stochastic process then T_n is a non-negative definite matrix for every $n = 0, 1, 2, \cdots, \infty.$ Similarly the converse also follows from (1-64). (see section 9.4, Text)

If $X(nT)$ represents a wide-sense stationary input to a discrete-time system $\{h(nT)\}$, and $Y(nT)$ the system output, then as before the cross correlation function satisfies

$$R_{XY}(n) = R_{XX}(n) * h^*(-n) \quad (1\text{-}65)$$

and the output autocorrelation function is given by

$$R_{YY}(n) = R_{XY}(n) * h(n) \quad (1\text{-}66)$$

or

$$R_{YY}(n) = R_{XX}(n) * h^*(-n) * h(n). \quad (1\text{-}67)$$

Thus wide-sense stationarity from input to output is preserved for discrete-time systems also.

Auto Regressive Moving Average (ARMA) Processes

Consider an input – output representation

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + \sum_{k=0}^{q} b_k W(n-k), \tag{1-68}$$

where $X(n)$ may be considered as the output of a system $\{h(n)\}$ driven by the input $W(n)$.

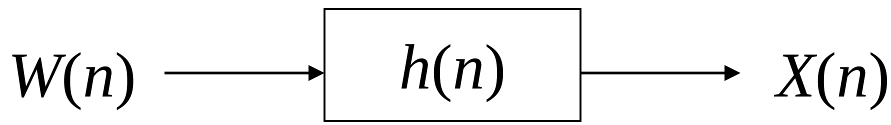


Fig.1.12

Z – transform of (1-68) gives

$$X(z)\sum_{k=0}^{p} a_k z^{-k} = W(z)\sum_{k=0}^{q} b_k z^{-k}, \quad a_0 \equiv 1 \tag{1-69}$$

or

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k} = \frac{X(z)}{W(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}} \triangleq \frac{B(z)}{A(z)} \tag{1-70}$$

represents the transfer function of the associated system response $\{h(n)\}$ in Fig 1.12 so that

$$X(n)=\sum_{k=0}^{\infty} h(n-k)W(k). \tag{1-71}$$

Notice that the transfer function $H(z)$ in (1-70) is rational with p poles and q zeros that determine the model order of the underlying system. From (1-68), the output undergoes regression over p of its previous values and at the same time a moving average based on $W(n), W(n-1),$ $\cdots, W(n-q)$ of the input over $(q + 1)$ values is added to it, thus generating an **Auto Regressive Moving Average** (ARMA (p, q)) process $X(n)$. Generally the input $\{W(n)\}$ represents a sequence of uncorrelated random variables of zero mean and constant variance σ_W^2 so that

$$R_{WW}(n)=\sigma_W^2\delta(n). \tag{1-72}$$

If in addition, $\{W(n)\}$ is normally distributed then the output $\{X(n)\}$ also represents a strict-sense stationary normal process.

If $q = 0$, then (1-68) represents an AR(p) process (all-pole process), and if $p = 0$, then (1-68) represents an MA(q)

process (all-zero process). Next, we shall discuss AR(1) and AR(2) processes through explicit calculations.

AR(1) process: An AR(1) process has the form (see (1-68))

$$X(n) = aX(n-1) + W(n) \qquad (1\text{-}73)$$

and from (1-70) the corresponding system transfer

$$H(z) = \frac{1}{1 - az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n} \qquad (14\text{-}74)$$

provided $|a| < 1$. Thus

$$h(n) = a^n, \qquad |a| < 1 \qquad (14\text{-}75)$$

represents the impulse response of an AR(1) stable system. Using (1-67) together with (1-72) and (1-75), we get the output autocorrelation sequence of an AR(1) process to be

$$R_{XX}(n) = \sigma_W^2 \delta(n) * \{a^{-n}\} * \{a^n\} = \sigma_W^2 \sum_{k=0}^{\infty} a^{|n|+k} a^k = \sigma_W^2 \frac{a^{|n|}}{1-a^2} \qquad (1\text{-}76)$$

where we have made use of the discrete version of (1-46). The normalized (in terms of $R_{XX}(0)$) output autocorrelation sequence is given by

$$\rho_X(n) = \frac{R_{XX}(n)}{R_{XX}(0)} = a^{|n|}, \quad |n| \geq 0. \tag{1-77}$$

It is instructive to compare an AR(1) model discussed above by superimposing a random component to it, which may be an error term associated with observing a first order AR process $X(n)$. Thus

$$Y(n) = X(n) + V(n) \tag{1-78}$$

where $X(n) \sim$ AR(1) as in (1-73), and $V(n)$ is an uncorrelated random sequence with zero mean and variance σ_V^2 that is also uncorrelated with $\{W(n)\}$. From (1-73), (1-78) we obtain the output autocorrelation of the observed process $Y(n)$ to be

$$\begin{aligned} R_{YY}(n) &= R_{XX}(n) + R_{VV}(n) = R_{XX}(n) + \sigma_V^2 \delta(n) \\ &= \sigma_W^2 \frac{a^{|n|}}{1-a^2} + \sigma_V^2 \delta(n) \end{aligned} \tag{1-79}$$

so that its normalized version is given by

$$\rho_Y(n) \triangleq \frac{R_{YY}(n)}{R_{YY}(0)} = \begin{cases} 1 & n = 0 \\ c\, a^{|n|} & n = \pm 1, \ \pm 2, \cdots \end{cases} \tag{1-80}$$

where

$$c = \frac{\sigma_W^2}{\sigma_W^2 + \sigma_V^2 (1 - a^2)} < 1. \tag{1-81}$$

Eqs. (1-77) and (1-80) demonstrate the effect of superimposing an error sequence on an AR(1) model. For non-zero lags, the autocorrelation of the observed sequence $\{Y(n)\}$ is reduced by a constant factor compared to the original process $\{X(n)\}$.

From (1-78), the superimposed error sequence $V(n)$ only affects the corresponding term in $Y(n)$ (term by term). However, a particular term in the "input sequence" $W(n)$ affects $X(n)$ and $Y(n)$ as well as *all* subsequent observations.

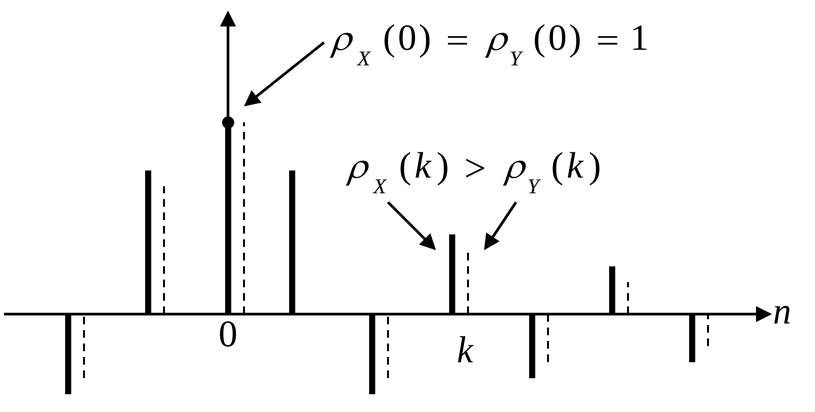


Fig. 1.13

AR(2) Process: An AR(2) process has the form

$$X(n) = a_1 X(n-1) + a_2 X(n-2) + W(n) \qquad \text{(1-82)}$$

and from (14-70) the corresponding transfer function is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_1}{1 - \lambda_1 z^{-1}} + \frac{b_2}{1 - \lambda_2 z^{-1}} \qquad \text{(1-83)}$$

so that

$$h(0) = 1,\ h(1) = a_1,\ h(n) = a_1 h(n-1) + a_2 h(n-2), \quad n \geq 2 \qquad \text{(1-84)}$$

and in term of the poles λ_1 and λ_2 of the transfer function, from (1-83) we have

$$h(n) = b_1 \lambda_1^n + b_2 \lambda_2^n, \qquad n \geq 0 \qquad \text{(1-85)}$$

that represents the impulse response of the system.
From (1-84)-(1-85), we also have $b_1 + b_2 = 1, \quad b_1 \lambda_1 + b_2 \lambda_2 = a_1.$
From (1-83),

$$\lambda_1 + \lambda_2 = a_1, \qquad \lambda_1 \lambda_2 = -a_2, \qquad \text{(1-86)}$$

and $H(z)$ stable implies $|\lambda_1|<1, \;\; |\lambda_2|<1.$
Further, using (1-82) the output autocorrelations satisfy the recursion

$$
\begin{aligned}
R_{XX}(n) &= E\{X(n+m)X^*(m)\} \\
&= E\{[a_1 X(n+m-1)+a_2 X(n+m-2)]X^*(m)\} \\
&\quad + E\{W(n+m)X^*(m)\} \quad (\to 0) \\
&= a_1 R_{XX}(n-1)+a_2 R_{XX}(n-2)
\end{aligned}
\qquad (1\text{-}87)
$$

and hence their normalized version is given by

$$
\rho_X(n) \triangleq \frac{R_{XX}(n)}{R_{XX}(0)} = a_1\rho_X(n-1)+a_2\rho_X(n-2). \qquad (1\text{-}88)
$$

By direct calculation using (1-67), the output autocorrelations are given by

$$
\begin{aligned}
R_{XX}(n) &= R_{WW}(n) * h^*(-n) * h(n) = \sigma_W^2 h^*(-n) * h(n) \\
&= \sigma_W^2 \sum_{k=0}^{\infty} h^*(n+k) * h(k) \\
&= \sigma_W^2 \left(\frac{|b_1|^2 (\lambda_1^*)^n}{1-|\lambda_1|^2} + \frac{b_1^* b_2 (\lambda_1^*)^n}{1-\lambda_1^*\lambda_2} + \frac{b_1 b_2^* (\lambda_2^*)^n}{1-\lambda_1\lambda_2^*} + \frac{|b_2|^2 (\lambda_2^*)^n}{1-|\lambda_2|^2} \right)
\end{aligned}
\qquad (1\text{-}89)
$$

where we have made use of (1-85). From (1-89), the normalized output autocorrelations may be expressed as

$$\rho_X(n) = \frac{R_{XX}(n)}{R_{XX}(0)} = c_1 \lambda_1^{*n} + c_2 \lambda_2^{*n} \tag{1-90}$$

where c_1 and c_2 are appropriate constants.

Damped Exponentials: When the second order system in (1-83)-(1-85) is real and corresponds to a damped exponential response, the poles are complex conjugate which gives $a_1^2 + 4a_2 < 0$ in (1-83). Thus

$$\lambda_1 = r\, e^{-j\theta}, \qquad \lambda_2 = \lambda_1^*, \qquad r < 1. \tag{1-91}$$

In that case $c_1 = c_2^* = c\, e^{j\varphi}$ in (1-90) so that the normalized correlations there reduce to

$$\rho_X(n) = 2\,\mathrm{Re}\{c_1 \lambda_1^{*n}\} = 2cr^n \cos(n\theta + \varphi). \tag{1-92}$$

But from (1-86)

$$\lambda_1 + \lambda_2 = 2r\cos\theta = a_1, \qquad r^2 = -a_2 < 1, \tag{1-93}$$

and hence $2r\sin\theta = \sqrt{-(a_1^2 + 4a_2)} > 0$ which gives

$$\tan\theta = \frac{\sqrt{-(a_1^2 + 4a_2)}}{a_1}. \tag{1-94}$$

Also from (1-88)

$$\rho_X(1) = a_1\rho_X(0) + a_2\rho_X(-1) = a_1 + a_2\rho_X(1)$$

so that

$$\rho_X(1) = \frac{a_1}{1-a_2} = 2cr\cos(\theta+\varphi) \tag{1-95}$$

where the later form is obtained from (14-92) with n = 1. But $\rho_X(0) = 1$ in (14-92) gives

$$2c\cos\varphi = 1, \quad \text{or} \quad c = 1/2\cos\varphi. \tag{1-96}$$

Substituting (1-96) into (1-92) and (1-95) we obtain the normalized output autocorrelations to be

$$\rho_x(n) = (-a_2)^{n/2} \frac{\cos(n\theta + \varphi)}{\cos\varphi}, \qquad -a_2 < 1 \tag{1-97}$$

where φ satisfies

$$\frac{\cos(\theta + \varphi)}{\cos\theta} = \frac{a_1}{1 - a_2} \frac{1}{\sqrt{-a_2}}. \tag{1-98}$$

Thus the normalized autocorrelations of a damped second order system with real coefficients subject to random uncorrelated impulses satisfy (1-97).

More on ARMA processes

From (1-70) an ARMA (p, q) system has only $p + q + 1$ independent coefficients, $(a_k, k = 1 \to p,\ b_i, i = 0 \to q)$, and hence its impulse response sequence $\{h_k\}$ also must exhibit a similar dependence among them. In fact according to P. Dienes (*The Taylor series*, 1931), an old result due to Kronecker[1] (1881) states that the necessary and sufficient condition for $H(z)=\sum_{k=0}^{\infty} h_k z^{-k}$ to represent a rational system (ARMA) is that

$$\det H_n = 0, \qquad n \geq N \qquad \text{(for all sufficiently large } n\text{)}, \qquad (1\text{-}99)$$

where

$$H_n \triangleq \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_n \\ h_1 & h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & & & & \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n} \end{pmatrix}. \qquad (1\text{-}100)$$

i.e., In the case of rational systems for all sufficiently large n, the Hankel matrices H_n in (1-100) all have the same rank.

The necessary part easily follows from (14-70) by cross multiplying and equating coefficients of like powers of z^{-k}, $k = 0, 1, 2, \cdots$.

---

[1]Among other things "*God created the integers and the rest is the work of man.*" (Leopold Kronecker)

This gives

$$
\begin{aligned}
b_0 &= h_0 \\
b_1 &= h_0 a_1 + h_1 \\
&\vdots \\
b_q &= h_0 a_q + h_1 a_{q-1} + \cdots + h_m
\end{aligned}
\tag{1-101}
$$

$$
0 = h_0 a_{q+i} + h_1 a_{q+i-1} + \cdots + h_{q+i-1} a_1 + h_{q+i}, \quad i \geq 1. \tag{1-102}
$$

For systems with $q \leq p-1$, letting $i = p-q, p-q+1, \cdots, 2p-q$ in (1-102) we get

$$
\begin{aligned}
h_0 a_p + h_1 a_{p-1} + \cdots + h_{p-1} a_1 + h_p &= 0 \\
&\vdots \\
h_p a_p + h_{p+1} a_{p-1} + \cdots + h_{2p-1} a_1 + h_{2p} &= 0
\end{aligned}
\tag{1-103}
$$

which gives det $H_p = 0$. Similarly $i = p-q+1, \cdots$ gives

$$h_0 a_{p+1} + h_1 a_p + \cdots + h_{p+1} = 0$$

$$h_1 a_{p+1} + h_2 a_p + \cdots + h_{p+2} = 0$$

$$\vdots$$

$$h_{p+1} a_{p+1} + h_{p+2} a_p + \cdots + h_{2p+2} = 0, \qquad (1\text{-}104)$$

and that gives det H_{p+1} = 0 etc. (Notice that $a_{p+k} = 0, \; k = 1, 2, \cdots$)
(For sufficiency proof, see Dienes.)
It is possible to obtain similar determinantial conditions for ARMA systems in terms of Hankel matrices generated from its output autocorrelation sequence.

Referring back to the ARMA (p, q) model in (1-68), the input white noise process $w(n)$ there is uncorrelated with its own past sample values as well as the past values of the system output. This gives

$$E\{w(n)w^*(n-k)\} = 0, \quad k \geq 1 \qquad (1\text{-}105)$$

$$E\{w(n)x^*(n-k)\} = 0, \quad k \geq 1. \qquad (1\text{-}106)$$

Together with (1-68), we obtain

$$r_i = E\{x(n)x^*(n-i)\}$$
$$= -\sum_{k=1}^{p} a_k \{x(n-k)x^*(n-i)\} + \sum_{k=0}^{q} b_k \{w(n-k)w^*(n-i)\}$$
$$= -\sum_{k=1}^{p} a_k r_{i-k} + \sum_{k=0}^{q} b_k \{w(n-k)x^*(n-i)\} \qquad (1\text{-}107)$$

and hence in general

$$\sum_{k=1}^{p} a_k r_{i-k} + r_i \neq 0, \qquad i \leq q \qquad (1\text{-}108)$$

and

$$\sum_{k=1}^{p} a_k r_{i-k} + r_i = 0, \qquad i \geq q+1. \qquad (1\text{-}109)$$

Notice that (14-109) is the same as (1-102) with $\{h_k\}$ replaced by $\{r_k\}$ and hence the Kronecker conditions for rational systems can be expressed in terms of its output autocorrelations as well.

Thus if $X(n) \sim$ ARMA (p, q) represents a wide sense stationary stochastic process, then its output autocorrelation sequence $\{r_k\}$ satisfies

$$\text{rank}\, D_{p-1} = \text{rank}\, D_{p+k} = p, \qquad k \geq 0, \tag{1-110}$$

where

$$D_k \triangleq \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_k \\ r_1 & r_2 & r_3 & \cdots & r_{k+1} \\ \vdots & & & & \\ r_k & r_{k+1} & r_{k+2} & \cdots & r_{2k} \end{pmatrix} \tag{1-111}$$

represents the $(k+1) \times (k+1)$ Hankel matrix generated from $r_0, r_1, \cdots, r_k, \cdots, r_{2k}$. It follows that for ARMA (p, q) systems, we have

$$\det D_n = 0, \qquad \text{for all sufficiently large } n. \tag{1-112}$$

Reference and sourse:

1. Multivariate Time Series Analysis: With R and Financial Applications by Ruey S. Tsay
2. Time Series Analysis by James Douglas Hamilton
3. The Analysis of Time Series: An Introduction with R Chapman & Hall/CRC Texts in Statistical Science)
4. Machine Learning for Time Series Forecasting with Python by Francesca Lazzeri
5. Time Series Analysis for the Social Sciences (Analytical Methods for Social Research) Part of: Analytical Methods for Social Research (14 Books)
6. Introduction to Probability, Statistics, and Random Processes by Hossein Pishro-Nik
7. Introduction to Time Series and Forecasting (Springer Texts in Statistics) Part of: Springer Texts in Statistics