Well-posedness; separation of variables; transforms; similarity solutions; nonlinear equilibria; linear stability;

Separation of variables

We will consider a method for solving the diffusion equation that can be used in simple cases. Specifically we will look at the case of no heat generation, where ρ, c_p and k are constants, the boundary conditions are linear, and the geometry of the region is particularly simple. The method can be extended in some cases and these are left as exercises.

The main idea of separation of variables is to seek a solution that is a linear sum of functions of a separated form. Hence if we look for solutions $u(x,t)$ to the PDE

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

we seek them in the form of a sum of solutions of the type

$$u(x,t) = X(x)T(t). \tag{1.1}$$

Furthermore we need the region for the solution to be bounded by surfaces each defined by one of the independent variables being constant (for Cartesian coordinates this requires regions with boundaries that are stationary eg ($x = a$, $x = b$). In higher dimensions Cartesian coordinates can be used for rectangular region while different coordinate systems eg cylindrical or spherical regions can be used for rods and balls). Note that there are problems where other special geometries allow separation of variables (such as a wedge when studying small water waves) but these are not encountered very often. Finally the boundary conditions must be linear and homogeneous (if there are forcing terms at the boundaries then these need to be removed by finding a suitable particular integral for the problem. Typically this is done by seeking a steady state solution, independent of t, that satisfies the boundary conditions). For the case here we take

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0$$

with initial conditions

$$u(x,t) = g(x)\ .$$

Putting the assumed form (1.1) for the solution into the equation gives us

$$X\frac{dT}{dt} = \kappa \frac{d^2X}{dx^2}T$$

and we can then separate this equation to find that

$$\frac{1}{\kappa}\frac{dT}{dt}\frac{1}{T} = \frac{d^2X}{dx^2}\frac{1}{X}$$

and since each side of this equation is a function of only one independent variable the two sides much be constant. Hence we write

$$\frac{1}{\kappa}\frac{dT}{dt}\frac{1}{T} = \frac{d^2X}{dx^2}\frac{1}{X} = \nu\ .$$

where ν is any constant. This gives us two ODE's for the functions $T(t)$ and $X(x)$ which in this simple case are

$$\frac{dT}{dt} - \nu\kappa T = 0 \qquad \text{and} \qquad \frac{d^2X}{dx^2} - \nu X = 0\,. \tag{1.2}$$

For the boundary conditions to hold we require

$$X(0)T(t) = 0 \qquad \text{and } X(L)T(t) = 0$$

and since we need $T(t) \not\equiv 0$ in order to avoid the trivial solution $u \equiv 0$ we require

$$X(0) = 0 \qquad \text{and } X(L) = 0.$$

This then creates an eigenvalue of problem for $X(x)$ where there are a set of eigenvalues ν_j, $j = 1, 2, \ldots$ for which there exists eigensolutions $X_j(x)$. In this case we find

$$X_j(x) = \sin\left(x\sqrt{-\nu_j}\right) \qquad \text{where} \qquad \nu_j = -(j\pi/L)^2 \qquad j = 1, 2, 3, \ldots$$

Solving (1.2) for $T(t)$ we find that we can write

$$u(x,t) = \sum_{j=1}^{\infty} A_j \sin\left(2j\pi x/L\right) \exp(-(j\pi/L)^2\kappa t)$$

where the A_j are constants still to be found. The final step is to satisfy the initial condition and here we find this requires

$$g(x) = \sum_{j=1}^{\infty} A_j \sin\left(2j\pi x/L\right)$$

Because of orthogonality conditions satisfied by trigonometric functions this can be solved using the usual Fourier series methods to find that the coefficients are given by

$$A_n = \frac{2}{L}\int_0^L g(x)\sin\left(n\pi x/L\right)\,dx$$

The method outlined above can be used in other coordinate systems, for different linear boundary conditions and in each case there is a linear ODE eigenvalue problem to be solved that may require some numerical approaches in order to find solutions.

Note that this method gives the solution as a infinite sum of eigenfunctions. In many cases these are difficult to find, particularly if they need calculating numerically and so the series is truncated, sometimes to include just the first term! Such approximations can be very useful, for example, since the first term dominates as t get large due to the exponential decay of the $T(t)$ functions,

Particular coordinate systems

Note it is often useful to exploit some geometric symmetry of a problem to reduce the number of dimensions that need to be analysed or to easily define a particular boundary. Two common examples are cylindrical polar coordinates and spherical polar coordinates where radial symmetry might be able to be exploited. Note that in these cases

Cylindrical polar coordinates (r, θ, z)

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$

and separation of variables in this coordinate system requires analysis of various Bessel functions. Useful for modelling wires, cables, pipes etc.

Spherical polar coordinates (r, θ, ϕ) (beware there are different similar confusing notations used for these coordinates - here $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$ so that θ is the polar angle $0 \le \theta \le \pi$, and ϕ is the azimuthal angle $0 \le \phi \le 2\pi$)

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

and separation of variables in this coordinate system requires analysis of various Legendre functions. Useful for modelling bubbles, droplet, particles, the “spherical cow”, etc.

Note there are just 11 different coordinate systems in which the three dimensional operator Laplace operator is separable. Cartesian, cylindrical and spherical are the most common but occasionally others can be useful eg: prolate or oblate spherical coordinates.

Properties of the heat equation and solution methods

Maximum principle

A very powerful property of diffusion problems can be found by considering the heat diffusion equation with constant coefficients and no heat generation, in a finite region, with given initial temperature and given temperature along the boundary ie.

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \qquad \text{for } t \ge 0, \quad 0 \le x \le L$$

with $u(0,t) = u_r(t)$, $u(L,t) = u_L(t)$, $u(x,0) = u_0(x)$. We then find that if we consider the solution $u(x,\tau)$ at a time $\tau \geq 0$ then this solution is bounded by the maximum value that occurred either initially or on either the boundary in the interval $0 \leq t \leq \tau$. Similarly the minimum of $u(x,\tau)$ is also bounded by the minimum initially or on the boundary. Specifically

$$u_{min}(\tau) \leq u(x,\tau) \leq u_{max}(\tau) \qquad \text{for } \tau \geq 0 \text{ and } 0 \leq x \leq L$$

where

$$\begin{aligned} u_{max} &= \max\left\{ \max_{0\leq x\leq L} u_0(x),\ \max_{0\leq t\leq \tau} u_r(t),\ \max_{0\leq t\leq \tau} u_L(t) \right\} \\ u_{min} &= \min\left\{ \min_{0\leq x\leq L} u_0(x),\ \min_{0\leq t\leq \tau} u_r(t),\ \min_{0\leq t\leq \tau} u_L(t) \right\}. \end{aligned}$$

This property can be extended readily to multiple spatial dimensions and to more general diffusion problems and is a useful tool in determining bounds on the solution. Note that the idea also extends to the steady problem where the solution of the steady state heat equation is then bounded by the values of the function on the surface (this is a well known property of Laplace's equation).

Note that one immediate consequence of the maximum principle is that the solution to the heat equation is unique. We can show this by contradiction by considering that there are two different two solutions for $u(x,t)$ given by $U(x,t)$ and $W(x,t)$. Since each of these satisfies the problem it follows that the problem for $U(x,t) - W(x,t)$ is

$$\frac{\partial(U-W)}{\partial t} = \kappa\frac{\partial^2(U-W)}{\partial x^2} \qquad \text{for} t \geq 0, \quad 0 \leq x \leq L$$

with $U(0,t) - W(0,t) = 0$, $U(L,t) - W(L,t) = 0$, $U(x,0) - W(x,0) = 0$. By the maximum principle it follows that $0 \leq U(x,\tau) - W(x,\tau) \leq 0$ for $\tau \geq 0$ so the two functions are identical and the solution $u(x,t)$ is therefore unique.

Wellposedness

In studying PDE problems we are interested in determining if the problem is "well-posed" since we would hope that a model of a physical problem should sufficiently well defined for the behaviour of the solution to represent the behaviour of the physical system. Hence we explore whether the PDE problem (the PDE and its associated region of solution and the specified boundary and initial conditions) is well posed. One common assessment of wellposedness is consider it in the sense of Hadamard and to ask the following the three questions

- Does a solution to the problem exist?
- Is a solution unique?
- Does the solution depend continuously on the problem data (the initial data, the boundary conditions and any forcing terms)?

The previous section showed one method for determining uniqueness an alternative method of showing the function $\phi(x,t) = U(x,t) - W(x,t)$ must be zero, and hence the solution $u(x,t)$ is unique is to use an energy method. Multiplying the PDE for ϕ by ϕ and integrating over the region $0 \leq x \leq L$ gives, after integrating by parts and using the boundary conditions,

$$\frac{\partial}{\partial t}\left(\int_0^L (\phi(x,t))^2 dx\right) = -2\kappa \int_0^L \left(\frac{\partial \phi(x,t)}{\partial x}\right)^2 dx \ . \tag{1.3}$$

Since $\int(\phi(x,0))^2 dx$is obviously non-negative and (1.3) implies that it is nonincreasing we can use the initial data

$$\int_0^L (\phi(x,0))^2 dx = 0$$

to conclude that

$$\int_0^L (\phi(x,t))^2 dx = 0 \ .$$

This then shows how the L_2 norm of the function is a natural measure in assessing how close two functions are in this case. The method of separation of variables or other steps can be use to demonstrate a solution exists.

The final "wellposedness" question is commonly answered using the L_2 norm to measure how sensitive the solution is to the problem data. One simple method of assessment valid for linear equations is to consider a Fourier representation of the solution. Hence solving in the region $0 \leq x \leq L$ take

$$u(x,t) = \sum_{k=-\infty}^{k=+\infty} e^{i2\pi kx/L} u_k(t)$$

Here k is the wave number and indicates one mode of the spatial distribution of the solution. Putting this into the PDE and using the fact that the Fourier modes, $\exp(i2\pi kx/L)$, are linearly independent then this indicates that the solution can be written in the form

$$u(x,t) = \sum_{k=-\infty}^{k=+\infty} A_k e^{i2\pi kx/L} e^{i\omega t}$$

where ω must be related to k by

$$i\omega = \kappa(2\pi ik/L)^2 \qquad \rightarrow \qquad \omega = -i(2\pi/L)^2k^2 \ . \tag{1.4}$$

and the constants A_k can be determined by initial data. This relation (1.4) between ω and k is called the "dispersion relation" of the PDE. In this case it shows that all the Fourier modes of the solution will decay as time increases (taking k to be real the imaginary part of ω is negative) except the mode $k = 0$ which will remain constant in time. Because modes decay the PDE is called dissipative and we can conclude that small changes to the initial data will result in small (decaying) changes in the solution. Hence this Fourier analysis indicates that the heat diffusion PDE is well posed (note that this method only considers wellposedness due to initial data and makes no statement about the wellposedness of the boundary conditions). Note a problem is not well posed (ie it is illposed) if there are Fourier modes that grow with time (ie the imaginary part of ω is positive for some real value of k). A PDE which has the imaginary part of ω equal to zero for all k is called conservative.

The energy method can also be use to determine the sensitivity of the solution to the problem data. The ODE (1.3) for the L_2 norm of ϕ shows that if the initial value of the solution is changed by a small amount (as measured by it L_2 norm) then the later solution will also have a small change. Similarly the method can account for different boundary conditions and assess their effect on the wellposedness.

When considering the heat diffusion problem on an infinite spatial region it is necessary to impose conditions at infinity to ensure the solution exists and is unique. In particular a condition that ensures grow is less than $\exp x^2$ is typically sufficient and unfortunately seldom indicated in many practical problems, so care should be taken.

Non- linear equlibria and linear stability. An introduction to dynamical systems

The theory and applications of nonlinear differential equations form an important part of modern nonlinear dynamics. These equations are natural mathematical models of various real-life phenomena, such as population dynamics and ecology, physiology and medicine, economics and other natural sciences. For this reason, the study of the stability of such models is extremely important.

One of the first methods to study the stability of zero solution to nonlinear systems is the method of linearization and stability analysis, based on the stability of the linear approximation system. Such types of work were done in the second half of the last century, for example [6], [8], [9]. The basic idea is: "If zero solution of the linear approximation is asymptotically stable, then in a sufficiently small neighbourhood of the equilibrium, the trivial solution of the original nonlinear system will be also stable."

If the system is asymptotically stable, then it comes to zero equilibrium position in an infinite time interval. An important characteristic of stability is the time for which the solution of the system goes into an ε neighbourhood of the origin of the system and will not leave it, this is, $\|x(t)\| \leq \varepsilon$. Using the obtained convergence estimate, it is possible to find the time for which the solution of the system from a position $x(t_0)$ falls into the ε neighbourhood of the zero position.

For systems of linear stationary differential equations

$$x'(t) = Ax(t), \quad t \geq t_0 \,, \tag{1}$$

where A is an $n \times n$ constant matrix, the estimates of the above type were obtained in [3]. In order to formulate the results obtained there, we introduce the notations associated with the Lyapunov matrix equation

$$A^T H + HA = -C. \tag{2}$$

If the matrix A is asymptotically stable, then for any positive definite $n \times n$ matrix C there exists a unique solution to (2). Such a solution is a positive definite $n \times n$ matrix H, see in [4]. Matrices C and H from the Lyapunov matrix equation (2) play an important role in estimating the convergence of solutions to an equilibrium. Namely, the following estimate of the exponential convergence of solutions to linear system (1) is derived in [4]

$$\|x(t)\| \leq \left[\varphi(H)\,\|x(t_0)\|\right] e^{-\frac{1}{2}\gamma(H)(t-t_0)}, \tag{3}$$

where

$$\varphi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \gamma(H) = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}, \tag{4}$$

$\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalues of the corresponding symmetric matrix, and

$$\|x(t)\| = \sqrt{\sum_{i=1}^{n} x_i^2(t)} \tag{5}$$

denotes the vector norm.

In this paper we deal with nonlinear systems in order to estimate the convergence of their solutions to stable singular points. The stability domain of the zero equilibrium of the systems of nonlinear differential equations with quadratic part and a fractional part is estimated. As a method of investigation such systems, the second Lyapunov method with quadratic Lyapunov functions is used [2]. The second Lyapunov method using the Lyapunov function of Lurje-Postnikov type was used in [5] to obtain an estimation of a solution to a control equation with nonlinearity of a sector form.

It should be noted that the Lyapunov function in quadratic form is determined by a symmetric, positive definite matrix H, which is the solution to the Lyapunov matrix equation (2) in the case, when the matrix A in the linear part of nonlinear system is a constant matrix.

Systems with the quadratic right-hand side

In this section we consider systems with special form of nonlinearity, namely, systems with the quadratic right-hand side, written in a vector-matrix form [1, 4, 7],

$$x'(t) = Ax(t) + X^T(t)Bx(t), \tag{6}$$

where A is an $n \times n$ constant matrix, $B = (B_1, B_2, \ldots, B_n)^T$, $B_i, i = 1, 2, \ldots, n$ are $n \times n$ constant matrices,

$$B_i = \begin{pmatrix} b^i_{11} & b^i_{12} & \ldots & b^i_{1n} \\ b^i_{21} & b^i_{22} & \ldots & b^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b^i_{n1} & b^i_{n2} & \ldots & b^i_{nn}, \end{pmatrix},$$

and $X^T = \big(X_1(t), X_2(t), \ldots, X_n(t)\big)$, $X_i(t), i = 1, 2, \ldots, n$ are $n \times n$ matrices in which only the i-th row is nonzero,

$$X_1(t) = \begin{pmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \ \ldots, \ X_n(t) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t) & x_2(t) & \ldots & x_n(t) \end{pmatrix}.$$

Recall that if the matrix A of the linear part in (6) be asymptotically stable, that is, all its eigenvalues have negative real parts, $\mathrm{Re}\lambda_i(A) < 0, i = 1, \ldots, n$, then, as follows from the stability theory of linear approximation, the zero solution of the corresponding nonlinear system is also asymptotically stable.

The following form of matrix norm will be used in our considerations

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)}. \tag{7}$$

Theorem 2.1. *Suppose that the matrix A in (6) is asymptotically stable. Then the trivial solution to (6) is asymptotically stable. Moreover, the domain*

$$G_{r_0} = \max_{r>0} \{G_r \colon G_r \subset G_0\}, \tag{8}$$

where

$$G_r = \{x \in R^n \colon x^T H x < r^2\}, \quad G_0 = \left\{x \in \mathbb{R}^n \colon \|x\| < \frac{\lambda_{\min}(C)}{2\|H\| \, \|B\|}\right\},$$

is the domain of stability.

Proof. We calculate the total derivative of the Lyapunov function in the quadratic form, $V(x) = x^T Hx$, along the trajectories of system (6)

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} = x^T(t)\big[\big(A^T H + HA\big) + \big(B^T X(t)H + HX^T B\big)\big]x(t). \tag{9}$$

Taking into account (2), equation (9) can be rewritten into the form

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} = -x^T(t)\big[C - \big(B^T X(t)H + HX^T(t)B\big)\big]x(t).$$

Therefore, the stability domain is the interior of the level surface of the Lyapunov function, which lies within the domain

$$G_0 = \big\{x \in \mathbb{R}^n \colon C - B^T XH - HX^T B > \Theta\big\},$$

where Θ denotes the zero matrix, and the expression inside denotes that the relevant matrix is positive definite. Since, in view of the vector and matrix norms defined by (5) and (7), we have $\|X(t)\| = \|x(t)\|$, thus the total derivative of the Lyapunov function can be estimated as

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} < -\big[\lambda_{\min}(C) - 2\|H\|\,\|B\|\,\|x(t)\|\,\big]\,\|x(t)\|^2. \tag{10}$$

Therefore, if the inequality

$$\|x(t)\| < \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|}$$

is satisfied, then the total derivative of the Lyapunov function is negative. □

Remark 2.2. Obviously, to obtain the "maximum" domain of stability, the sphere G_0 should have the radius

$$R = \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|},$$

and the r should "stretch" as long as the ellipse $x^T Hx = r^2$ touches the sphere.

Theorem 2.3. *Suppose that the matrix A in (6) is asymptotically stable. Then for any solution to (6) satisfying the initial condition*

$$\|x(0)\| < \frac{\gamma(H)}{2\|B\|\varphi(H)}, \tag{11}$$

the following estimate

$$\|x(t)\| \le \frac{\gamma(H)\|x(0)\|}{\big[\gamma(H) - 2\|B\|\,\varphi(H)\,\|x(0)\|\,\big]\,e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x(0)\|} \tag{12}$$

for the convergence of solutions to the zero singular point holds.

P r o o f. The total derivative of the Lyapunov function $V(x) = x^T Hx$ along trajectories of system (6) is given by (9). Since for $V(x)$ two-sided inequality,

$$\lambda_{\min}(H)\|x\|^2 \le V(x) \le \lambda_{\max}(H)\|x\|^2, \tag{13}$$

is satisfied, then the estimate (10) of the Lyapunov function can be rewritten as

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} \le -\frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}V(x(t)) + 2\lambda_{\max}(H)\|B\|\frac{V^{\frac{3}{2}}(x(t))}{\lambda_{\min}(H)},$$

or, taking into account (4), we get

$$\frac{\mathrm{d}\,\mathrm{V(x(t))}}{\mathrm{d}\,t} \le -\gamma(H)V(x(t)) + 2\|B\|\frac{V^{\frac{3}{2}}(x(t))\varphi(H)}{\sqrt{\lambda_{\min}(H)}}.$$

Dividing by $V^{\frac{3}{2}}(x)$ and denoting

$$V^{-\frac{1}{2}}(x(t)) = z(t), \tag{14}$$

we obtain

$$-2\frac{\mathrm{d}\,z(t)}{\mathrm{d}\,t} \le -\gamma(H)z(t) + 2\|B\|\frac{\varphi(H)}{\sqrt{\lambda_{\min}(H)}},$$

and from here

$$\frac{\mathrm{d}\,z(t)}{\mathrm{d}\,t} \ge \frac{1}{2}\gamma(H)z(t) - \frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}.$$

Solving the inequality by analogy with a linear nonhomogeneous equation, we have

$$\|z(t)\| \ \ge \ \left[z(0) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}.$$

Since (14), we get

$$V^{-\frac{1}{2}}(x(t)) \ \ge \ \left[V^{-\frac{1}{2}}(x(0)) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}},$$

or

$$V^{\frac{1}{2}}(x(t)) \ge \left(\left[V^{-\frac{1}{2}}(x(0)) - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right]e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right)^{-1}.$$

Consequently, using two-sided inequality (13), we obtain

$$\sqrt{\lambda_{\min}(H)}\|x(t)\| \le \frac{\gamma(H)\sqrt{\lambda_{\min}(H)}\|x(0)\|}{\big(\gamma(H) - 2\|B\|\varphi(H)\|x(0)\|\big)e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x(0)\|}.$$

Therefore, any solution $x(t)$ to (6) satisfying the initial condition (11) under the assumption $x(0) \in G_0$, is estimated by (12). $\square$

Remark 2.4. In applications one can find nonlinear systems with a quadratic part in the form

$$x_i'(t) = \left[-a_i + \sum_{j=1}^{n} b_{ij} x_j(t) \right] x_i(t), \quad i = 1, 2, \dots, n, \tag{15}$$

where $a_i, b_{ij} \in \mathbb{R}^+,\ i, j = 1, 2, \dots, n$.
If we denote $A = \operatorname{diag}(a_1, a_2, \dots, a_n)$, $B = (B_1, B_2, \dots, B_n)^T$, $B_i, i = 1, 2, \dots, n$ are $n \times n$ constant matrices in which only the i-th column is nonzero,

$$B_i = \begin{pmatrix} 0 & \dots & b_{i2} & \dots & 0 \\ 0 & \dots & b_{i2} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & b_{in} & \dots & 0 \end{pmatrix},$$

and X^T as in system (6), then system (15) can be written in the form

$$x'(t) = -Ax(t) + X^T(t) B x(t). \tag{16}$$

Since $a_i > 0, i = 1, 2, \dots, n$, all eigenvalues are negative, which means that the trivial solution to (15) is asymptotically stable, and the result of Theorem 2.3 can be applied to estimate solutions to system (16) as well as (15).

The second singular point $x_0 = (x_1^0, x_2^0, \dots, x_n^0)^T$ to system (16) is solution to algebraic system

$$B_0 x = a, \tag{17}$$

where

$$B_0 = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}, \quad a = \left(a_1, a_2, ..., a_n\right)^T,$$

under assumption that $\det B_0 \neq 0$.
Then, using substitution $y(t) = x(t) - x_0$, we obtain the transformed system with the zero equilibrium in the form

$$y'(t) = \bar{A} y(t) + Y^T(t) B y(t), \tag{18}$$

where

$$\bar{A} = \begin{pmatrix} b_{11}x_1^0 & b_{12}x_1^0 & \dots & b_{1n}x_1^0 \\ b_{21}x_2^0 & b_{22}x_2^0 & \dots & b_{2n}x_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}x_n^0 & b_{n2}x_n^0 & \dots & b_{nn}x_n^0 \end{pmatrix}.$$

If the matrix $\bar{A}$ is asymptotically stable, then the result of Theorem 2.3 can be applied to estimate solutions to system (18) as well as (15).

Example 2.1. We illustrate the result obtained on a scalar equation

$$x'(t) = -ax(t) + bx^2(t), \quad a, b > 0. \tag{19}$$

Since $\lambda = -1$, the trivial solution to equation (19) is stable. Any solution to (19), satisfying the initial condition $x(0) = x_0$, can be determined by the formula

$$x(t) = \frac{ax(0)e^{-at}}{a - bx(0)\left[1 - e^{-at}\right]}.$$

To estimate these solutions in a neighbourhood of the trivial solution, we take the Lyapunov function in the form $V(x) = x^2$. So, $H = 1$, $\lambda_{\min}(H) = \lambda_{\max}(H) = 1$, $C = 1$, $\varphi(H) = 1$, $\gamma(H) = 2a$. In accordance with the result of Theorem 2.3, convergence to the zero singular point of any solution to (19), satisfying initial condition $x(0) < \frac{a}{b}$, is estimated as follows

$$x(t) \le \frac{ax(0)}{\left[a - bx(0)\right]e^{at} + bx(0)}.$$

As a result, the exact solution to equation (19) coincides with the obtained estimate, using by quadratic Lyapunov function.

It should be noted, the second equilibrium $x = \frac{a}{b}$ of equation (19) is unstable.

Remark 2.5. Interesting results of estimating the convergence of solutions to the zero singular point using the Lyapunov function can be obtained for the planar system with the quadratic right-hand side, this is, for system

$$\begin{aligned} x_1'(t) &= a_{11}x_2(t) + a_{12}x_2(t) + b_{11}^1 x_1^2(t) + 2b_{12}^1 x_1 x_2 + b_{22}^1 x_2^2(t), \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_{11}^2 x_1^2(t) + 2b_{12}^2 x_1 x_2 + b_{22}^2 x_2^2(t). \end{aligned} \tag{20}$$

Using notations

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{12}^1 & b_{22}^1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11}^2 & b_{12}^2 \\ b_{12}^2 & b_{22}^2 \end{pmatrix}, \quad B = \left(B_1, B_2\right)^T$$

system (20) can be rewritten into the vector-matrix form (6). Under the assumption that $\operatorname{Re}\lambda_{1,2}(A) < 0$, Theorem 2.3 can be applied to estimate solutions to (20). Thus, the total derivative of the Lyapunov function along trajectories of system (20) can be estimated as in (10), where

$$\begin{aligned} \|H\| = \lambda_{\max}(H) &= \frac{1}{2}\left(h_{11} + h_{22} + \sqrt{(h_{11} - h_{22})^2 + 4h_{12}^2}\right), \\ \lambda_{\min}(C) &= \frac{1}{2}\left(c_{11} + c_{22} - \sqrt{(c_{11} - c_{22})^2 + 4c_{12}^2}\right), \\ \|B\| &= \lambda_{\max}(B^T B). \end{aligned}$$

Therefore, in view of (8), the interior of the ellipse

$$h_{11}x^2 + 2h_{12}xy + h_{22}y^2 < r_0^2, \quad r_0 = \frac{\lambda_{\min}(C)}{2\|H\|\,\|B\|}$$

is the guaranteed domain of stability.

References and further readings

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