# Diffusion problems arising in heat flow, chemical reactions, pattern formation, and thermal runaway. Conservation laws

# Introduction

## 1 Diffusion problems

Physical problems that can be described by a diffusion type equation are many and varied. These include heat flow in a solid, movement of populations across a surface, transport of a chemical species in a gas all of which have some underlying random walk phenomena. However, equations of this type can also arise from different physics such as spreading viscous films. In this section we will give a derivation of the underlying equation, give some of its properties, identify various methods of solution. We then explore behaviour of solutions to systems of such equations and to special nonlinear problems.

### 1.1 The heat equation and separation of variables

#### 1.1.1 Derivation of the equation

We start by considering the flow of heat through a solid material in which a chemical reaction is creating heat. Such a situation might arise in a large pile of sawdust where the heat is generated by slow oxidation of the wood. We will derive a simple model of this behaviour. We derive the underlying equation in two ways to illustrate possible approaches, one that considers an infinitesimal region of the sawdust pile and the other that considers an arbitrary region.

In the simplest case consider a single spatial dimension, $x$ (measured in $m$), and with time $t$ (measured in $s$). The temperature, which we seek to find, is then $T(x,t)$ (measured in $K$) and the flux of heat, which we also seek, is $q(x,t)$ (measured in $J/m^2/s$). We also presume that heat is generated by the chemical reaction that creates heat at a rate given by $f(x,t,T)$ (measured in $J/m^3/s$). The total heat per unit volume, or enthalpy, at any point in the pile is defined to be

$$H(T) = \int_0^T \rho c_p(T)\, dT$$

where $\rho$ (measured in $kg/m^3$) is the density of the sawdust and $c_p(T)$ (measured in $J/kg/K$ is the specific heat at constant pressure of the sawdust. Note that typically $\rho$ is nearly constant, and $c_p(T)$ is constant over large ranges of temperature but can varying dramatically near temperatures where there are changes in phase of a material.

To derive the governing equation we need to first consider conservation of heat in a very small region extending from $x$ to $x+\Delta x$. The total heat per crossectional area in this region is then

$$\int_x^{x+\Delta x} H(T)\; dx\ .$$

The balance equation is then derived by requiring that the change in this total heat can only be due either to flow of heat into the region from either boundary of the region or by generation of heat within the region. This gives us

$$\frac{\partial}{\partial t}\left(\int_x^{x+\Delta x} H(T)\; dx\right) = q(x+\Delta x,t) - q(x,t) + \int_x^{x+\Delta x} f(x,t,T)\; dx$$

Change in heat with time = heat in from left - heat out to right + heat generated

Using the definition of $H$ this gives us

$$\frac{\partial T}{\partial t}\left(\int_x^{x+\Delta x} \rho c_p(T(x,t))\; dx\right) = q(x,t) - q(x+\Delta x,t) + \int_x^{x+\Delta x} f(x,t,T)\; dx$$

To complete the derivation we now take the limit $\Delta x \rightarrow 0$. In doing this we make the assumption that both $c_p(T(x,t))$ and $f(x,t)$ are continuous so that

$$\frac{\partial T}{\partial t}\Delta x \rho c_p(T(x,t)) = q(x,t) - q(x+\Delta x,t) + \Delta x f(x,t,T)$$

and then assume that $q(x,t)$ is differentiable so that

$$\rho c_p(T(x,t))\frac{\partial T}{\partial t} = -\frac{\partial q(x,t)}{\partial x} + f(x,t,T)$$

This is the equation of conservation of heat and is central to all such modelling. Note the assumptions of continuity and differentiability are not trivial aspects and there are many common situations where these may not be valid (eg models of melting ice) and great care must then be taken.

To complete the model of the temperature movement within the sawdust pile we need to add an equation that determines $q(x,t)$. To make the model simple we first note that the sawdust is not moving and also will use an empirical law that states that heat travels from hot regions to cold regions. Specifically we write

$$q(x,t) = -k\frac{\partial T}{\partial x}$$

where $k$ (measured in $J/s/m/K$) is the thermal diffusivity of the sawdust. This empirical law is called "Fick's law" and there are similar laws for other physical situations. In practice $k$ may depend on: $x$ due to inhomogeneities in the sawdust pile; $t$ due to changes in the pile structure over time; and $T$ due changes in transport mechanisms such as radiation at high temperatures. Hence we have that the temperature within the sawdust pile is governed by

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + f(x,t,T)$$

In the simplest case where the density, the specific heat and the diffusivity can adequately be assumed to be constant and no heat is generated this gives the classical diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

where $\kappa = k/\rho/c_p$ (measured in $m^2/s$) is the thermal diffusivity.

The mathematical problem is finally completed by specifying the region in which the problem is to be solved, imposing conditions at the boundary of this region which describe the interaction of the system with the outside world and what the temperature in the region is initially. Hence for example the sawdust pile may be taken to extend from $x - 0$ to $x = L$ with both of these surfaces held at a constant, due to surrounding air, with $T(0,t) = T_a$ and $T(L,t) = T_a$. In addition we will assume that the sawdust pile was put in place at some time $t - 0$ and that it was at a spatially uniform temperature then $T(x,0) = T_0$.

An alternative derivation of the diffusion equation is to consider an arbitrary fixed region $V$ of the sawdust pile where the outer surface of this region is denoted by $\partial V$. We consider the three dimension problem with the spatial variable $\mathbf{x}$, and introduce $\mathbf{q}(\mathbf{x},\mathbf{t})$ as a vector quantity describing the flux of heat. Conservation of heat then requires

$$\frac{\partial}{\partial t} \iiint_V H(T)\, dV = \iint_{\partial V} -\mathbf{q} \cdot d\mathbf{S} + \iiint_V f(\mathbf{x},t,T)\, dV$$

where $d\mathbf{S}$ is a surface area vector pointing out of $V$. Using the previous definition of $H(T)$ and making assumptions about continuity of $H$ and $f$ and differentiability of $\mathbf{q}$ we can now exploit the divergence theorem to find that

$$\iiint_V \rho c_p \frac{\partial T}{\partial t}\, dV = \iiint_V -\nabla \cdot \mathbf{q}\, dV + \iiint_V f(\mathbf{x},t,T)\, dV\ .$$

Since this holds for all possible $V$ it follows that

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + f(\mathbf{x},t,T)\ .$$

To this we then add Fick's law that

$$\mathbf{q} = -k \nabla T$$

and this gives the three dimensional heat diffusion equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + f(\mathbf{x}, t, T) \; .$$

A condition then needs to be given at all points of the outer surface of the sawdust pile to describe the interaction with the outside world and a condition to describe the temperature distribution at some initial time.

### 1.1.2 Chemical reactions

When modelling chemical reactions it is common to consider the various species to be dilute in a fluid or gas. A good example of such a dilute system is the movement and reaction of oxygen and a fuel in air. Here these species are in relatively small concentrations compared to nitrogen which is dominant and in this case nonreacting. For dilute gases the species can be taken to have concentration $c_i(\mathbf{x}, t)$ for various $i$ and a simple model of their movement is to take the underlying dominant gas to move with velocity $\mathbf{v}$ and for the species to diffusive relative to this gas. Hence the flux of each species is given by

$$\mathbf{q}_i = \mathbf{v} c_i + D_i \nabla c_i$$

where $D_i$ is the diffusivity of that species in the dominant background gas. If there are no reactions the governing equation for each species comes from considering conservation of that species and is

$$\frac{\partial c_i}{\partial t} + \nabla \cdot (c_i \mathbf{v}) = \nabla \cdot (D_i \nabla c_i) \; .$$

If there are reactions in the bulk of the fluid, as opposed to reactions at a surface, then these can be included by a source term. If the reaction is

$$c_1 + c_2 \leftrightharpoons c_3$$

then typically the reaction rate depends on the local concentration of the species and the local temperature. For a simple reaction this might be modelled using a simple probability argument so that the forward reaction would depend linearly on concentrations $c_1$ and on $c_2$, while the reverse reaction would be proportional to the concentration of $c_3$. For more complex reaction, where there are intermediate species that are not modelled, the effective reaction rates are usually made proportional to some power of the various species. The reaction rate is typically highly dependent on the temperature and a common model is to use a Arrhenius form for this dependence. The reactions may also release or absorb heat as they progress and hence may alter the temperature. Hence a model of the reaction above might take the form

$$\frac{\partial c_1}{\partial t}+\nabla\cdot c_1\mathbf{v}=\nabla\cdot(D_1\nabla c_1)-k_1c_1c_2e^{-E_1/(RT)}+k_2c_3e^{-E_2/(RT)}\ .$$

$$\frac{\partial c_2}{\partial t}+\nabla\cdot c_2\mathbf{v}=\nabla\cdot(D_2\nabla c_2)-2k_1c_1c_2e^{-E_1/(RT)}+2k_2c_3e^{-E_2/(RT)}$$

$$\frac{\partial c_3}{\partial t}+\nabla\cdot c_3\mathbf{v}=\nabla\cdot(D_3\nabla c_3)+k_1c_1c_2e^{-E_1/(RT)}-k_2c_3e^{-E_2/(RT)}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+\nabla\cdot T\mathbf{v}\right)=\nabla\cdot(k\nabla T)+\Delta H_1\left(k_1c_1c_2e^{-E_1/(RT)}+k_2c_3e^{-E_2/(RT)}\right)\ .$$

where the constants $E_1$ and $E_2$ are the activation energies of the forward and reverse reactions, $k_1$ and $k_2$ are prefactors of the reaction rates and, $\Delta H_1$ is the heat of the reaction. Such a model requires initial data for all the species and the temperature and one boundary condition on each boundary for each of the dependent variables.

Some common approximations to simplify this model are:

1) No flow of the dominant gas ($\mathbf{v}\approx 0$), no heat of reaction ($\Delta H_1\approx 0$), and all chemical species diffuse at the same rate ($D_1=D_2=D_3$, the "Lewis number" is unity). Here there is no need to solve for the temperature and the equation for the third chemical species can be replaced by an equations for the variable $C=c_1+c_3$ (or some other appropriate combination of species) which is given by

$$\frac{\partial C}{\partial t}=\nabla\cdot(D_3\nabla C)$$

(note the lack of a source term). This can then be solved and subsequently the other species solved for. For the simplest case where the boundary conditions are such that $C$ is a constant the nondimensional problem is then

$$\frac{\partial c_1}{\partial t}=\nabla^2c_1-\alpha c_1c_2e^{E_1/(RT)}+\beta(1-c_1)$$

$$\frac{\partial c_2}{\partial t}=\nabla^2c_2-2\alpha c_1c_2e^{E_1/(RT)}+2\beta(1-c_1)$$

which is pair of diffusion equations coupled by a nonlinear source term (in this case it may also be possible to further simplify the problem by considering the variable $(c_1-c_2)$ and, if the boundary conditions for this quantity are constant, then the entire dynamics is governed by a single diffusion equation. Similar problems appear in many situations such as population dynamics.

2) No flow of the dominant gas ($\mathbf{v} \approx 0$), no reverse reaction ($k_2 = 0$) large concentrations of the two reacting species so we can take $c_1$, $c_2$ both as constants. The problem is then simply one of finding the temperature which is governed by the single equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \Delta H_1 k_1 c_1 c_2 e^{-E_1/(RT)} \,.$$

The nondimensional version of this equation is:

$$\frac{\partial T}{\partial t} = \nabla^2 T + \nu e^{-\gamma/T} \,.$$

This nonlinear diffusion equation is commonly approximated because the main issue is about how the temperature moves from its initial condition (such as modelling ignition of a reaction). Typically $\gamma$ is a relatively large number and we take the initial temperature to be $T = 1$. we scale the problem with $T = 1 + \hat{T}/\gamma$ so that

$$\frac{\partial \hat{T}}{\partial t} = \nabla^2 \hat{T} + \gamma \nu e^{-\gamma/(1+\hat{T}/\gamma)} \,.$$

We now expand the function in the exponential, in the limit $\gamma \to \infty$, assuming that $\hat{T} = \mathcal{O}(1)$ so that

$$\frac{\gamma}{(1 + \hat{T}/\gamma)} \sim \gamma(1 - \hat{T}/\gamma + (\hat{T}/\gamma)^2 + \ldots) \sim \gamma - \hat{T} + \ldots$$

The approximate equation is therefore

$$\frac{\partial \hat{T}}{\partial t} = \nabla^2 \hat{T} + \lambda \nu e^{\hat{T}}$$

where $\lambda = \nu\gamma \exp(\gamma)$. This simple model is used to explore ignition phenomena and shows that the temperature dependence of the reaction rate can be approximated, at least in the early stages of a reaction, by a simple exponential.

## Conservation Laws in Continuum Modeling.

# 1 Introduction.

In formulating a mathematical model for a continuum physical system, there are three basic steps that are often used:

**A.** Identify appropriate conservation laws (e.g. mass, momentum, energy, etc) and their corresponding densities and fluxes.

**B.** Write the corresponding equations using conservation.

**C.** Close the system of equations by proposing appropriate relationships between the fluxes and the densities.

Of these steps, the mathematical one is the second. While it involves some subtlety, once you understand it, its application is fairly mechanical. The first and third steps involve physical issues, and (generally) the third one is the hardest one, where all the main difficulties appear in developing a new model. In what follows we will go through these steps, using some practical examples to illustrate the ideas.

Of course, once a model is formulated, a **fourth step** arises, which is that of analyzing and validating the model, comparing its predictions with observations ... and correcting it whenever needed. This involves simultaneous mathematical and physical thinking. You should never forget that a model is no better than the approximations (explicit and/or implicit) made when deriving it. It is never a question of just "solving" the equations, forgetting what is behind them.

# 2 Continuum Approximation; Densities and Fluxes.

The modeling of physical variables as if they were a continuum field is almost always an approximation. For example, for a gas one often talks about the density $\rho$, or the flow velocity $\mathbf{u}$, and thinks of them as functions of space and time: $\rho = \rho(\mathbf{x}, t)$ or $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. But the fact is that a gas is made up by very many discrete molecules, and the concepts of density, or flow velocity, only make sense as local averages. These averages must be made over scales large enough that the discreteness of the gas becomes irrelevant, but small enough that the notion of these local averages varying in space and time makes sense.

Thus, **in any continuum modeling there are several scales.** On the one hand one has the **"visible" scales,** which are the ones over which the mathematical variables in the model vary (densities, fluxes). On the other hand, there are the **"invisible" scales,** that pertain to the micro-scales that have been averaged in obtaining the model. **The second set of scales must be much smaller than the first set for the model to be valid.** Unfortunately, this is not always the case, and whenever this fails all sort of very interesting (and largely open) problems in modern science and engineering arise.

Note that the reason people insist on trying to use continuum type models, even in situations where one runs into the difficulties mentioned at the end of the last paragraph, is that continuum models are often much simpler (both mathematically and computationally) than anything else, and supply general understanding that is often very valuable.

The first step in the modeling process is to identify conserved quantities (e.g. mass) and define the appropriate densities and fluxes — as in the following examples.

## 2.1 Examples

**Example 2.1 River Flow** (a one dimensional example).

*Consider a nice river (or a channel) flowing down a plain (e.g. the Mississippi, the Nile, etc.). Let* $x$ *be the length coordinate along the river, and at every point (and time) along the river let* $A = A(x, t)$ *be the filled (by water) cross-section of the river bed.*

*We note now that* $A$ *is the* **volume density** *(volume per unit length) of water along the river. We also note that, since water is incompressible,* **volume is conserved**.[1] *Finally, let* $Q = Q(x, t)$ *be the* **volume flux** *of water down the river (i.e.: volume per unit time). Notice that, if* $u = u(x, t)$ *is the average* **flow velocity** *down the river, then* $\boxed{Q = uA}$ *(by* definition *of* $u$*).*

*Thus, in this case, an appropriate conservation law is the* **conservation of volume**, *with corresponding density* $A$ *and flux* $Q$*. We note that both* $A$ *and* $Q$ *are regularly measured at various points along important rivers.*

**Example 2.2 Traffic Flow** (a one dimensional example).

*Consider a one lane road, in a situation where there are no cross-roads (e.g.: a tunnel, such as the Lincoln tunnel in NYC, or the Summer tunnel in Boston). Let* $x$ *be length along the road. Under "heavy" traffic conditions,*[2] *we can introduce the notions of* **traffic density** $\rho = \rho(x, t)$ *(cars per unit length) and* **traffic flow** $q = q(x, t)$ *(cars per unit time). Again, we have* $\boxed{q = u\rho,}$ *where* $u$ *is the average* **car flow velocity** *down the road.*

*In this case, the appropriate conservation law is, obviously, the* **conservation of cars.** *Notice that this is one example where the continuum approximation is rather borderline (since, for example, the local averaging distances are almost never much larger than a few car separation lengths). Nevertheless, as we will see, one can gain some very interesting insights from the model we will develop (and some useful practical facts).*

**Example 2.3 Heat Conductivity.**

*Consider the thermal energy in a chunk of solid material (such as, say, a piece of copper). Then the* **thermal energy density** *(thermal energy per unit volume) is given by* $e = c\,\rho\,T(\mathbf{x}, t)$*, where* $T$ *is the temperature,* $c$ *is the specific heat per unit mass, and* $\rho$ *is the density of the material (for simplicity we will assume here that both* $c$ *and* $\rho$ *are constants). The* **thermal energy flow**, $\mathbf{Q} = \mathbf{Q}(\mathbf{x}, t)$ *is now a vector, whose magnitude gives the energy flow across a unit area normal to the flow direction.*

*In this case, assuming that heat is not being lost or gained from other energy forms, the relevant conservation law is the* **conservation of heat energy.**

**Example 2.4 Steady State (dry) Granular Flow.**

*Consider steady state (dry) granular flow down some container (e.g. a silo, containing some dry granular material, with a hole at the bottom). At every point we characterize the flow in terms of two velocities: an* **horizontal (vector) velocity** $\mathbf{u} = \mathbf{u}(x, y, z, t)$*, and a* **vertical (scalar) velocity** $v = v(x, y, z, t)$*, where* $x$ *and* $y$ *are the horizontal length coordinates, and* $z$ *is the vertical one.*

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[1] We are neglecting here such things as evaporation, seepage into the ground, etc. This cannot always be done.

[2] Why must we assume "heavy" traffic?

*The* **mass flow rate** *is then given by* $\mathbf{Q} = \rho\,[\mathbf{u}, v]$*, where* $\rho$ *is the* **mass density** *— which we will assume is nearly constant. The relevant conservation is now the* **conservation of mass.**

*This example is different from the others in that we are looking at a steady state situation. We also note that this is another example where the continuum approximation is quite often "borderline", since the scale separation between the grain scales and the flow scales is not that great.*

# 3 Conservation Laws in Mathematical Form.

In this section we assume that we have identified some conservation law, with conserved density $\rho = \rho(\mathbf{x}, t)$, and flux $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$, and derive mathematical formulations for the conservation hypothesis. In other words, we will just state in mathematical terms the fact that $\rho$ is the density for a conserved quantity, with flux $\mathbf{F}$.

**First consider the one dimensional case** (where the flux $F$ is a scalar, and there is only one space coordinate: $x$). In this case, consider some (fixed) arbitrary interval in the line $\Omega = \{a \leq x \leq b\}$, and let us look at the evolution in time of the conserved quantity inside this interval. At any given time, the total amount of conserved stuff in $\Omega$ is given by (this by definition of density)

$$M(t) = \int_a^b \rho(x, t)\, dx\,. \tag{3.1}$$

Further, the net rate at which the conserved quantity enters $\Omega$ is given by (definition of flux)

$$R(t) = F(a, t) - F(b, t)\,. \tag{3.2}$$

It is also possible to have **sources and sinks** for the conserved quantity.[5] In this case let $\boxed{s = s(x, t)}$ be the total net amount of the conserved quantity, per unit time and unit length, provided by the sources and sinks. For the interval $\Omega$ we have then a net rate of added conserved stuff, per unit time, given by

$$S(t) = \int_a^b s(x, t)\, dx\,. \tag{3.3}$$

The conservation law can now be stated in the mathematical form

$$\frac{d}{dt} M = R + S\,, \tag{3.4}$$

which **must apply for any choice of interval** $\Omega$**.** Since this equation involves only integrals of the relevant densities and fluxes, it is known as the **Integral Form of the Conservation Law.**

**Assume now that the densities and fluxes are nice enough to have nice derivatives.** Then we can write:

$$\frac{d}{dt} M = \int_a^b \frac{\partial}{\partial t} \rho(x, t)\, dx \quad \text{and} \quad R = -\int_a^b \frac{\partial}{\partial x} F(x, t)\, dx\,. \tag{3.5}$$

Equation (3.4) can then be re-written in the form

$$\int_a^b \left( \frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} F(x, t) - s(x, t) \right) dx = 0\,, \tag{3.6}$$

which must apply for any choice of the interval $\Omega$. It follows that the integrand above in (3.6) must vanish identically. This then yields the following partial differential equation involving the density, flux and source terms:

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} F(x, t) = s(x, t)\,. \tag{3.7}$$

This equation is known as the **Differential Form of the Conservation Law.**

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[5] As an illustration, in the inviscid fluid flow case of example 2.5, the effects of gravity translate into a vertical source of momentum, of strength $\rho g$ per unit volume — where $g$ is the acceleration of gravity. Other body forces have similar effects.

**Remark 3.1** *You may wonder why we even bother to give a name to the form of the equations in (3.4), since the differential form in (3.7) appears so much more convenient to deal with (it is just one equation, not an equation for every possible choice of $\Omega$). The reason is that it is not always possible to assume that the densities and fluxes have nice derivatives. Oftentimes the physical systems involved develop, as they evolve,*[6] *short enough scales that force the introduction of discontinuities into the densities and fluxes — and then (3.7) no longer applies, but (3.4) still does.* **Shock waves** *are the best known example of this situation. Examples of shock waves you may be familiar with are: the sonic boom produced by a supersonic aircraft; the hydraulic jump occurring near the bottom of the discharge ramp in a large dam; the wave-front associated with a flood moving down a river; the backward facing front of a traffic jam; etc. Some shock waves can cause quite spectacular effects, such as those produced by supernova explosions.*

**Now let us consider the multi-dimensional case**, when the flux $\mathbf{F}$ is a vector. In this case, consider some (fixed but arbitrary) region in space $\Omega$, with boundary $\partial\Omega$, and inside unit normal along the boundary $\hat{\mathbf{n}}$. We will now look at the evolution in time of the conserved quantity inside this region. At any given time, the total amount of conserved stuff in $\Omega$ is given by

$$M(t) = \int_\Omega \rho(\mathbf{x}, t)\, dV\,. \tag{3.8}$$

On the other hand, the net rate at which the conserved quantity enters $\Omega$ is given by

$$R(t) = \int_{\partial\Omega} \mathbf{F}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}\, dS\,. \tag{3.9}$$

Let also $\boxed{s = s(\mathbf{x}, t)}$ be the total net amount of conserved quantity, per unit time and unit volume, provided by any sources and/or sinks. For the region $\Omega$ we have then a net rate of added conserved stuff, per unit time, given by

$$S(t) = \int_\Omega s(\mathbf{x}, t)\, dV\,. \tag{3.10}$$

The conservation law can now be stated in the mathematical form (compare with equation (3.4)) — **Integral Form of the Conservation Law:**

$$\frac{d}{dt} M = R + S\,, \tag{3.11}$$

which **must apply for any choice of the region $\Omega$.**

**If the densities and fluxes are nice enough to have nice derivatives,** we can write:

$$\frac{d}{dt} M = \int_\Omega \frac{\partial}{\partial t}\rho(\mathbf{x}, t)\, dV \quad \text{and} \quad R = -\int_\Omega \mathsf{div}(\mathbf{F}(\mathbf{x}, t))\, dV\,, \tag{3.12}$$

where we have used the Gauss divergence theorem for the second integral. Equation (3.11) can then be re-written in the form

$$\int_\Omega \left( \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \mathsf{div}(\mathbf{F}(\mathbf{x}, t)) - s(\mathbf{x}, t) \right) dV = 0\,, \tag{3.13}$$

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[6] Even when starting with very nice initial conditions.

which must apply for any choice of the region $\Omega$. It follows that the integrand above in (3.13) must vanish identically. This then yields the following partial differential equation involving the density, flux and source terms (compare with equation (3.7))

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \mathsf{div}(\mathbf{F}(\mathbf{x},t)) = s(\mathbf{x},t)\,. \tag{3.14}$$

This equation is known as the **Differential Form of the Conservation Law.**

**Remark 3.2** *In the case of a* **vector conservation law***, the density $\rho$ and the source term $s$ will both be vectors, while the flux* $\mathbf{F}$ *will be a rank two tensor (each row being the flux for the corresponding element in the density vector $\rho$). In this case equation (3.14) is valid component by component, but can be given a vector meaning if we define the divergence for a rank two tensor* $\mathbf{F} = [F_{ij}]$ *as follows:*

$$\mathsf{div}(\mathbf{F}) = \left[\sum_j \frac{\partial}{\partial x_j} F_{ij}\right] ,$$

*so that* $\mathsf{div}(\mathbf{F})$ *is a vector (each element corresponding to a row in* $\mathbf{F}$*). You should check that this is correct.*[7]

References and further readings

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