Introduction to fuzzy set theory

Fuzzy set

In mathematics, **fuzzy sets** (also referred to sometimes as **uncertain sets**) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh and Dieter Klaua in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an *L*-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are now used throughout fuzzy mathematics and have applications in areas such as linguistics, decision-making and clustering, are special cases of *L*-relations when *L* is the unit interval [0, 1]. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions (aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called *crisp sets*. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

Fuzzy logic is being developed as a discipline to meet two objectives:

- As a professional subject for building systems of high utility - for example fuzzy control.
- As a theoretical subject - fuzzy logic is "symbolic logic with a comparative notion of truth developed fully in the spirit of classical logic. It is a branch of many-valued logic based on the paradigm of inference under vagueness."

What is Fuzzy Logic?

Fuzzy Logic is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than precise. Fuzzy logic is **not a vague logic system**, but a system of logic for dealing with vague concepts. As in fuzzy set theory the set membership values can range (inclusively) between 0 and 1, in fuzzy logic the degree of truth of a statement can range between 0 and 1 and is not constrained to the two truth values true/false as in classic predicate logic.

Example for Fuzzy Logic

Problem: A real estate owner wants to classify the houses he offers to his clients. One main indicator of comfort of these houses is the **number of bedrooms in them**. Let the available types of houses be represented by the following set.

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

The houses in this set are described by **U** number of bedrooms in a house. The realtor wants to describe a "comfortable house for a 4-person family," using a fuzzy set.

Solution: The fuzzy set "comfortable type of house for a 4-person family" may be described

using a fuzzy set in the following manner.

HouseForFour =FuzzySet [{{1, 0.2}, {2, .5}, {3, .8}, {4, 1}, {5, .7}, {6, .3}}, Universal Set—>{1,10}];

FuzzyPlot [HouseForFour, ShowDots –> True];

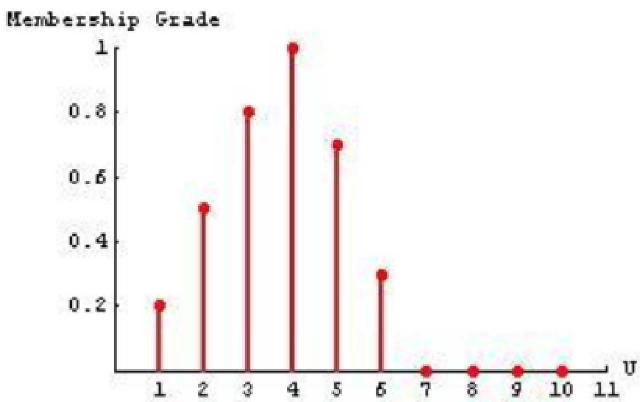

Fig 1. The Plot

Fuzzy System

A Fuzzy System can be contrasted with a conventional - crisp system in three main ways:

- A **Linguistic Variable** is defined as a variable whose values are sentences in a natural or artificial language. Thus, “if tall”, “not tall”, “very tall”, “very very tall”, etc. are values of height, then height is a linguistic variable.

- **Fuzzy Conditional Statements** are expressions of the form “If A THEN B”, where A and B have fuzzy meaning, e.g. “If x is small THEN y is large”, where small and large are viewed as labels of fuzzy sets.

- **A Fuzzy Algorithm** is an ordered sequence of instructions which may contain fuzzy assignment and conditional statements, e.g., “x = very small, IF x is small THEN y is large”. The execution of such instructions is governed by the compositional rule of inference and the rule of the preponderant alternative.

2.1 Fuzzy Sets

A Fuzzy set is a set whose elements have degrees of membership. Fuzzy sets are an extension of the classical notion of set (known as a Crisp Set). More mathematically, a fuzzy set is a pair (A, μA) where A is a set and $\mu A : A \to [0, 1]$. For all x A, μA(x) is called the grade of membership of x. If μA(x) = 1, we say that x is **Fully Included** in (A, μA), and if μA(x) = 0, we say that x is Not Included in (A, μA). If there exists some x A such that μA(x) = 1, we say that (A, μA) is **Normal**. Otherwise, we say that (A, μA) is **Subnormal**.

A fuzzy set is denoted as:

A = μA(x1)/x1 + · · · + μA(xn)/xn

that belongs to a finite universe of discourse: $A \subseteq \{x1, x2, \ldots, xn\} = X$

where μA(xi)/xi (a singleton) is a pair “grade of membership element”.

Simple Example:

Consider X = {1, 2, . . . , 10}.

Suppose a child is asked which of the numbers in X are “large” relative to the others. The child might come up with the following:

Number	Comment	Degree
10	Definitely	1
9	Definitely	1
8	Quite possible	0.8
7	May be	0.5
6	Not usually	0.2
5,4,3,2,1	Definitely Not	0

Fig 2. Possible solution given by the child

Definitions on Fuzzy Sets

Following are the definitions for two fuzzy sets (A, μA) and (B, μB), where A, B ⊆ X:

- **Equality:** A = B if μA(x) = μB(x) for all x ∈ X
- **Inclusion:** A ⊆ B iff μA(x) ≤ μB(x) for all x ∈ X
- **Cardinality:** |A| = Xn i=1 μA(xi)
- **Empty Set:** A is empty iff μA(x) = 0 for all x ∈ X.
- **α-Cut:** Given α ∈ [0, 1], the α-cut of A is defined by Aα = {x ∈ X | μA(x) ≥ α}

Operations on Fuzzy Sets

Let (A, μA),(B, μB) be a fuzzy sets.

- Complementation: (¬A, μ¬A), where μ¬A = 1 − μA
- Height: h(A) = max x∈X μA(x)
- Support: supp(A) = {x ∈ X | μA(x) > 0}

- Core: $C(A) = \{x \in X \mid \mu A(x) = 1\}$
- Intersection: $C = A \cap B$, where $\mu C = \min_{x \in X} \{\mu A, \mu B\}$
- Union: $C = A \cup B$, where $\mu C = \max_{x \in X} \{\mu A, \mu B\}$
- Bounded Sum: $C = A + B$, where $\mu C(x) = \min\{1, \mu A(x) + \mu B(x)\}$
- Bounded Difference: $C = A - B$, where $\mu C(x) = \max\{0, \mu A(x) - \mu B(x)\}$ 7

1. Exponentiation: $C = A\alpha$ where $\mu C = (\mu A)\ \alpha$ for $\alpha > 0$
2. Level Set: $C = \alpha A$ where $\mu C = \alpha \mu A$ for $\alpha \in [0, 1]$
3. Concentration: $C = A\alpha$ where $\alpha > 1$
4. Dilation: $C = A\alpha$ where $\alpha < 1$

Note that $A \cap \neg A$ is not necessarily the empty set, as would be the case with classical set theory. Also, if A is crisp, then $A\alpha = A$ for all α. We will define the Cartesian product $A \times B$ to be the same as $A \cap B$.

Membership Functions

A *membership function* (MF) is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) between 0 and 1. The input space is sometimes referred to as the *universe of discourse*, a fancy name for a simple concept.

The most commonly used MFs are

Triangles

Trapezoids

Bell Curves

Gaussian and

Sigmoidal

- **Fuzzy Rules**

Human beings make decisions based on rules. Although, we may not be aware of it, all the decisions we make are all based on computer like if-then statements. If the weather is fine, then we may decide to go out. If the forecast says the weather will be bad today, but fine tomorrow, then we make a decision not to go today, and postpone it till tomorrow. Rules associate ideas and relate one event to another.

Fuzzy machines, which always tend to mimic the behavior of man, work the same way. However, the decision and the means of choosing that decision are replaced by fuzzy sets and the rules are replaced by fuzzy rules. Fuzzy rules also operate using a series of if-then statements. For instance, if X then A, if y then b, where A and B are all sets of X and Y. Fuzzy rules define **fuzzy *patches***, which is the key idea in fuzzy logic. A machine is made smarter using a concept designed by Bart Kosko called the Fuzzy Approximation Theorem (FAT). The FAT theorem generally states a finite number of patches can cover a curve as seen in the figure below. If the patches are large, then the rules are sloppy. If the patches are small then the rules are fine.

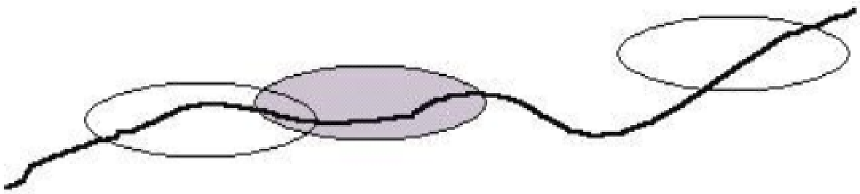

Fig 3. Fuzzy Patches

In a fuzzy system this simply means that all our rules can be seen as patches and the input and output of the machine can be associated together using these patches. Graphically, if the rule patches shrink, our fuzzy subset triangles get narrower. Simple enough? Yes, because even novices can build control systems that beat the best math models of control theory. Naturally, it is math-free system.

2.3 Fuzzy Reasoning

Single rule with single antecedent

Rule: if x is A then y is B

Fact: x is A'

Conclusion: y is B'

The i-th fuzzy rule from this rule-base

Ri : if x is Ai and y is Bi then z is Ci is implemented by a fuzzy relation Ri and is defined as

$$Ri(u, v, w) = (Ai \times Bi \rightarrow Ci)(u, w)$$

$$= [Ai(u) \times Bi(v)] \rightarrow Ci(w) \quad \text{for } i = 1,\ldots,n.$$

2.4 Fuzzy Inference (Expert) system

A fuzzy inference system (FIS) is a system that uses fuzzy set theory to map inputs (features in the case of fuzzy classification) to outputs (classes in the case of fuzzy classification).

Fuzzy inference systems have been successfully applied in fields such as automatic control, data classification, decision analysis, expert systems, and computer vision. Because of its multidisciplinary nature, fuzzy inference systems are associated with a number of names, such as fuzzy-rule-based systems, fuzzy expert systems, fuzzy modeling, fuzzy associative memory, fuzzy logic controllers, and simply (and ambiguously) fuzzy systems.

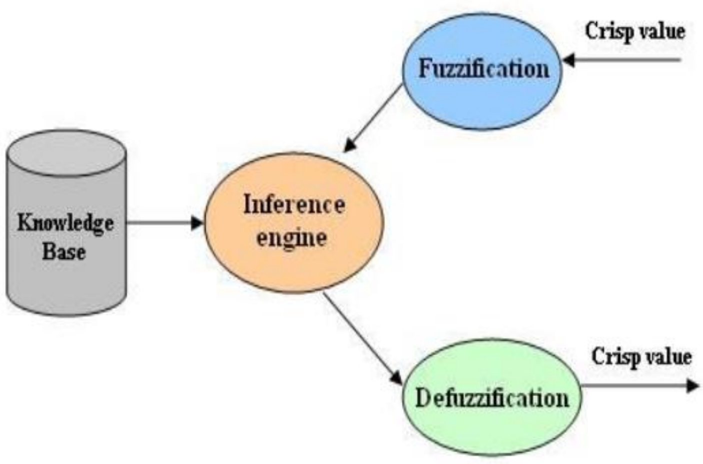

Fig 4. Structure of a Fuzzy Expert System

The rules in FIS (sometimes may be called as fuzzy expert system) are fuzzy production rules of the form:

- *if p then q,* where p and q are fuzzy statements.

For example, in a fuzzy rule

- if x is low and y is high then z is medium.
- Here x is low; y is high; z is medium are fuzzy statements; x and y are input variables; z is an output variable, low, high, and medium are fuzzy sets.

The antecedent describes to what degree the rule applies, while the conclusion assigns a fuzzy function to each of one or more output variables. Most tools for working with fuzzy expert systems allow more than one conclusion per rule.

The set of rules in a fuzzy expert system is known as ***knowledge base***.

The functional operations in fuzzy expert system proceed in the following steps.

- Fuzzification
- Fuzzy Inferencing (apply implication method)
- Aggregation of all outputs
- Defuzzification

References

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