

Time series analysis

Lecture 5. Non-stationary time series

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ARIMA models for time series data

- Autoregression

- Consider regression models of the form

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

- Define

$$x_1 = y_{t-1}$$

$$x_2 = y_{t-2}$$

$$\vdots$$

$$x_p = y_{t-p}$$

ARIMA models for time series data

- Then the previous equation becomes

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + \varepsilon_t$$

- The explanatory variables in this equations are time-lagged values of the variable y .
- Autoregression (AR) is used to describe models of this form.

ARIMA models for time series data

- Autoregression models should be treated differently from ordinary regression models since:
 - The explanatory variables in the autoregression models have a built-in dependence relationship.
 - Determining the number of past values of y_t to include in the model is not always straight forward.

ARIMA models for time series data

- Moving average model
 - A time series model which uses past errors as explanatory variable:

$$y_t = \beta_0 + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_p e_{t-p} + \varepsilon_t$$

is called moving average(MA) model

- Note that this model is defined as a moving average of the error series, while the moving average models we discussed previously are the moving average of the observations.

ARIMA models for time series data

- Autoregressive (AR) models can be coupled with moving average (MA) models to form a general and useful class of time series models called *Autoregressive Moving Average (ARMA)* models.
- These can be used when the data are stationary.

ARIMA models for time series data

- This class of models can be extended to non-stationary series by allowing the differencing of the data series.
- These are called *Autoregressive Integrated Moving Average (ARIMA)* models.
- There are a large variety of ARIMA models.

ARIMA models for time series data

- The general non-seasonal model is known as ARIMA (p, d, q):
 - p is the number of autoregressive terms.
 - d is the number of differences.
 - q is the number of moving average terms.

ARIMA models for time series data

- A white noise model is classified as ARIMA (0, 0, 0)
 - No AR part since y_t does not depend on y_{t-1} .
 - There is no differencing involved.
 - No MA part since y_t does not depend on e_{t-1} .

ARIMA models for time series data

- A random walk model is classified as ARIMA (0, 1, 0)
 - There is no AR part.
 - There is no MA part.
 - There is one difference.

ARIMA models for time series data

- Note that if any of p , d , or q are equal to zero, the model can be written in a shorthand notation by dropping the unused part.
- Example
 - $\text{ARIMA}(2, 0, 0) = \text{AR}(2)$
 - $\text{ARIMA}(1, 0, 1) = \text{ARMA}(1, 1)$

An autoregressive model of order one AR(1)

- The basic form of an ARIMA (1, 0, 0) or AR(1) is:

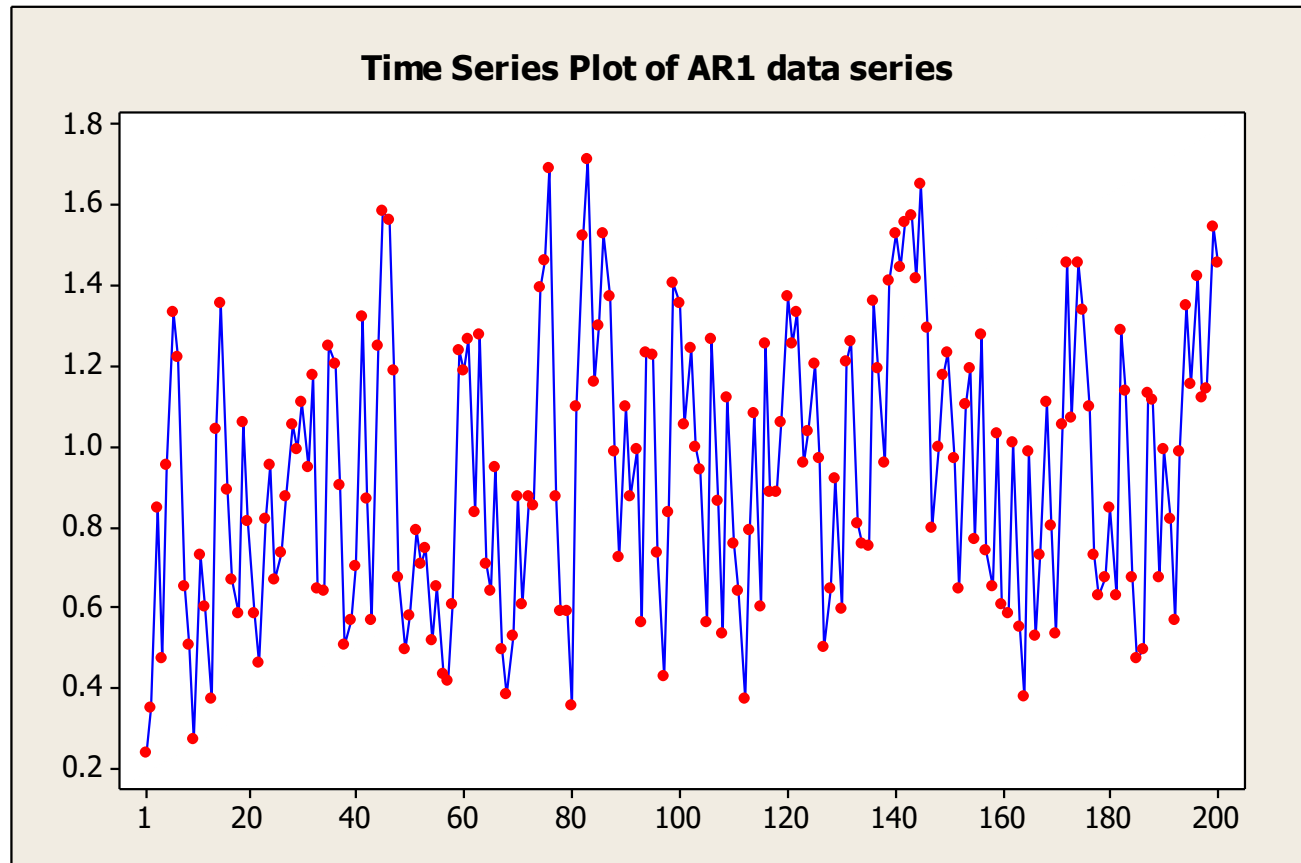
$$y_t = C + \phi_1 y_{t-1} + e_t$$

- Observation y_t depends on y_{t-1} .
- The value of autoregressive coefficient ϕ_1 is between -1 and 1 .

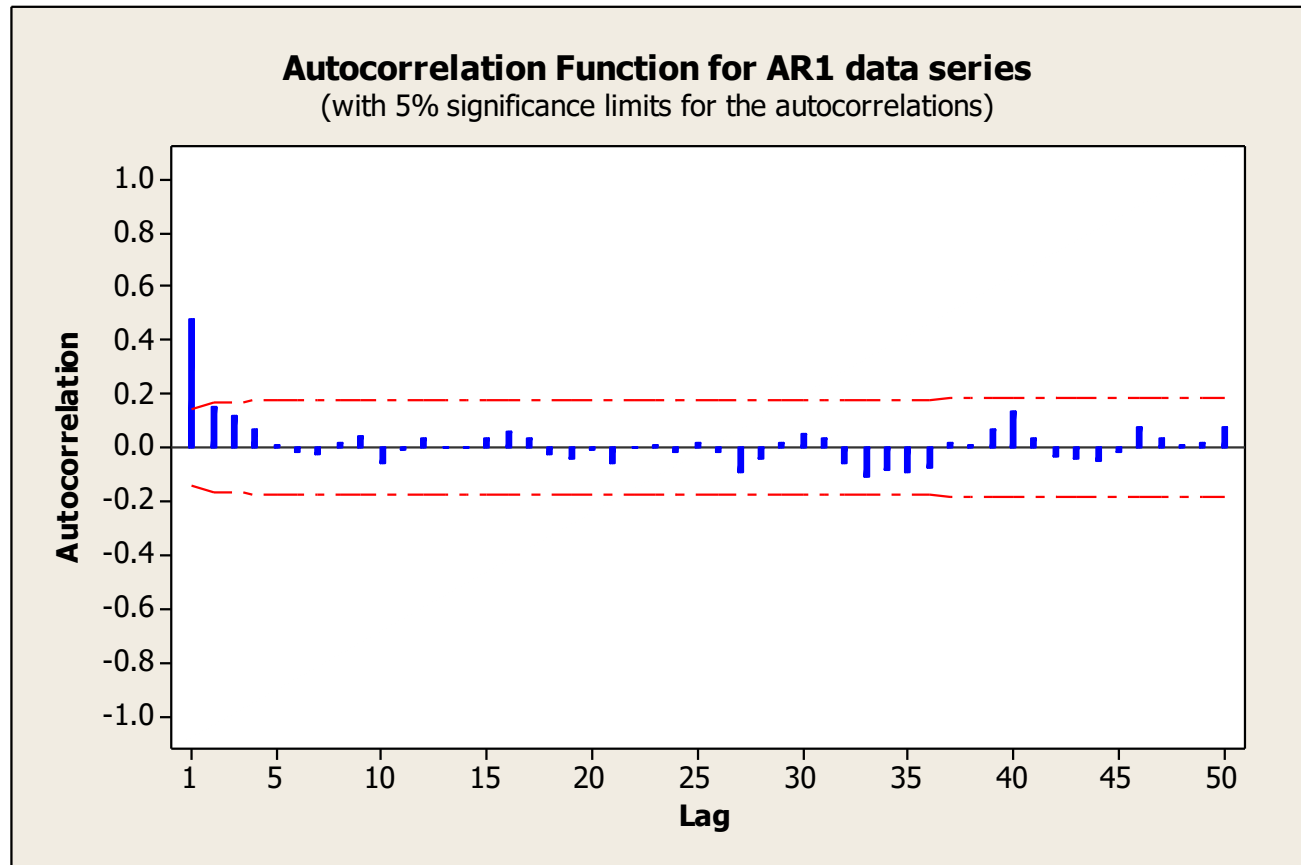
An autoregressive model of order one

- The time plot of an AR(1) model varies with the parameter ϕ_1 .
 - When $\phi_1 = 0$, y_t is equivalent to a white noise series.
 - When $\phi_1 = 1$, y_t is equivalent to a random walk series
 - For negative values of ϕ_1 , the series tends to oscillate between positive and negative values.
- The following slides show the time series, ACF and PACF plot for an ARIMA(1, 0, 0) time series data.

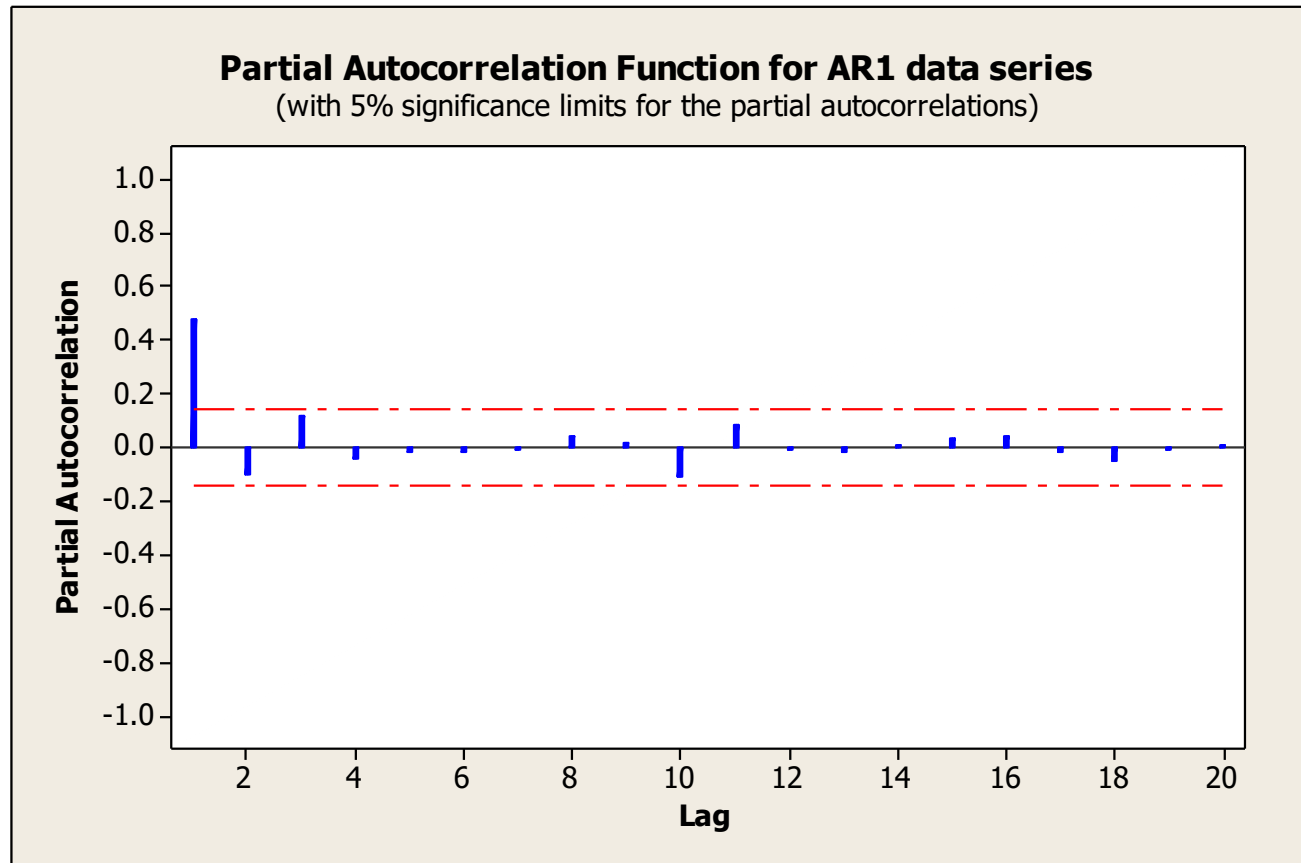
An autoregressive model of order one



An autoregressive model of order one



An autoregressive model of order one



An autoregressive model of order one

- The ACF and PACF can be used to identify an AR(1) model.
 - The autocorrelations decay exponentially.
 - There is a single significant partial autocorrelation.

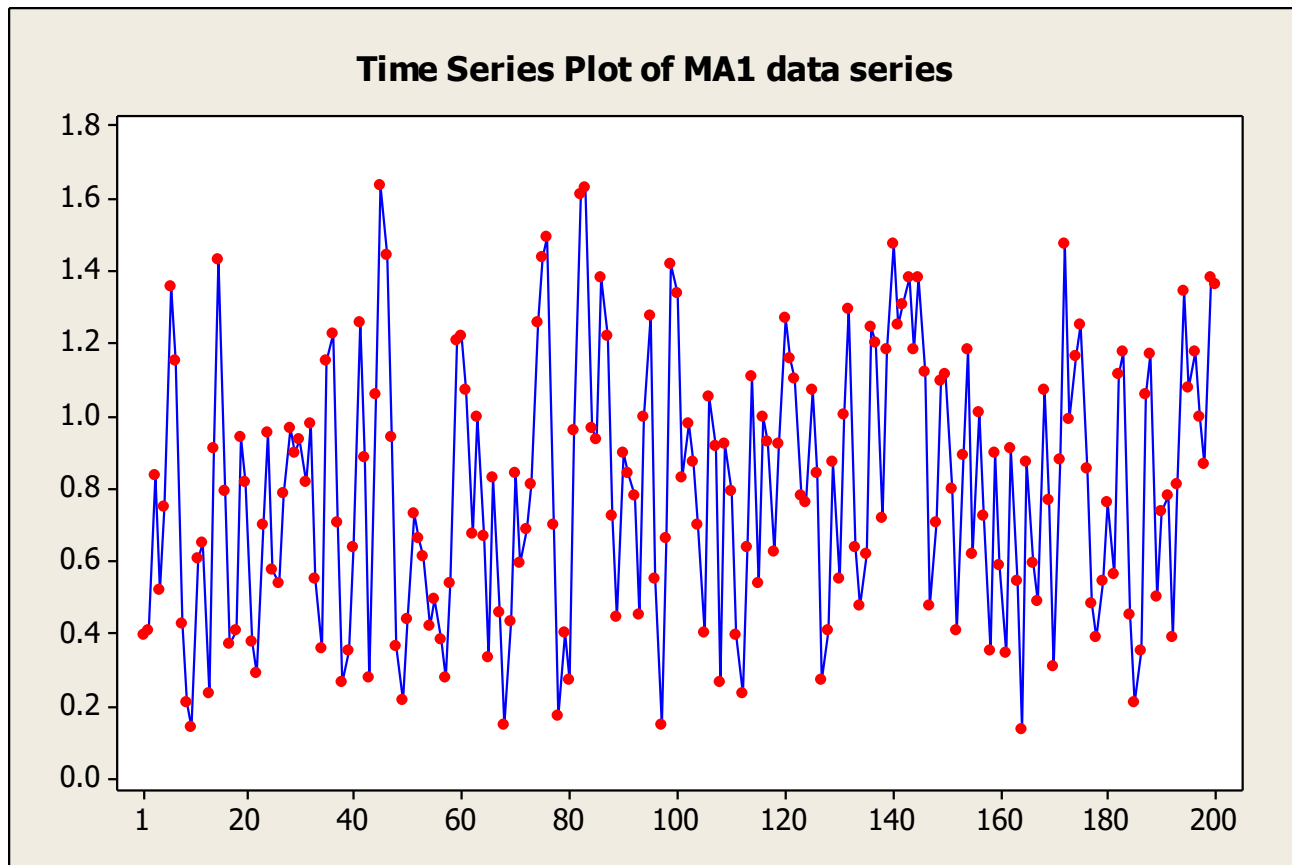
A moving average of order one MA(1)

- The general form of ARIMA (0, 0, 1) or MA(1) model is

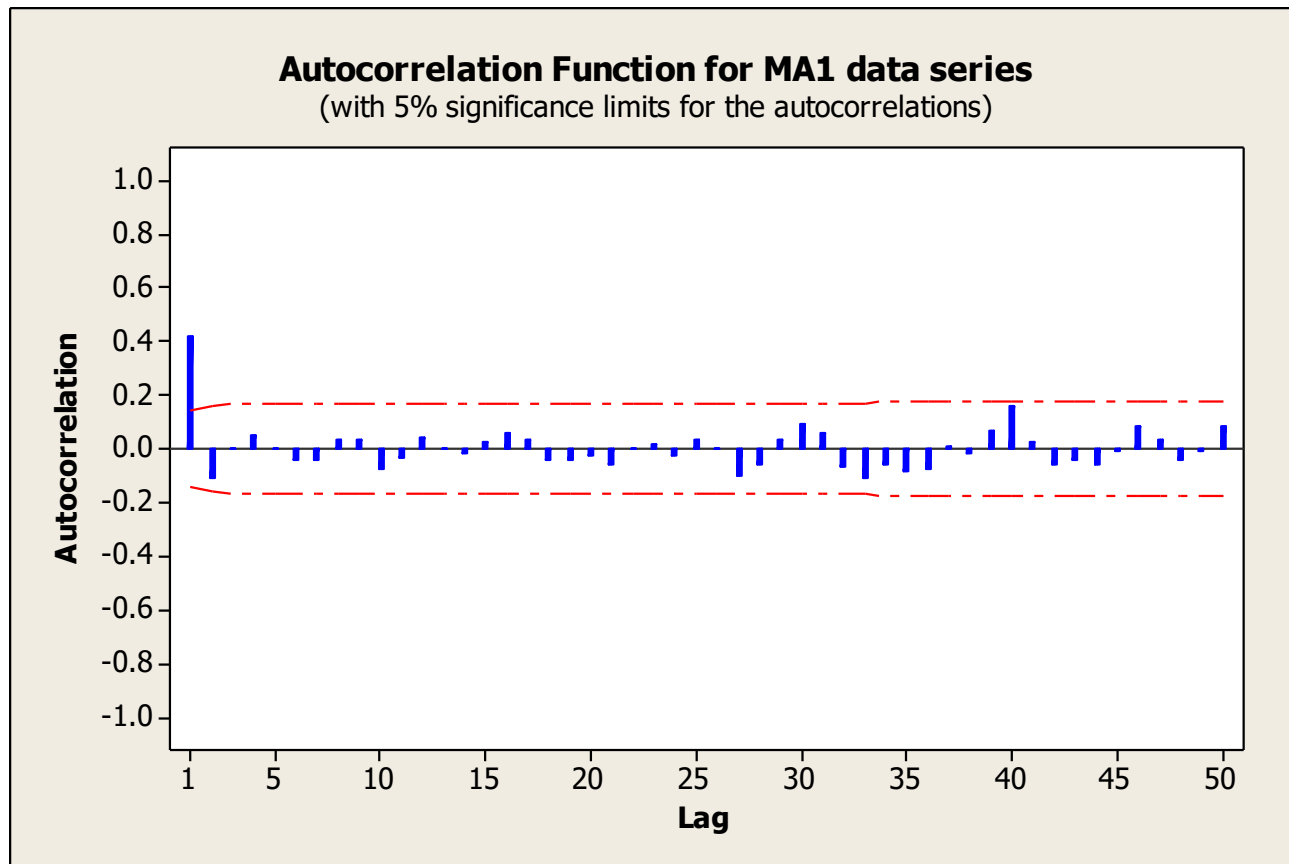
$$y_t = C + e_t - \theta_1 e_{t-1}$$

- Y_t depends on the error term e_t and on the previous error term e_{t-1} with coefficient $-\theta_1$.
- The value of θ_1 is between -1 and 1 .
- The following slides show an example of an MA(1) data series.

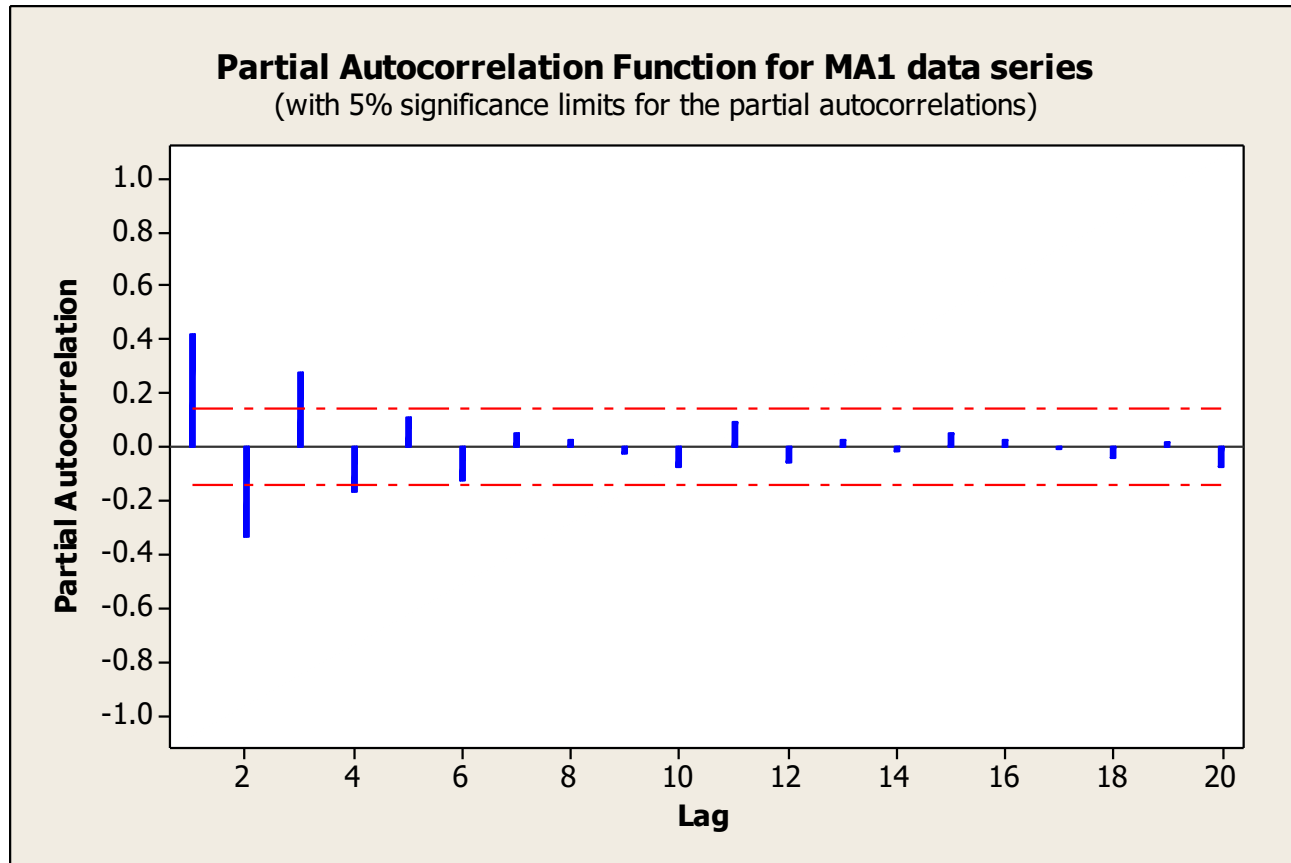
A moving average of order one MA(1)



A moving average of order one MA(1)



A moving average of order one MA(1)



A moving average of order one MA(1)

- Note that there is only one significant autocorrelation at time lag 1.
- The partial autocorrelations decay exponentially, but because of random error components, they do not die out to zero as do the theoretical autocorrelation.

Higher order auto regressive models

- A pth-order AR model is defined as

$$y_t = C + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t$$

- C is the constant term
- ϕ_j is the jth auto regression parameter
- e_t is the error term at time t.

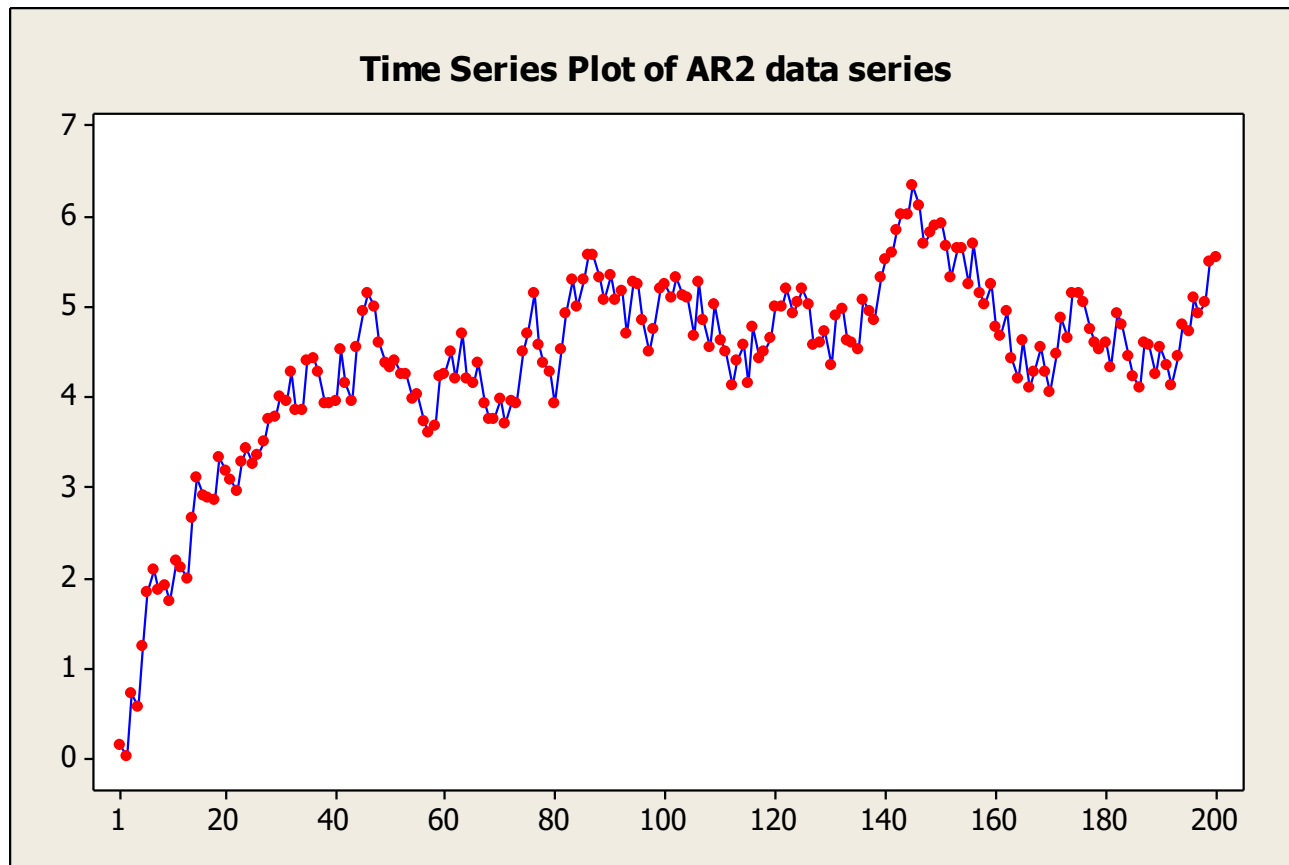
Higher order auto regressive models

- Restrictions on the allowable values of auto regression parameters
 - For $p = 1$
 - $-1 < \phi_1 < 1$
 - For $p = 2$
 - $-1 < \phi_2 < 1$
 - $\phi_1 + \phi_2 < 1$
 - $\phi_2 - \phi_1 < 1$

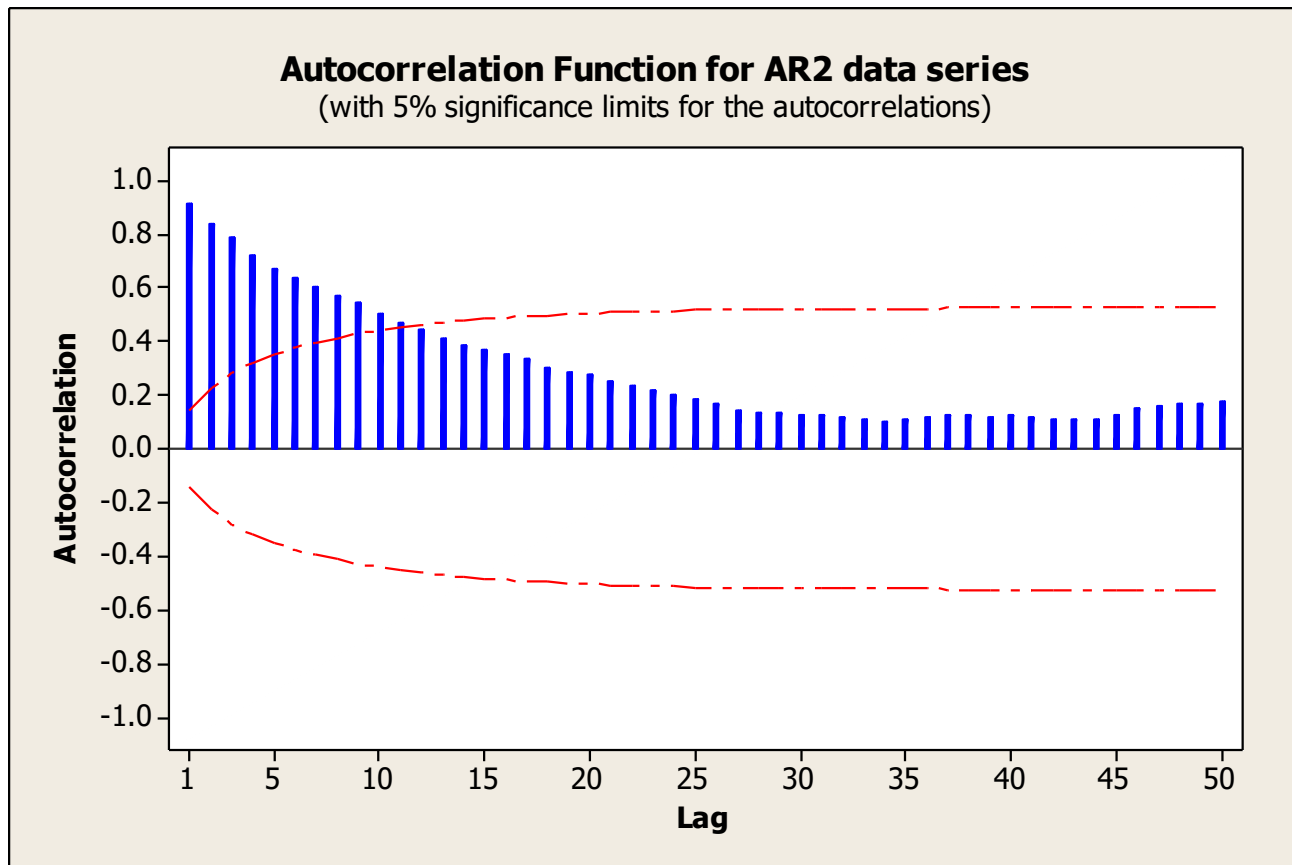
Higher order auto regressive models

- A great variety of time series are possible with autoregressive models.
- The following slides shows an AR(2) model.
- Note that for AR(2) models the autocorrelations die out in a damped Sine-wave patterns.
- There are exactly two significant partial autocorrelations.

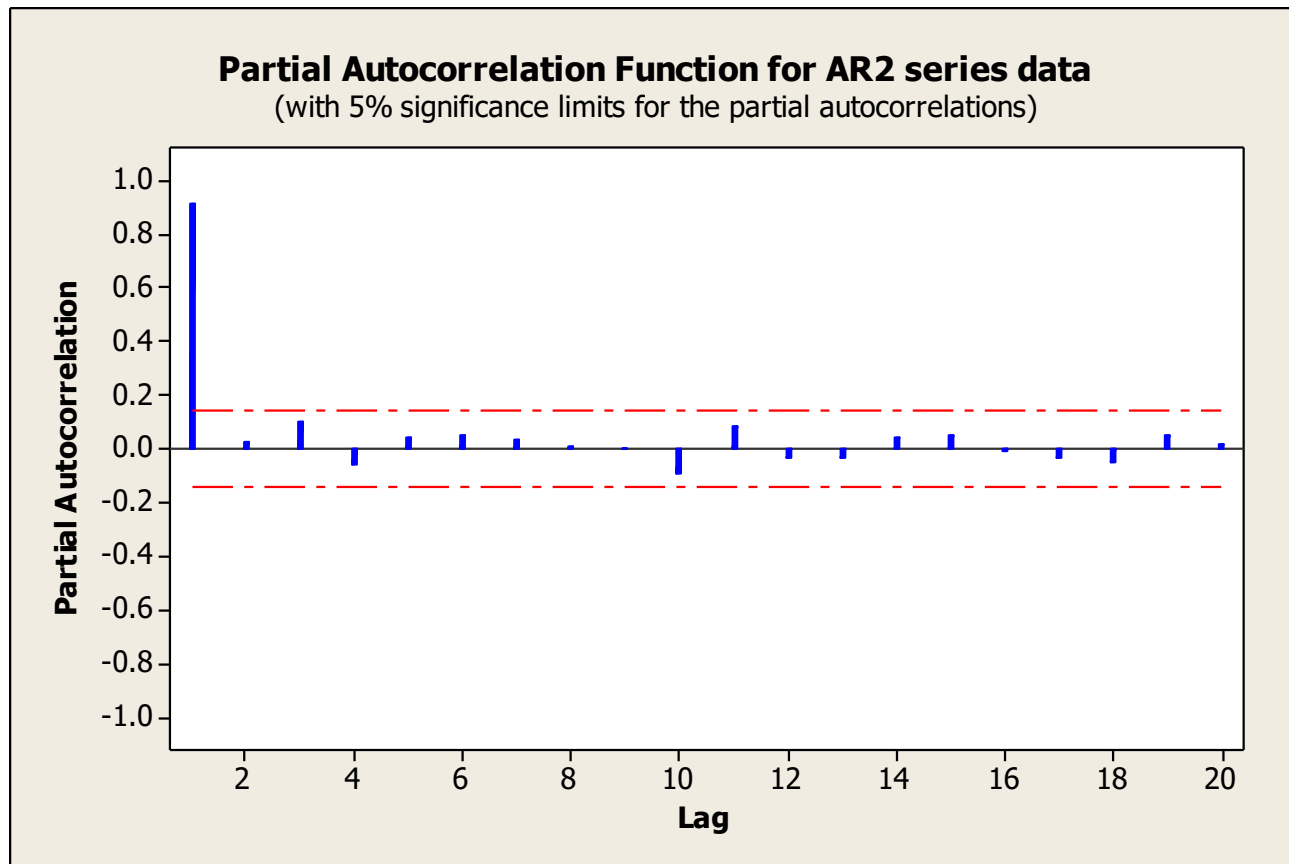
Higher order auto regressive models



Higher order auto regressive models



Higher order auto regressive models



Higher order moving average models

- The general MA model of order q can be written as

$$y_t = C + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} + \dots + \theta_q e_{t-q}$$

- C is the constant term
- θ_j is the j th moving average parameter.
- e_{t-k} is the error term at time $t-k$

Higher order moving average models

- Restrictions on the allowable values of the MA parameters.
 - For $q = 1$
 - $-1 < \theta_1 < 1$
 - For $q = 2$
 - $-1 < \theta_2 < 1$
 - $\theta_1 + \theta_2 < 1$
 - $\theta_2 - \theta_1 < 1$

Higher order moving average models

- A wide variety of time series can be produced using moving average models.
- In general, the autocorrelations of an MA(q) models are zero beyond lag q
- For $q \geq 2$, the PACF can show exponential decay or damped sine-wave patterns.

Reference and source:

1. Multivariate Time Series Analysis: With R and Financial Applications by Ruey S. Tsay
2. Time Series Analysis by James Douglas Hamilton
3. The Analysis of Time Series: An Introduction with R (Chapman & Hall/CRC Texts in Statistical Science)
4. Machine Learning for Time Series Forecasting with Python by Francesca Lazzeri
5. Time Series Analysis for the Social Sciences (Analytical Methods for Social Research) Part of: Analytical Methods for Social Research (14 Books)
6. Introduction to Probability, Statistics, and Random Processes by Hossein Pishro-Nik
7. Introduction to Time Series and Forecasting (Springer Texts in Statistics) Part of: Springer Texts in Statistics