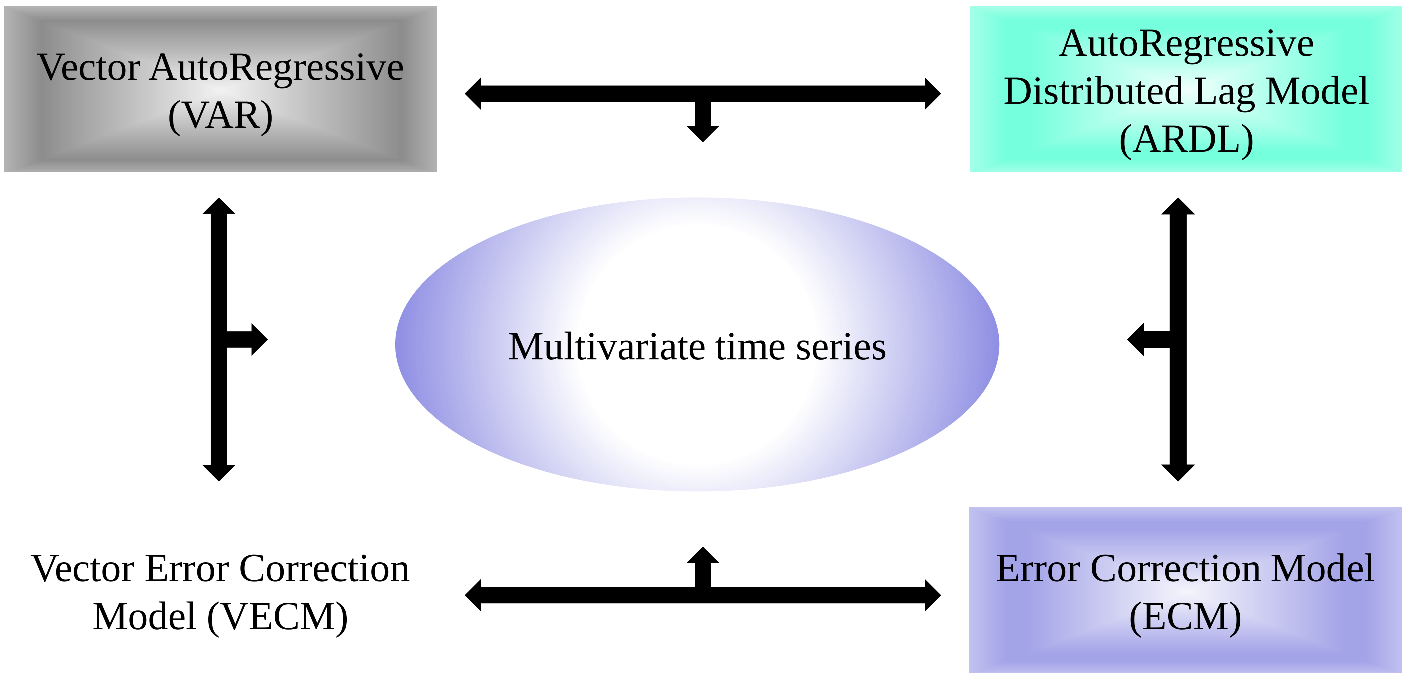
Time series analysis

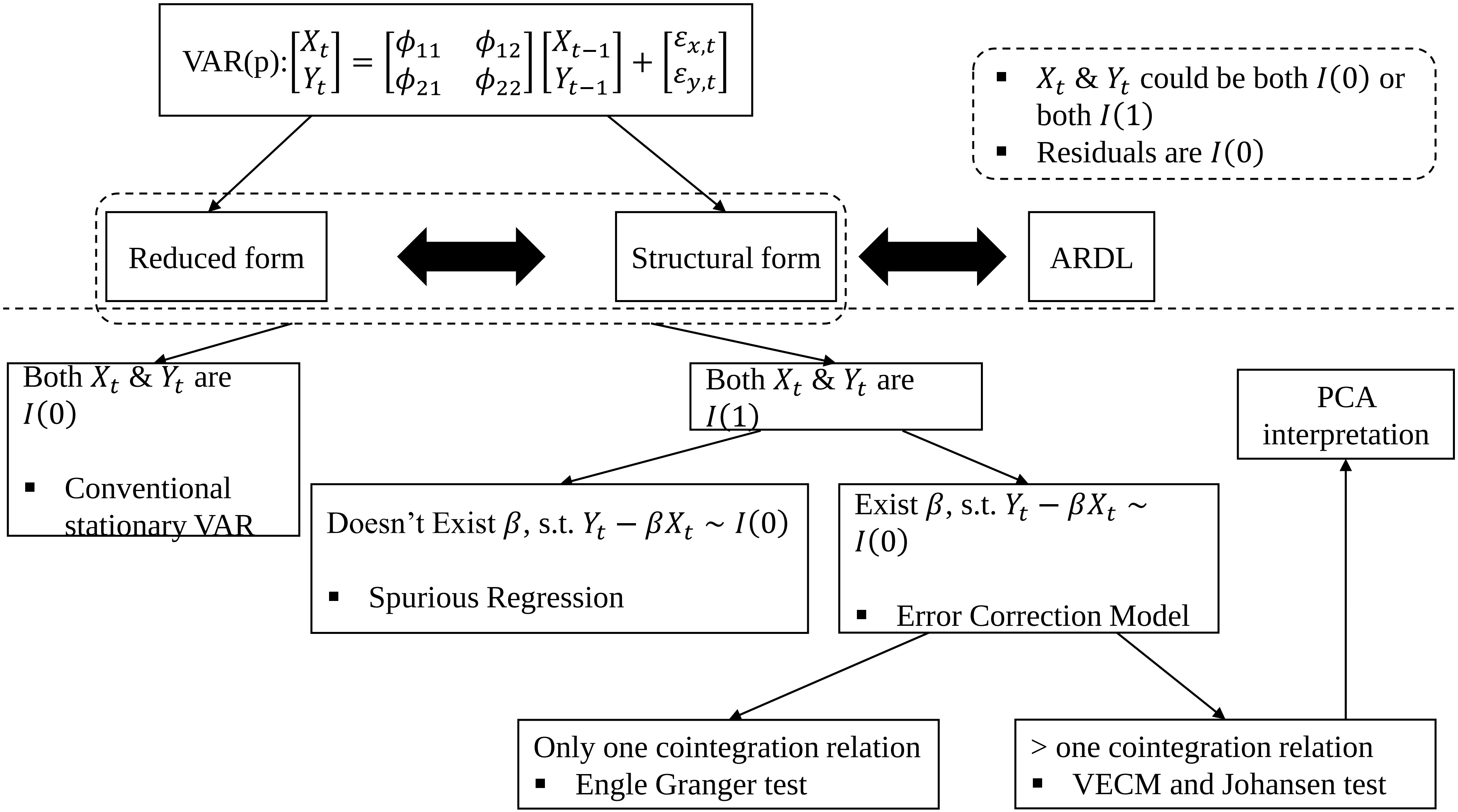
Lecture 9. Vector autoregression model and co-integration

Dr. Khamidov Obidjon

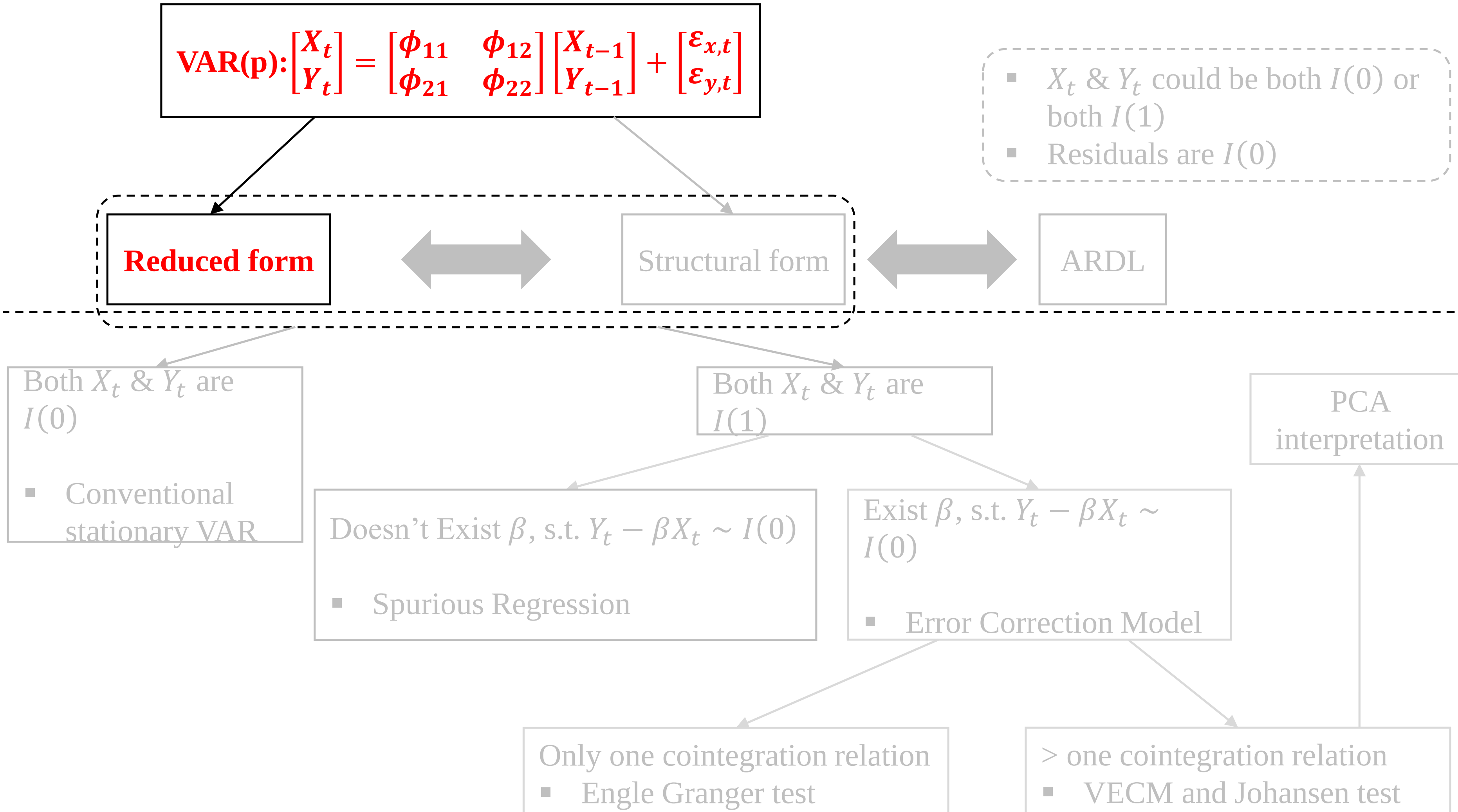
Analysis of multivariate time series

In this presentation, we study the inter-relationships between several multivariate time series regression methods to provide guidance on when to use what method, and how to implement it in SAS, R, or Matlab.


Vector AutoRegressive (VAR)
AutoRegressive Distributed Lag Model (ARDL)
Multivariate time series
Vector Error Correction Model (VECM)
Error Correction Model (ECM)



VAR(p): $\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$
▪ X_t & Y_t could be both $I(0)$ or both $I(1)$
▪ Residuals are $I(0)$
Reduced form
Structural form
ARDL
Both X_t & Y_t are $I(0)$
▪ Conventional stationary VAR
Both X_t & Y_t are $I(1)$
PCA interpretation
Doesn't Exist β, s.t. $Y_t - \beta X_t \sim I(0)$
▪ Spurious Regression
Exist β, s.t. $Y_t - \beta X_t \sim I(0)$
▪ Error Correction Model
Only one cointegration relation
▪ Engle Granger test
> one cointegration relation
▪ VECM and Johansen test



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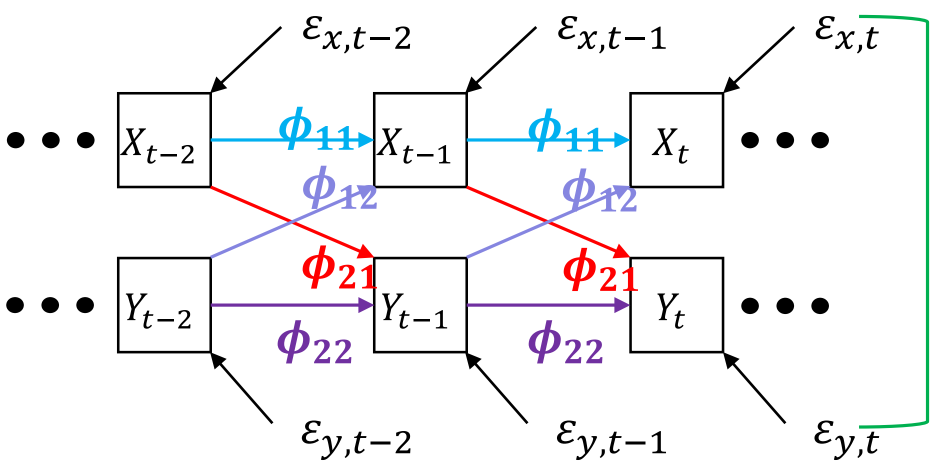
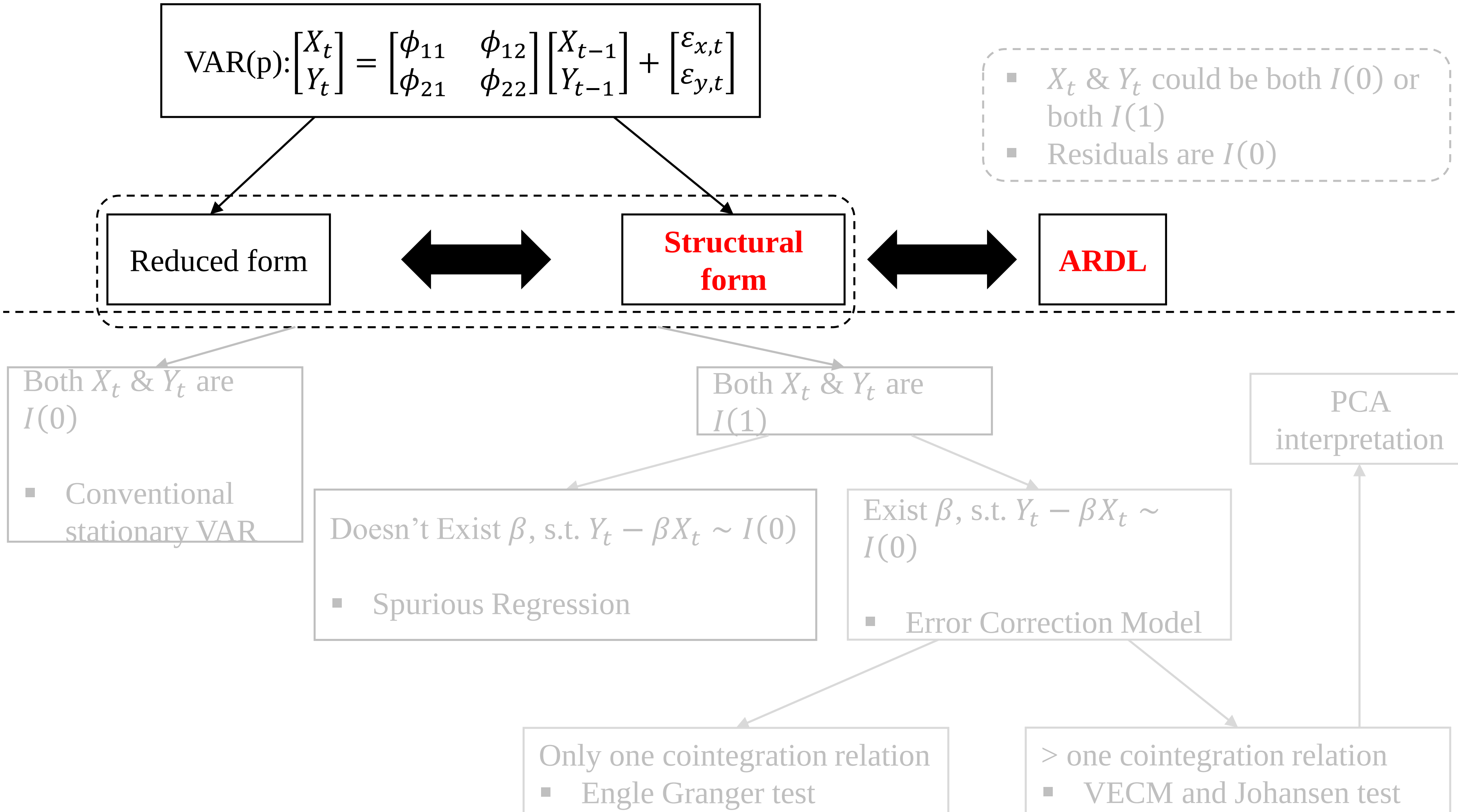
Reduced form of VAR

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix},$$

where $\begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix} \sim WN(0, \Sigma)$, and $\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$.



- Relation between X_t & Y_t are not directly shown
- Cholesky Decomposition: there exists $L = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, s.t. $\Sigma = LDL'$, where $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$
- $L^{-1}\Sigma L'^{-1} = D$, where $L^{-1} = \begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$
- $\begin{bmatrix} \phi_{11}^* & \phi_{12}^* \\ \phi_{21}^* & \phi_{22}^* \end{bmatrix} = L^{-1} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, $\begin{bmatrix} \varepsilon_{x,t}^* \\ \varepsilon_{y,t}^* \end{bmatrix} = L^{-1} \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$, then
  $$L^{-1} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11}^* & \phi_{12}^* \\ \phi_{21}^* & \phi_{22}^* \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t}^* \\ \varepsilon_{y,t}^* \end{bmatrix}$$
- $Cov\left(\begin{bmatrix} \varepsilon_{x,t}^* \\ \varepsilon_{y,t}^* \end{bmatrix}\right) = L^{-1}\Sigma L'^{-1} = D$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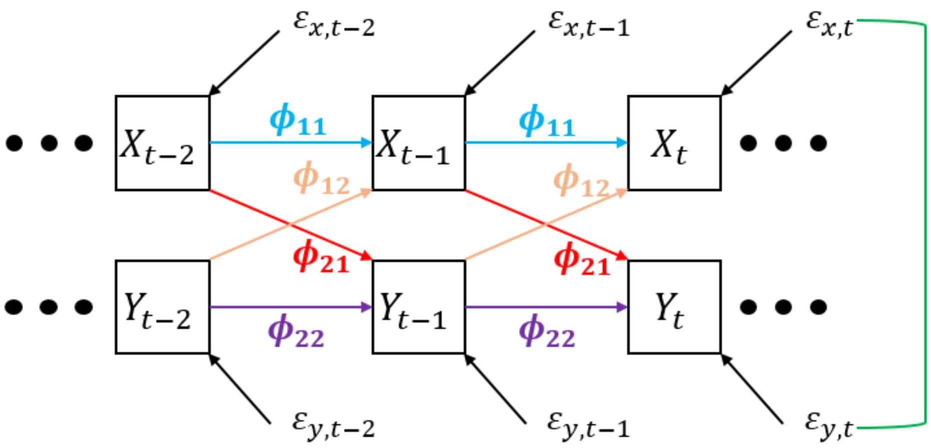
Structural form of VAR

$$L^{-1}\begin{bmatrix}X_t\\Y_t\end{bmatrix}=\begin{bmatrix}1 & 0\\-c & 1\end{bmatrix}\begin{bmatrix}X_t\\Y_t\end{bmatrix}=\begin{bmatrix}\phi_{11}^* & \phi_{12}^*\\\phi_{21}^* & \phi_{22}^*\end{bmatrix}\begin{bmatrix}X_{t-1}\\Y_{t-1}\end{bmatrix}+\begin{bmatrix}\varepsilon_{x,t}^*\\\varepsilon_{y,t}^*\end{bmatrix}$$

$$\begin{cases} X_t = \phi_{11}^* X_{t-1} + \phi_{12}^* Y_{t-1} + \varepsilon_{x,t}^* \\ Y_t = c X_t + \phi_{21}^* X_{t-1} + \phi_{22}^* Y_{t-1} + \varepsilon_{y,t}^* \end{cases}$$



ARDL

- Relation between X_t & Y_t is indicated by c, as a results, $\varepsilon_{x,t}^*$ & $\varepsilon_{y,t}^*$ are no longer correlated
- Reduced form and structural form are equivalent
- $Y_t = cX_t + \phi_{21}^* X_{t-1} + \phi_{22}^* Y_{t-1} + \varepsilon_{y,t}^*$ is **A**uto**R**egressive **D**istributed **L**ag Model –ARDL(1,1)
- $X_t = \phi_{11}^* X_{t-1} + \phi_{12}^* Y_{t-1} + \varepsilon_{x,t}^*$ $= 0Y_t + \phi_{11}^* X_{t-1} + \phi_{12}^* Y_{t-1} + \varepsilon_{x,t}^*$ is also ARDL

Fitting function of ARDL/ARX:

- Matlab:

```
Spec = vgxset('n',1,'nAR',1,'nX',1,'Constant',false);
estSpec = vgxvarx(Spec,Y(:,2),num2cell(Y(:,1)),[]);
```

- R Univariate: `fit = arima(y,c(1,0,0),xreg = x)`
- R Multivariate: `fit = VAR(cbind(y,z),p = 1,type = 'none',exogen = x)`

- SAS: `proc VARMAX;model y = x / p=1 noint;`

More than two variables

- Reduced form

$$\begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{Kt} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1K} \\ \vdots & \ddots & \vdots \\ \phi_{K1} & \cdots & \phi_{KK} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ \vdots \\ X_{K,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{K,t} \end{bmatrix}, \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{K,t} \end{bmatrix} \sim N(0, \Sigma).$$

- Structural form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21}^* & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{K1}^* & l_{K2}^* & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{Kt} \end{bmatrix} = \begin{bmatrix} \phi_{11}^* & \cdots & \phi_{1K}^* \\ \vdots & \ddots & \vdots \\ \phi_{K1}^* & \cdots & \phi_{KK}^* \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ \vdots \\ X_{K,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t}^* \\ \varepsilon_{2,t}^* \\ \vdots \\ \varepsilon_{K,t}^* \end{bmatrix}, cov\left(\varepsilon_{i,t}^*, \varepsilon_{j,t}^*\right) = 0$$

- ARDL

$$\begin{cases} X_{1t} = \phi_{11}^* X_{1,t-1} + \cdots + \phi_{1K}^* X_{K,t-1} + \varepsilon_{1,t}^* \\ X_{2t} = \phi_{21}^* X_{1,t-1} + \cdots + \phi_{2K}^* X_{K,t-1} - l_{21}^* X_{1t} + \varepsilon_{2,t}^* \\ X_{3t} = \phi_{31}^* X_{1,t-1} + \cdots + \phi_{3K}^* X_{K,t-1} - l_{31}^* X_{1t} - l_{32}^* X_{2t} + \varepsilon_{3,t}^* \\ \vdots \\ X_{Kt} = \phi_{K1}^* X_{1,t-1} + \cdots + \phi_{KK}^* X_{K,t-1} - \Sigma_{i=1}^{K-1} X_{it} + \varepsilon_{K,t}^* \end{cases}$$

What if VAR(p) instead of VAR(1)?

- VAR(2)

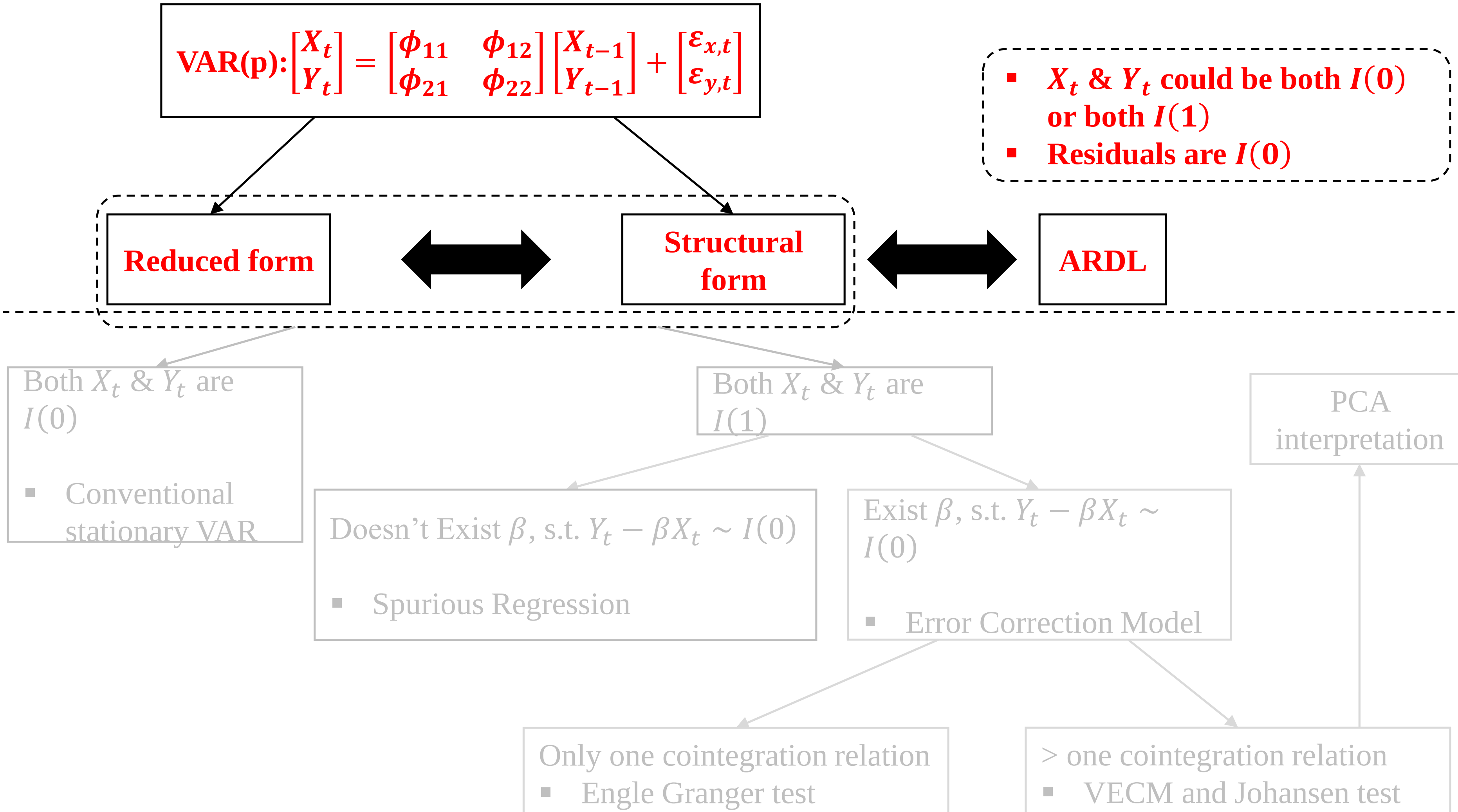
$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}^{(2)} & \phi_{12}^{(2)} \\ \phi_{21}^{(2)} & \phi_{22}^{(2)} \end{bmatrix} \begin{bmatrix} X_{t-2} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}, \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix} \sim WN(0, \Sigma)$$

- Transform to VAR(1)

$$\begin{bmatrix} X_{t-1} \\ Y_{t-1} \\ X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \phi_{11}^{(2)} & \phi_{12}^{(2)} & \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(2)} & \phi_{22}^{(2)} & \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} X_{t-2} \\ Y_{t-2} \\ X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$$

Companion matrix

- Any VAR(p) could be transformed to VAR(1)



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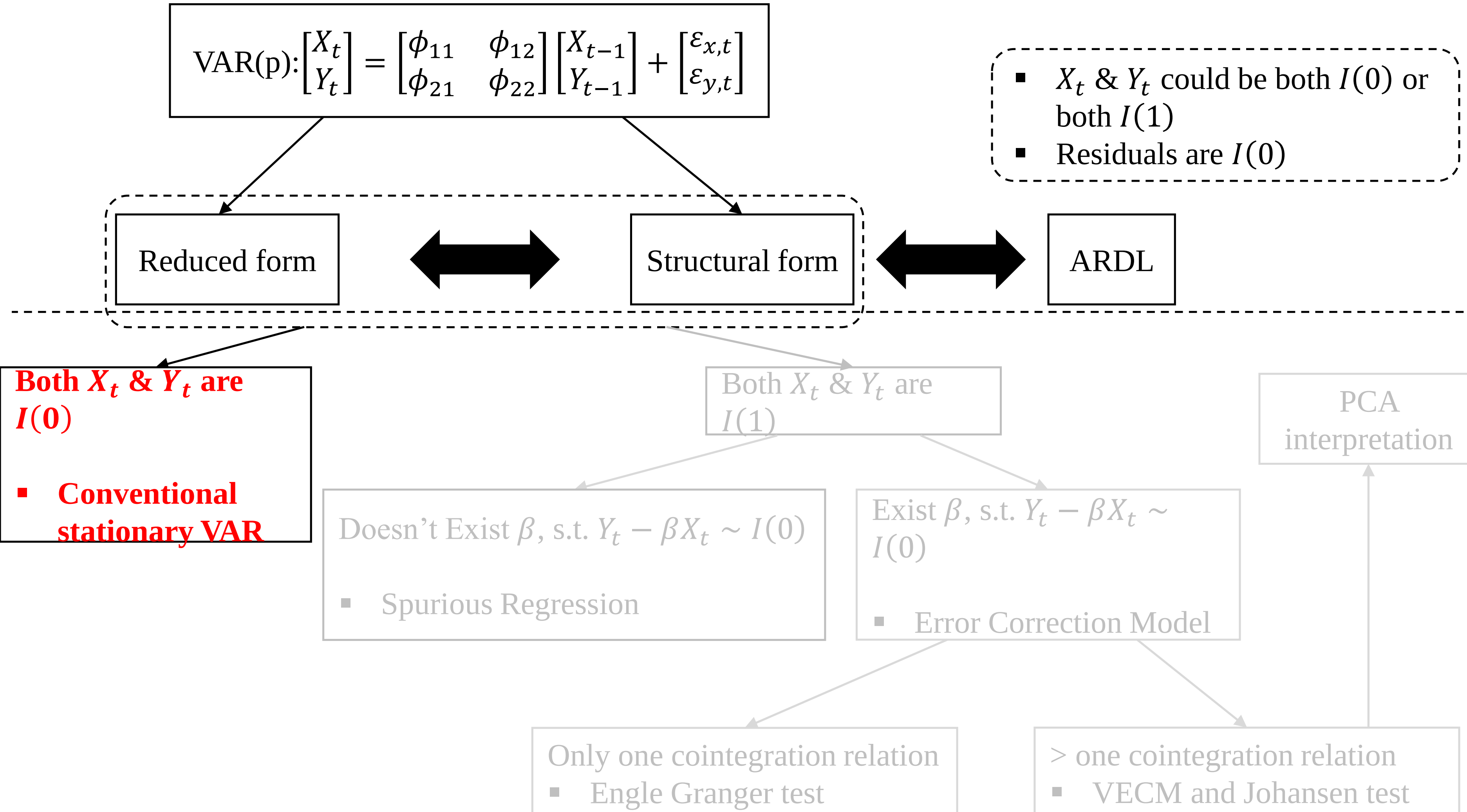

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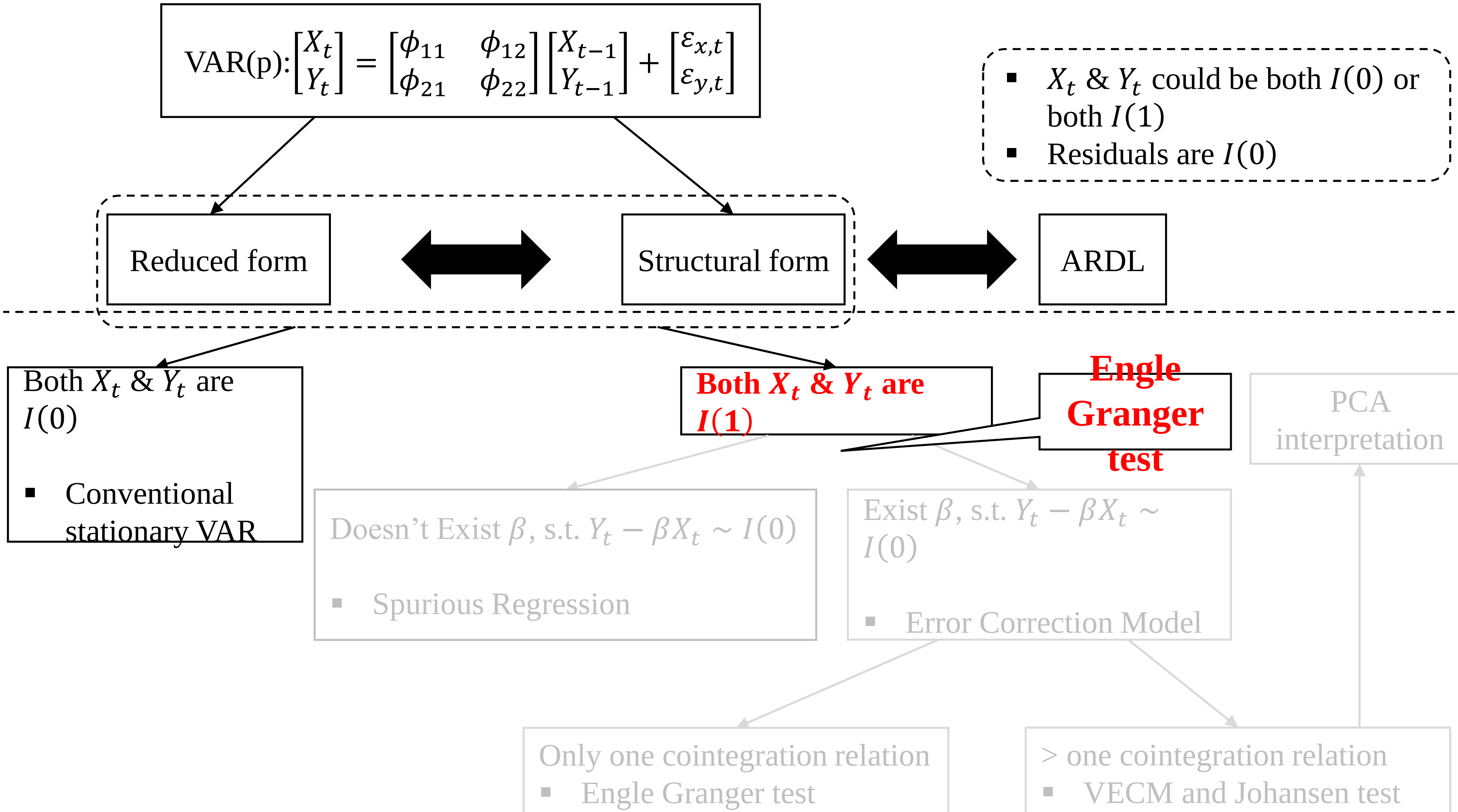
- $$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$$
- Roots of $\left| I - \begin{bmatrix} \phi_{11}B & \phi_{12}B \\ \phi_{21}B & \phi_{22}B \end{bmatrix} \right|$ have modulus greater than 1

Fitting function of stationary VAR:

- Matlab:

```
   Spec = vgxset('n',2,'nAR',1,'Constant',false);
estSpec = vgxvarx(Spec,Y);
```

- R Reduced form: `fit  = VAR(cbind(x,y), p = 1, type = 'none')`
- R Structural form: `fit2 = SVAR(fit,Amat = matrix(c(1,0,NA,1),2,2))` $\Leftarrow \mathbf{L} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$
- SAS: `proc VARMAX;model x y / p=1 noint;`



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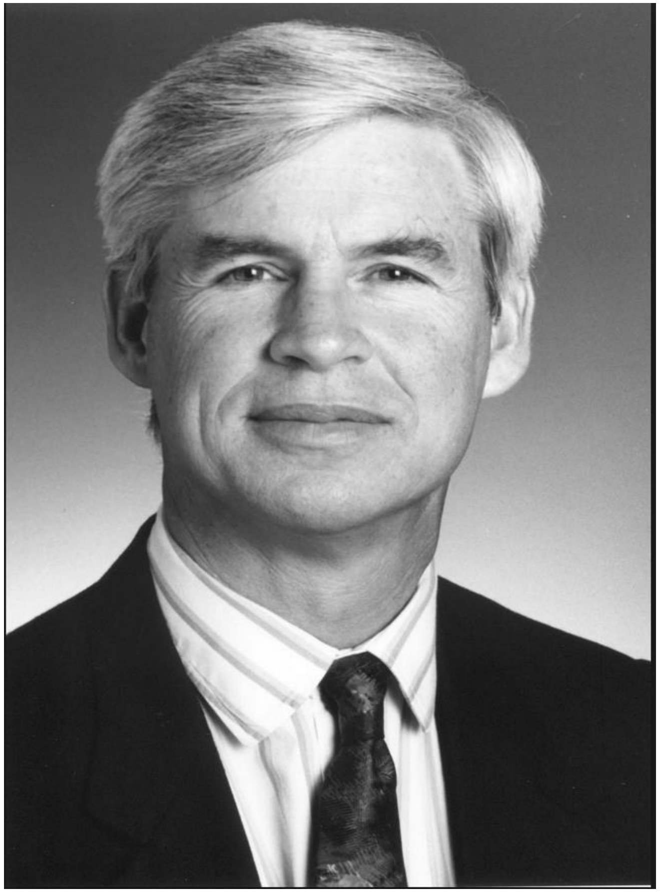
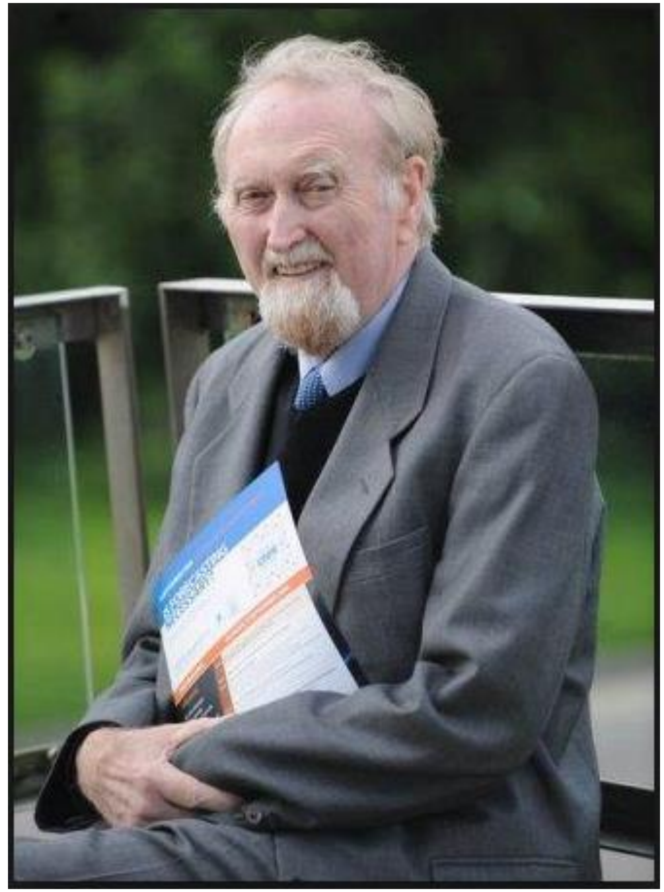


Robert F. Engle (born in 1942)
American economist, currently teaches at New York University, Stern School of Business



Clive Granger (1934 – 2009)
British economist, taught at University of Nottinghan in Britain & University of California, San Diego in US

They shared Nobel Memorial Prize in Economic Sciences in 2003 for the methods of ARCH (Engle), and Co-integration (Granger) respectively.

Cointegration:

There are two $I(1)$ series, X_t and Y_t, if there exists β s.t. $Y_t - \beta X_t \sim I(0)$, we say X_t and Y_t are cointegrated, or there is one cointegrating relation between X_t and Y_t, and the state of $Y_t - \beta X_t \sim I(0)$ is called the long-run equilibrium between X_t and Y_t

Test cointegration:

- Engle Granger test
  - Fit OLS – $Y_t = \hat{\alpha} + \hat{\beta} X_t + \hat{Z}_t$
  - Test $\hat{Z}_t$ is I(0) or I(1)

How to test I(0) v.s. I(1):

Dickey-Fuller test & Augmented Dickey-Fuller test

	Dickey Fuller test
▪ If there's no drift or linear trend	$Z_t = \phi Z_{t-1} + u_t$, where u_t is white noise
$\begin{cases} H_0: \phi = 1 \\ H_1: \lvert\phi\rvert < 1 \end{cases}$	Under H_0, Z_t is a random walk, while under H_1, Z_t is stationary AR
	$Z_t = \phi Z_{t-1} + u_t \Leftrightarrow \nabla Z_t = (\phi - 1)Z_{t-1} + u_t = \gamma Z_{t-1} + u_t$
$\begin{cases} H_0: \phi = 1 \\ H_1: \lvert\phi\rvert < 1 \end{cases} \Leftrightarrow \begin{cases} H_0: \gamma = 0 \\ H_1: \gamma < 0 \end{cases}$	Under H_0, $t_\gamma = \frac{\hat{\gamma}_{OLS}}{SE(\hat{\gamma}_{OLS})}$ is not standard normal, and its limiting distribution is called Dickey-Fuller distribution, which doesn't have closed form representation
▪ To capture the nonzero mean	$Z_t = c + \phi Z_{t-1} + u_t$, where u_t is white noise
	$Z_t = c + \phi Z_{t-1} + u_t \Leftrightarrow \nabla Z_t = c + (\phi - 1)Z_{t-1} + u_t = c + \gamma Z_{t-1} + u_t$
▪ To capture the nonzero mean & linear trend	$Z_t = c + \delta t + \phi Z_{t-1} + u_t$, where u_t is white noise
	$Z_t = c + \delta t + \phi Z_{t-1} + u_t \Leftrightarrow \nabla Z_t = c + \delta t + (\phi - 1)Z_{t-1} + u_t = c + \delta t + \gamma Z_{t-1} + u_t$
	○ Nonzero c indicates existence of drift ○ Nonzero δ indicates existence of linear trend ○ $\gamma = 0$ means non-stationarity, and rejection of $\gamma = 0$ indicates stationarity of Z_t

	Augmented Dickey Fuller test
▪ DF test assumes u_t is white noise, by adding enough lags of ∇Z_{t-j} on the right hand side, this property could be guaranteed approximately	
▪ $\nabla Z_t = c + \delta t + \gamma Z_{t-1} + \Sigma_{j=1}^{p} \phi_j \nabla Z_{t-j} + u_t$	
How to choose # of lags?	○ Ng and Perron (1995): Set a upper bound p_{max}, then decrease p till ϕ_p is significant ○ Schwert (1989): $p = \left[12 \cdot \left(\frac{T}{100}\right)^{1/4}\right]$, T is the number of data points ○ AIC, BIC, etc.
	○ Nonzero c indicates existence of drift ○ Nonzero δ indicates existence of linear trend ○ $\gamma = 0$ means non-stationarity, and rejection of $\gamma = 0$ indicates stationarity of Z_t

EG test

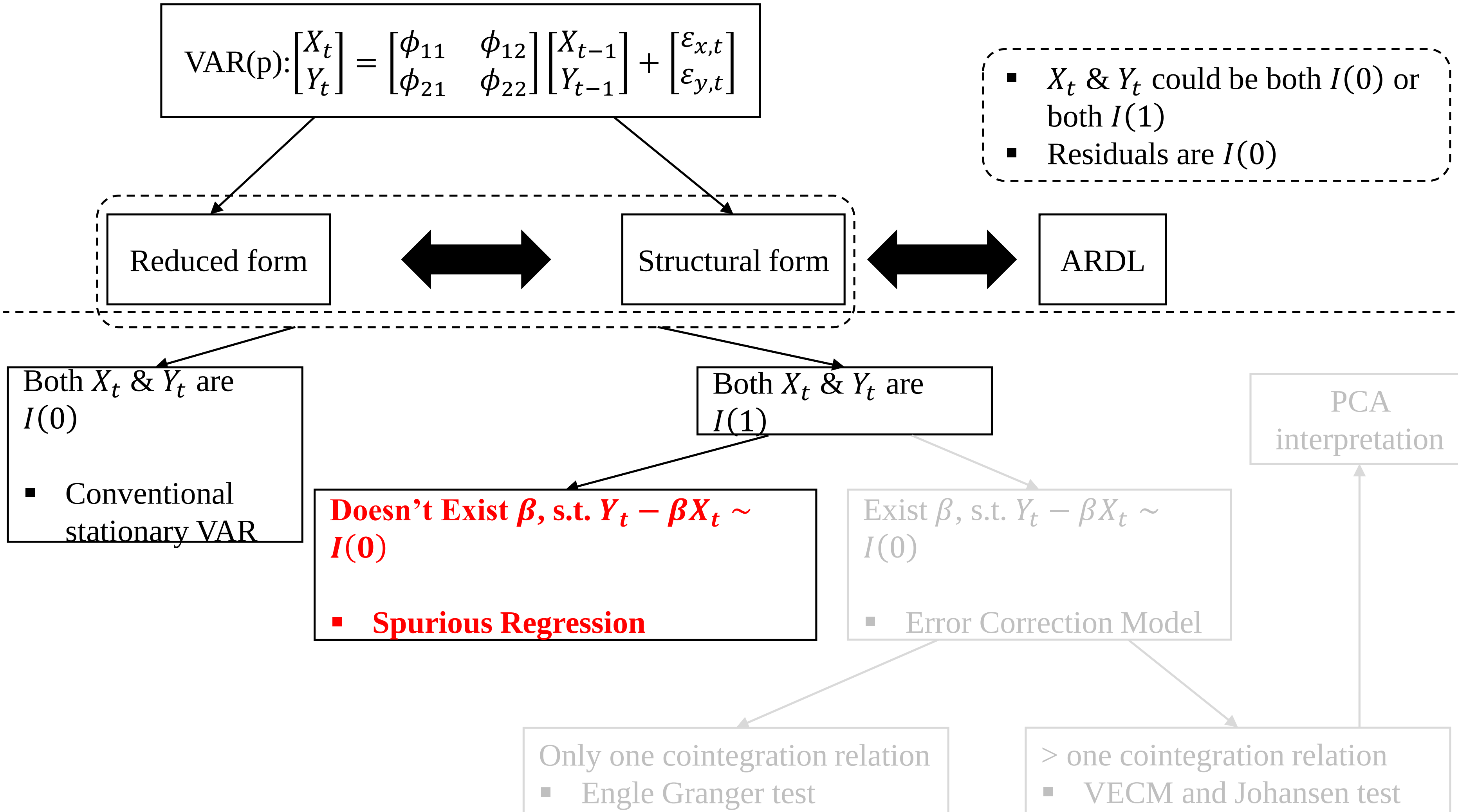
- Matlab:

```
[h,pvalue,stat,cValue,reg] =
egcitest(Y_new,'test','t2','creg','nc');
```

- R: `fit5 = egcm(x,y)`

- SAS:

```
proc AUTOREG; model y = x/ nlag=1
stationarity =(adf=1);
```



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Spurious Regression

- EG test indicates there's no β s.t. $Y_t - \beta X_t \sim I(0)$

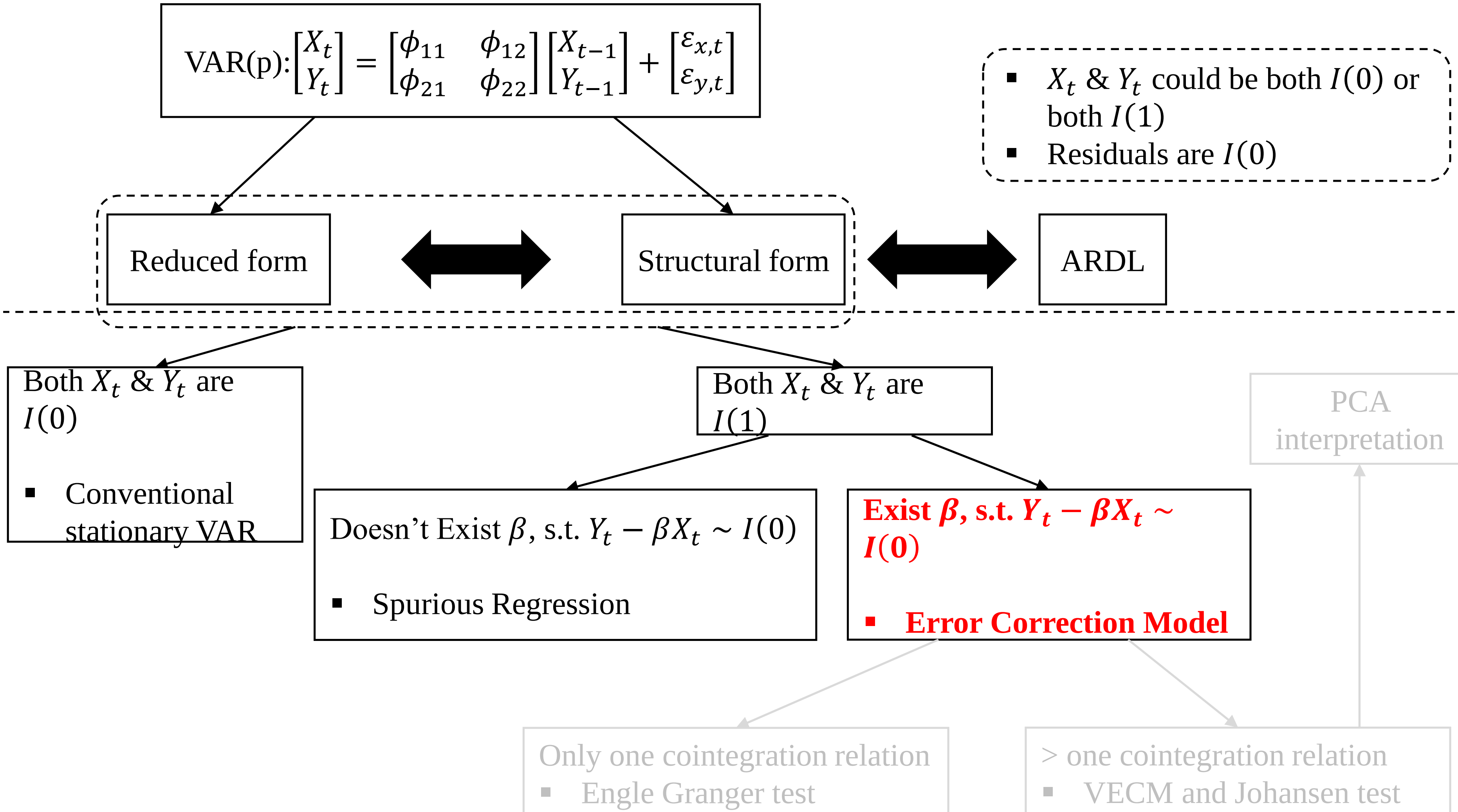
E.g. X_t: Monthly salary at Microsoft; Y_t: Population in an developing African city

- Two series are totally unrelated
- Increase in X_t & Y_t doesn't cause each other
- OLS of Y_t on X_t will generate high R^2 because $corr(X_t, Y_t)$ is high
- OLS is not reliable when spurious regression is present

Matlab:
```
ToEstMd = regARIMA(0,1,0);
ToEstMd.Intercept = 0;
EstMd = estimate(ToEstMd,Y(:,3),'X',Y(:,1),'Display','params');
```

R:
```
fit <- arima(z, c(0,1,0), xreg=x);
fit <- lm(diff(z) ~ diff(x));
```

SAS:
```
proc arima data=simul;
identify var=z(1) crosscorr=x(1);
estimate input=x;
run;
```



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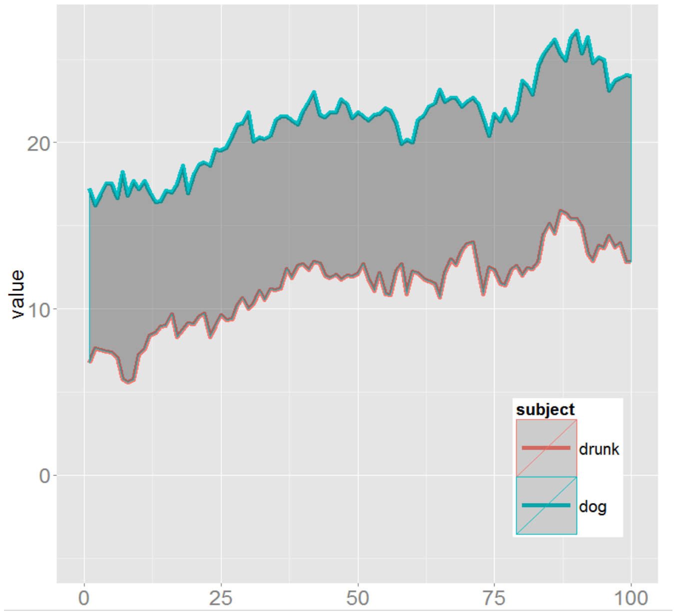
An example of cointegration

- **Drunk woman and a stray dog**

Drunk's position is a random walk along real line: $X_t = X_{t-1} + u_t$, where u_t is white noise
Her dog also wanders aimlessly as a random walk: $Y_t = Y_{t-1} + w_t$, where w_t is white noise

- **What if the dog belongs to the drunk?**
  - They wouldn't be far away from each other
  - Drunk's current position X_t is not only affected by her previous position X_{t-1}, but also affected her distance from her dog previously, i.e. $Y_{t-1} - X_{t-1}$
  - Dog's current position Y_t is not only affected by her previous position Y_{t-1}, but also affected her distance from her dog previously, i.e. $Y_{t-1} - X_{t-1}$



Error Correction Model (ECM)

- Reduced: $\begin{cases} X_t = \phi_{11} X_{t-1} + \phi_{12} Y_{t-1} + \varepsilon_{x,t} \\ Y_t = \phi_{21} X_{t-1} + \phi_{22} Y_{t-1} + \varepsilon_{y,t} \end{cases}$

- Structural: $\begin{cases} X_t = \phi_{11}^* X_{t-1} + \phi_{12}^* Y_{t-1} + \varepsilon_{x,t}^* \\ Y_t = c X_t + \phi_{21}^* X_{t-1} + \phi_{22}^* Y_{t-1} + \varepsilon_{y,t}^* \end{cases}$

- Substitute … $X_t + X_{t-1}$

- ∇X_t … $) + \varepsilon_{x,t}^*$; $\nabla Y_t = ($ … $X_{t-1}) + \varepsilon_{y,t}^*$

- If roots of $\left| I - \begin{bmatrix} \phi_{11} B & \phi_{12} B \\ \phi_{21} B & \phi_{22} B \end{bmatrix} \right| = 0$ are all outside of unit circle – stationary

- $\left| I - \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \right| = 0$, meaning $(1 - \phi_{11})(1 - \phi_{22}) = \phi_{12}\phi_{21}$

- From which it could be proven that $\frac{\phi_{11}-1}{\phi_{12}} = \frac{\phi_{21}}{\phi_{22}-1} = \frac{\phi_{11}^*-1}{\phi_{12}^*} = \frac{\phi_{21}^*}{\phi_{22}^*-1}$

EQ_{t-1} → ∇Y_t ← $\varepsilon_{y,t}$; $\phi_{22} - 1$

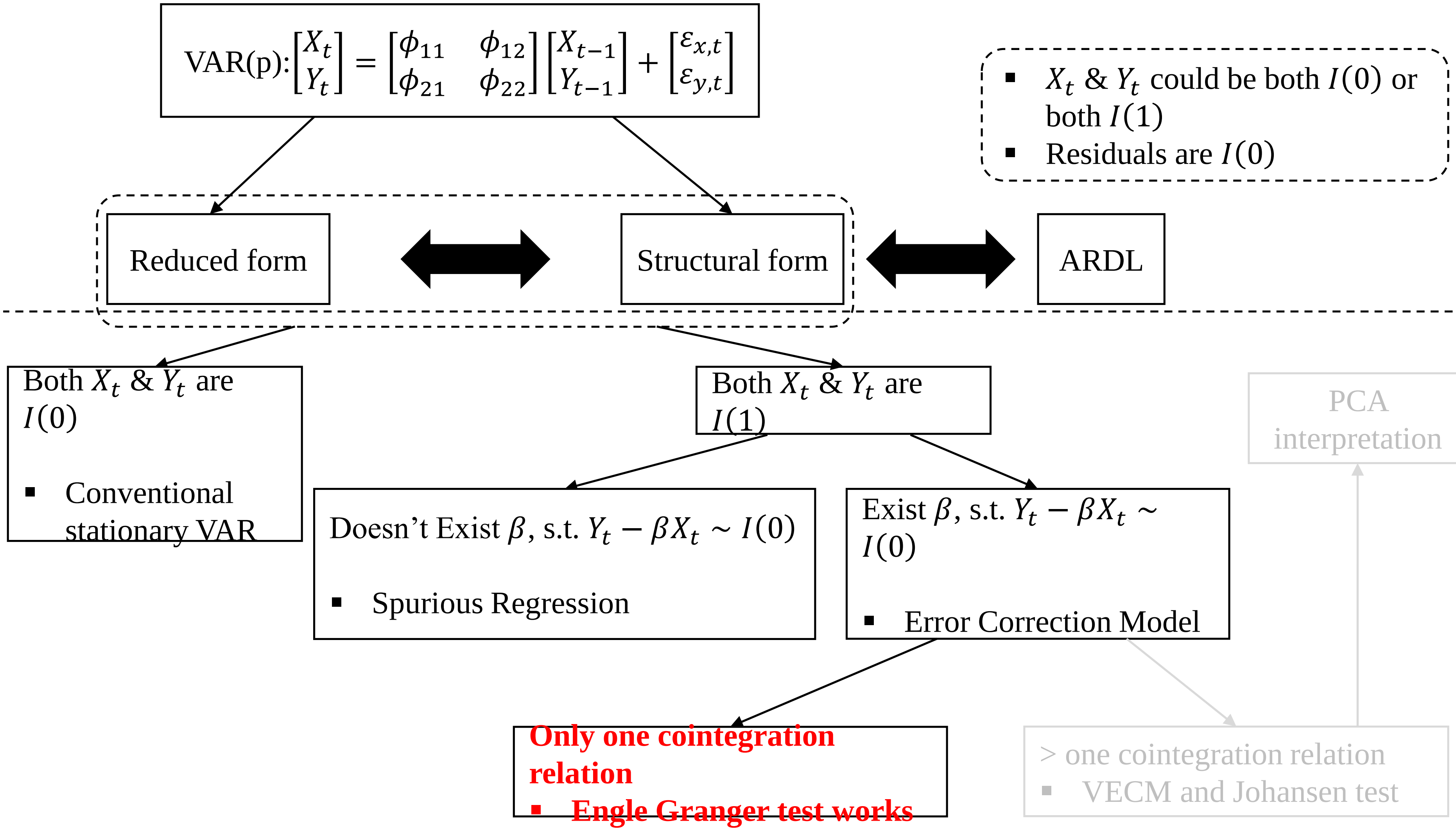
$\varepsilon_{x,t}^*$; EQ_{t-1} → ∇Y_t ← $\varepsilon_{y,t}^*$; c; $\phi_{22}^* - 1$

How to fit ECM model?

EG two-step fitting:

- Fit OLS: $Y_t = \hat{\alpha} + \hat{\beta} X_t + \hat{u}_t$
  (Since X_t & Y_t are cointegrated, $\hat{\alpha}$ & $\hat{\beta}$ are cointegrated)

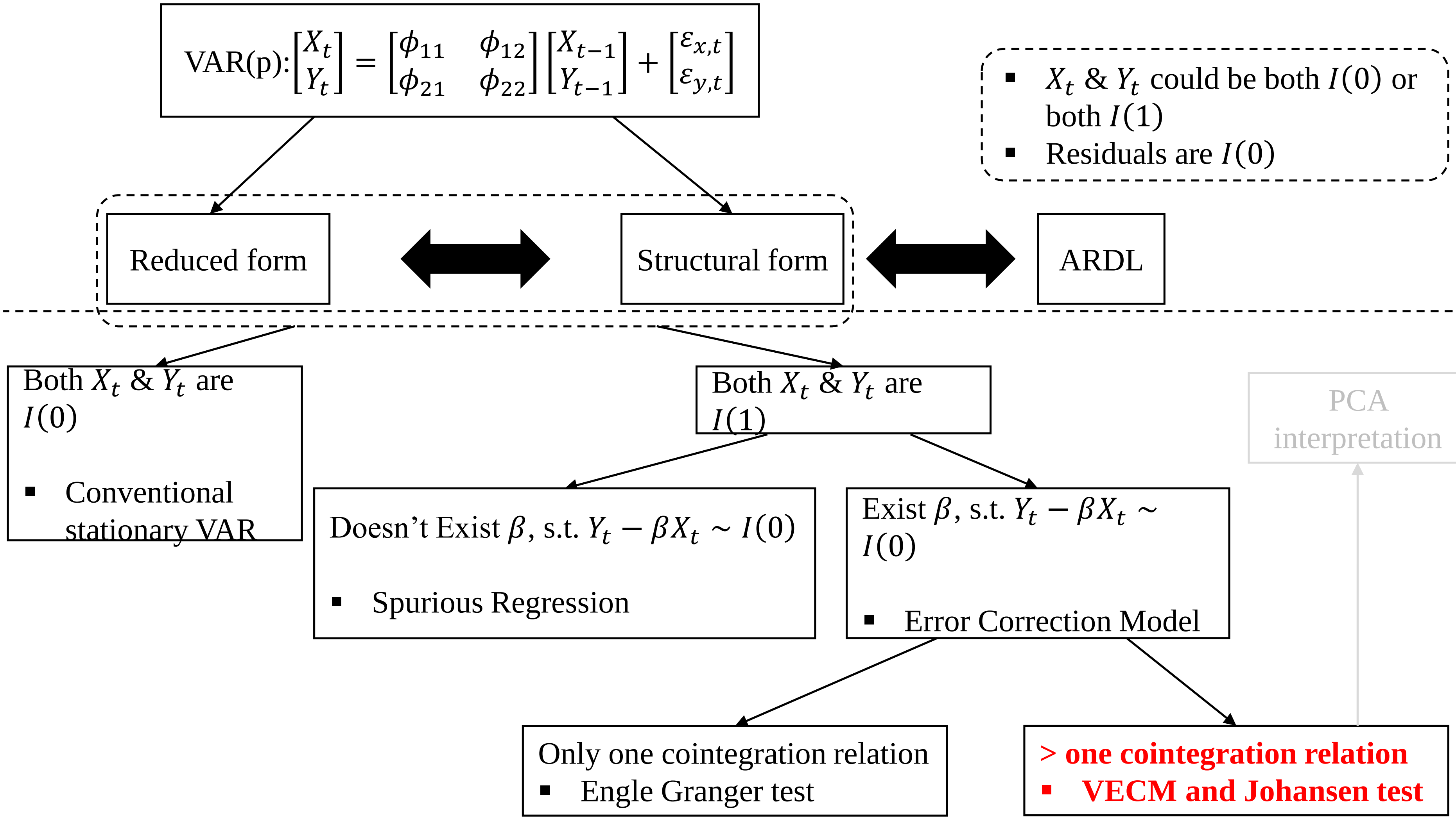
- Fit OLS: $\nabla Y_t = \hat{\theta} \nabla X_t + \hat{\phi}\left(Y_{t-1} - \hat{\alpha} - \hat{\beta} X_{t-1}\right) + \hat{\varepsilon}_t$



VAR(p): $\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$
X_t & Y_t could be both $I(0)$ or both $I(1)$
Residuals are $I(0)$
Reduced form
Structural form
ARDL
Both X_t & Y_t are $I(0)$
Conventional stationary VAR
Both X_t & Y_t are $I(1)$
PCA interpretation
Doesn't Exist β, s.t. $Y_t - \beta X_t \sim I(0)$
Spurious Regression
Exist β, s.t. $Y_t - \beta X_t \sim I(0)$
Error Correction Model
Only one cointegration relation
Engle Granger test works
> one cointegration relation
VECM and Johansen test

What if there're more than two variables?

- There're X_{1t}, X_{2t} and X_{3t}
  - If we're confident that there wouldn't be more than one cointegration relation among them, i.e. #cointegration = 0 or 1
  - Engle Granger test:
    Fit OLS: $X_{3t} = \hat{\beta}_0 + \hat{\beta}_1 X_{1t} + \hat{\beta}_2 X_{2t} + \hat{u}_t$
    ADF test on $\hat{u}_t$
  - If there're two cointegration relations: i.e. $X_{3t} - \beta_1 X_{1t} - \beta_2 X_{2t} \sim I(0)$ & $X_{3t} - \beta_1' X_{1t} - \beta_2' X_{2t} \sim I(0)$, it means $(\beta_1 - \beta_1')X_{1t} + (\beta_2 - \beta_2')X_{2t} \sim I(0)$
  - There is strong multicollinearity between X_{1t} & X_{2t}
  - Thus…


VAR(p): $\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{x,t} \\ \varepsilon_{y,t} \end{bmatrix}$
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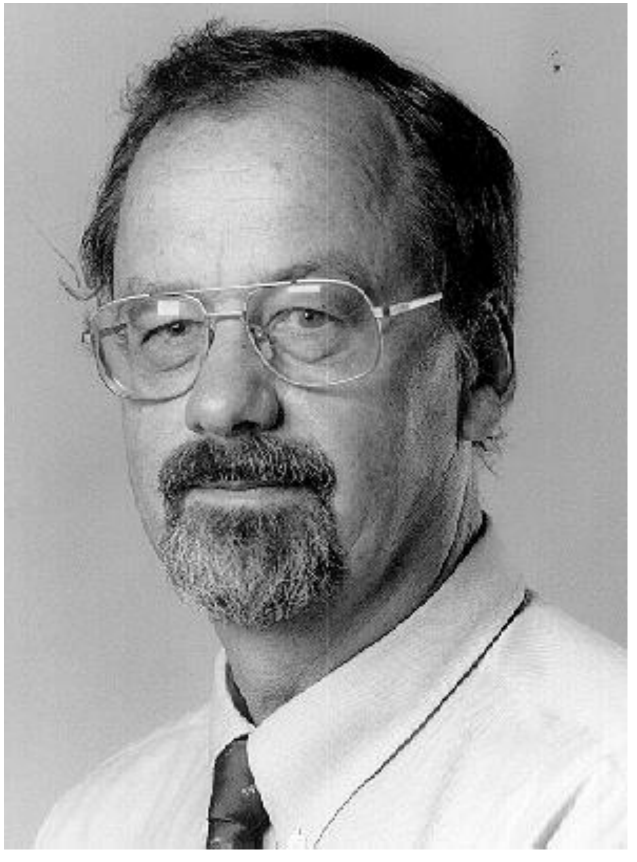
VECM for Cointegration Johansen Approach

Tingjun Ruan

Johansen Methodology (1988)

- The drawback of Engle Granger approach: it can only identify a ***single equilibrium*** relationship among the variables. If we have more than two variables in the model, then there is a possibility of having more than one cointegration relationships
- Johansen (1988) proposed a framework of estimating and testing of vector error correction model (VECM) *based on vector auto regressive (VAR)* equations with which we can find out how many cointegrating relationships exist among variables
- Johansen VECM is more general for testing ***multiple cointegrating relationships*** when there are more than two variables.
- For k variables, we can have up to k-1 cointegrations



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Danish Statistician and Econometrician, who is known for his research in time series analysis, in particular the theory of cointegration in collaboration with Katarina Juselius.

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Reference and sourse:

1. Multivariate Time Series Analysis: With R and Financial Applications by Ruey S. Tsay
2. Time Series Analysis by James Douglas Hamilton
3. The Analysis of Time Series: An Introduction with R Chapman & Hall/CRC Texts in Statistical Science)
4. Machine Learning for Time Series Forecasting with Python by Francesca Lazzeri
5. Time Series Analysis for the Social Sciences (Analytical Methods for Social Research) Part of: Analytical Methods for Social Research (14 Books)
6. Introduction to Probability, Statistics, and Random Processes by Hossein Pishro-Nik
7. Introduction to Time Series and Forecasting (Springer Texts in Statistics) Part of: Springer Texts in Statistics