Elastic waves, acoustics, Stokes waves, electromagnetism.

Linear waves

There are numerous physical situations in which waves form and propagate. In this chapter we examine some of the physical mechanisms underlying such behaviour and the mathematical model that have been used to describe them.

We will then explore methods for studying linear waves by studying the underlying properties of the PDE problems

Strings and membranes

The simplest wave equations arise in situations where a very thin materials, a string or a membrane which can sustain tension but requires negligible force to bend, is held taut at its ends (at the boundary) and allowed to vibrate.

The classical problem of the small transverse movement of a string can be modelled using Newton's law for a string with a linear density ρ (mass/unit length) and a constant tension T force) along the string. Taking x to be the distance along the string (in the case of small displacements this is same as the distance along the string when it is straight and hence undeformed) and u as the transverse displacement from a straight string we have

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right)$$

and for constant T and ρ this gives the classic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

In a similar manner we can consider the tangential displacement of a thin membrane with undeformed membrane in the x, y plane and u as the transverse displacement. Assuming the membrane is under uniform stress σ_0 in all direction we find

$$\frac{\partial^2 u}{\partial t^2} = \frac{h\sigma_0}{\rho}\nabla^2\mathbf{u}.$$

where h is the membrane thickness and ρ is the area density (mass /unit area).

Elastic waves

We now consider the motion of a block of elastic material and how deformations within such a material can propagate as waves. In deriving the equations we shall assume that the deformations of the material from its initial state are small (nonlinear theories that account for large displacements, or for large strains, or for growth of a material can be found but are much more complicated)

Newton's law ($F = ma$) can be applied to a small element of an elastic material V with the assumption that the only forces acting are those on the surface to give

$$\frac{\partial^2}{\partial t^2} \int_V \rho \mathbf{u} \, \mathrm{d}V = \int_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{d}S$$

where ρ is the density, $\mathbf{u}$ the displacement of a point in the material from its original position, and $\boldsymbol{\sigma}$ the stress tensor. Assuming that $\boldsymbol{\sigma}$ is differentiable and using Green's theorem (divergence theorem) it then follows that

$$\frac{\partial^2}{\partial t^2} \int_V \rho \mathbf{u} \, \mathrm{d}V = \int_V \nabla \cdot \boldsymbol{\sigma} \, \mathrm{d}V,$$

Hence since V is arbitrary we have

$$\rho \frac{\partial \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}.$$

We now need a constitutive law describing how a particular material acts by prescribing the relationship between the stress and the strain. For a simple linear elastic material (acting like a linear spring) we take Hooke's law where

$$\boldsymbol{\sigma} = \lambda \, \mathrm{tr}(\boldsymbol{\epsilon}) \mathbf{I} + 2\mu \boldsymbol{\epsilon}, \qquad \sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij},$$

and $\boldsymbol{\epsilon}$ is the Strain tensor,

$$\boldsymbol{\epsilon} = (\nabla \mathbf{u})^S, \qquad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Note in these we have used summation convention over repeated indices, we have taken the superscript S to indicate symmetric part of of the strain, and the constants λ and μ are the Lamé constants which can be determined by measurements on any particular material. Thus (again using summation convention)

$$\begin{aligned}\rho\frac{\partial u_i}{\partial t^2} &= \frac{\partial \sigma_{ij}}{\partial x_j}\\ &= \frac{\partial}{\partial x_j}\left(\lambda\delta_{ij}\epsilon_{kk}+2\mu\epsilon_{ij}\right)\\ &= \frac{\partial}{\partial x_j}\left(\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right)\\ &= (\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k}+\mu\frac{\partial^2 u_i}{\partial x_j^2}.\end{aligned}$$

In vectorial form this is

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}$$

which is commonly referred to as Navier's equation.

Now we can use the vector identity

$$\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\wedge\nabla\wedge\mathbf{u},$$

to rewrite Navier's equation as

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\wedge\nabla\wedge\mathbf{u}.$$

We now examine properties of this Navier equation. Start by noting that the dilation (the volume change from the initial state) of the material is given by $\Delta = \nabla\cdot\mathbf{u}$ and the rotation of the material from its initial state is given by $\mathbf{r} = \frac{1}{2}\nabla\times\mathbf{u}$. Here $\mathbf{r}$ might be thought of as the elastic equivalent of vorticity in fluid dynamics.

By taking the divergence and then the crossproduct of the governing Navier equation we now find that

$$\begin{aligned}\rho\frac{\partial^2\Delta}{\partial t^2} &= (\lambda+2\mu)\nabla^2\Delta,\\ \rho\frac{\partial^2\mathbf{r}}{\partial t^2} &= -\mu\nabla\wedge\nabla\wedge\mathbf{r} = \mu\nabla^2\mathbf{r}.\end{aligned}$$

Thus both Δ and $\mathbf{r}$ satisfy a wave equation, but with different wavespeeds. The function Δ is a P-wave and represents a dilation or primary wave (also called a pressure wave or longitudinal wave in special circumstances) and the corresponding wavespeed is denoted c_P. The function $\mathbf{r}$ is an S-wave and represents a rotational or secondary wave (also called a transverse or shear wave in special circumstances), and the corresponding wavespeed is denoted c_s. Thus

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_s = \sqrt{\frac{\mu}{\rho}}.$$

A very useful physical property of these waves comes from noting that since λ and μ are both positive, $c_p > c_s$ and so those deformations described by P-waves always travel faster than the deformations related to S-waves.

Acoustic waves

Next we consider the motion of a compressible gas and explore how the resulting behaviour gives rise to acoustic waves. The underlying model considers the gas to be homentropic (the entropy at any point or at any time remains constant) and that it is inviscid (no significant viscous dissipation).

Conservation of mass gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

where ρ is the fluid density and $\mathbf{u}$ is the fluid velocity (note the easy confusion between $\mathbf{u}$ as used in elasticity and $\mathbf{u}$ as used in fluid!!). Conservation of momentum gives

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p.$$

For conservation of energy our assumption of homentropic flow gives

$$p = p(\rho)$$

where the precise form of this relationship depends on a constitutive relation describing the gas properties. We now linearise about a uniform steady the ambient state by setting

$$\rho = \rho_0 + \delta \bar{\rho}, \qquad p = p(\rho_0) + \delta \bar{p}, \qquad \mathbf{u} = \delta \bar{\mathbf{u}},$$

where $0 < \delta \ll 1$. Keeping only leading order terms in δ gives (dropping the overbar notation),

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} &= 0, \\ \rho_0 \frac{\partial \mathbf{u}}{\partial t} &= -\nabla p, \\ p &= \frac{\mathrm{d}p}{\mathrm{d}\rho} \rho, \end{aligned}$$

where $\mathrm{d}p/\mathrm{d}\rho$ in the last equation is evaluated at $\rho = \rho_0$. Eliminating ρ and $\mathbf{u}$ gives

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \nabla^2 p,$$

where

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}$$

and c is the wave speed,

Stokes waves

We now consider a large class of problems that arise in situations where there are waves at the interface between a dense incompressible inviscid fluid (eg water) and a gas. Typical examples are waves on the sea, ripples on a pond, waves in a teacup etc. In these situations it is reasonable to neglect the role of the gas except that this imposes a constant pressure and no tangential stress on the interface and to assume the the only external force on the system is gravity which acts on the dense fluid (extensions are well known that include other mechanisms such as surface tension etc.). The conventional terminology used for such waves on an inviscid irrotational incompressible fluid is Stokes waves or surface gravity waves.

Consider an inviscid, irrotational incompressible fluid occupying the region $-H < z < h(x, y, t)$, where the bottom surface, $z = -H$, is rigid, but the top surface, $z = h(x, y, t)$, is a free boundary. Irrotationality implies $\nabla \wedge \mathbf{u} = \mathbf{0}$, which implies the existence of a velocity potential ϕ such that

$$\mathbf{u} = \nabla \phi.$$

Conservation of mass along with incompressibility (ρ =constant) implies $\nabla \cdot \mathbf{u} = 0$, giving

$$\nabla^2 \phi = 0,$$

in the fluid. Since the bottom surface is impermeable, $\mathbf{u} \cdot \mathbf{n} = 0$ there, i.e.

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -H.$$

On the free surface we have the kinematic condition that particles on the free surface must remain there, giving

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - h) = 0 \qquad \text{on } z = h(x, y, t),$$

where $\mathrm{D}/\mathrm{D}t$ is the convective derivative. Thus this implies

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)(z - h) = 0 \qquad \text{on } z = h(x, y, t),$$

and hence

$$-\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad \text{on } z = h(x, y, t).$$

On the free surface we also have the dynamic condition, which arises from Bernoulli's equation. Bernoulli's equation gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 = -gz - \frac{p}{\rho}.$$

On the free surface the pressure is atmospheric (which we may take to be zero without loss of generality), giving the condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 = -gh \qquad \text{on } z = h(x, y, t).$$

This system of equations and boundary conditions now describes the behaviour of the fluid and the interface.

Linearisation

We now consider a special limit of the Stokes wave problem and look at very small waves on a very deep stationary pool of water (other limits give various different equations for the surface with equally interesting behaviour).

Consider small perturbations to a flat free surface and a static fluid by setting

$$\phi = \delta\bar{\phi}, \qquad h = \delta\bar{h}.$$

Keeping only leading order terms in δ, and dropping the overbar notation, gives

$$\begin{aligned} \nabla^2\phi &= 0 && \text{in } -H < z < 0, \\ \frac{\partial \phi}{\partial z} &= 0 && \text{on } z = -H, \\ -\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial z} &= 0 && \text{on } z = 0, \\ \frac{\partial \phi}{\partial t} &= -gh && \text{on } z = 0. \end{aligned}$$

(note care has to be taken particularly when linearising the boundary condition on $z = \delta h$)

We now consider the behaviour of waves in this linear system by separating the variables. Hence we take

$$\phi = Z(z)\psi(x, y, t)$$

which gives

$$\frac{Z''}{Z} = -\frac{\nabla^2\psi}{\psi} = \lambda^2.$$

where λ is an arbitrary constant. Using this equation and the boundary condition on $z = -H$ we find

$$Z = \cosh(\lambda(z + H)).$$

The conditions on $z = 0$ then gives

$$\begin{aligned} -\frac{\partial h}{\partial t} + \psi\lambda \sinh \lambda H &= 0, \\ \frac{\partial \psi}{\partial t} \cosh(\lambda H) &= -gh. \end{aligned}$$

along with

$$\nabla^2 \psi = -\lambda \psi$$

(where the operator ∇^2 is only in the plane x, y).

This gives wave phenomena which we can easily see by looking for wave solutions (or doing separation of variables)

$$h = A(x, y)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \psi = B(x, y)\mathrm{e}^{\mathrm{i}\omega t}.$$

Substituting these into the governing equations gives

$$\begin{aligned} -A\mathrm{i}\omega + B\lambda \sinh \lambda H &= 0, \\ B\mathrm{i}\omega \cosh(\lambda H) &= -gA. \end{aligned}$$

Hence a nonzero solution for A and B exists providing λ satisfies

$$\omega^2 = g\lambda \tanh(\lambda H).$$

The actual amplitude of the waves that occur then depend on solving

$$\nabla^2 A = -\lambda A$$

with suitable boundary conditions.

Note that if we just consider this problem in one dimension then the amplitude equation is

$$A = \mathrm{e}^{\mathrm{i}kx}$$

and substituting this into the governing equations gives

$$\lambda = k$$

so that we require

$$\omega = \sqrt{gk \tanh(kH)}.$$

This is the "dispersion relation" of the waves. It relates the frequency of the wave, ω to the wavelength, k. From this we can see that waves of different wavelength travel at different speeds.

Electromagnetism

Finally we consider the behaviour of electromagnetic waves by briefly deriving Maxwell's equations. These are given by Gauss's law, Gauss's law for magnetism, Faraday's law, Ampere's Law, conservation of charge and some constitutive equations to describe the material that the electromagnetic waves are moving through.

Gauss's Law

Gauss's law says that the integral of the electric displacement field over a surface S is given by the total net charge in the volume V enclosed within that surface:

$$\int_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_V \rho \, \mathrm{d}V.$$

By Green's theorem there is an equivalent differential form

$$\nabla \cdot \mathbf{D} = \rho. \tag{2.1}$$

Gauss's Law for magnetism

Gauss's law for magnetism says that magnetic monopoles (the magnetic equivalent of electric charge) do not exist. Thus the integral of the magnetic induction around any closed surface is zero:

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

By Green's theorem there is an equivalent differential form

$$\nabla \cdot \mathbf{B} = 0. \tag{2.2}$$

Faraday's Law

Faraday's Law states that the line integral of the electric field $\mathbf{E}$ around a closed curve C equals the negative of the integral of the time variation of the magnetic induction $\mathbf{B}$ through the surface S enclosed by C:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\boldsymbol{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}.$$

By Stokes Theorem there is an equivalent differential form

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{2.3}$$

Conservation of charge

An electric current is a flux of charge. Conservation of charge gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Combining this with Gauss's law gives

$$\nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) = 0.$$

This implies the existence of a field $\mathbf{H}$ (the magnetic field) such that

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla \wedge \mathbf{H}. \tag{2.4}$$

This equation is known as Ampere's law.

The set of equations (2.1)-(2.4) are collectively referred to as Maxwell's equations.

Constitutive equations

It remains to relate the fields $\mathbf{J}$, $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$ to each other. These relations depend on the particular material under consideration, and can be viewed as constitutive relations. Usually the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ are proportional to each other, and the electric displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$ are proportional to each other.

In free space we assume the linear relations between the fields but also assume no current can flowing and there is no charge. Hence have

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mathbf{H}, \qquad \mathbf{J} = \mathbf{0}, \qquad \rho = 0,$$

where $\epsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{F\,m^{-1}}$ is the vacuum permittivity and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{V\,s\,A^{-1}\,m^{-1}}$ is the vacuum permeability.

Then Maxwell's equations can be manipulated to give

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\nabla \wedge \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\mu_0 \epsilon_0} \nabla \wedge \nabla \wedge \mathbf{B} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \mathbf{B}.$$

$$\nabla^2 \mathbf{E} = -\nabla \wedge \nabla \wedge \mathbf{E} = \frac{\partial}{\partial t} \nabla \wedge \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

This indicates that the components of **E** and **B** all satisfy the same wave equation. Hence in free space electromagnetic waves travel at a speed given by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3.00 \times 10^8 \mathrm{m\,s^{-1}}.$$

Note for more complex materials or situations the electric and magnetic fields may satisfy vector wave equations which can result in different types of behaviour.

Characteristics, Group and Phase velocities

Second order linear PDEs in two independent variables are often classified as elliptic, parabolic or hyperbolic based on the reduction of the PDE to a standard, canonical form. In this course we will address such classification later and by an alternative method. However we make the following observation based on the canonical approach. If we start with the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then the change of variables from x, t to

$$\eta = x + ct, \qquad \xi = x = ct$$

resulting in the PDE becoming

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = 0.$$

Hence the general solution to the problem is

$$u - F(\xi) + G(\eta) = F(x - ct) + G(x + ct)$$

for arbitrary functions F and G (this is commonly called “D’Alembert’s solution”). These special variables, η and ξ, are called characteristic variables with the curves in x, t space given by ξ=constant and η =constant representing two families of projected characteristics. Physically we see that “information” in the problem propagates along these characteristics with one representing waves travelling to the left and the other waves travelling to the right.

We now consider more general wave equations and some properties of the behaviour that help interpret observations. The role of the dispersion relation in wave equations can be given a special interpretation. Consider an initial shape of the wave given by a "wave packet" with a wave form $u(x,t)$ as a function of position x and time t. Let $A(k)$ be its Fourier transform at time $t=0$:

$$u(x,0) = \int_{-\infty}^{\infty} A_k e^{ikx}\, dk,$$

where k is the wave number of the modes. The wavepacket at any time t is then

$$u(x,t) = \int_{-\infty}^{\infty} A_k e^{i(kx-\omega t)}\, dk,$$

where ω and k are related by the dispersion relation of the governing equation. Now assume that the wave packet has a special form where the only modes in the wave are all almost at the same frequency (this corresponds to $A(k)$ being nonzero only in the vicinity of a central wavenumber k_0). We can then linearise the dispersion relation about this central wavenumber to give

$$\omega(k) \approx \omega_0 + (k-k_0)\omega_0'$$

where $\omega_0 = \omega(k_0)$ and $\omega_0' = \frac{\partial \omega(k)}{\partial k}|_{k=k_0}$. It then follows that

$$u(x,t) = e^{it(\omega_0' k_0 - \omega_0)} \int_{-\infty}^{\infty} A_k e^{ik(x-\omega_0' t)}\, dk.$$

and hence

$$|u(x,t)| = |u((x-\omega_0' t), 0)|.$$

Hence the wave packet, moves at speed ω_0' and this is called the "group velocity". If the dispersion relation has k linearly proportional to ω, such as in the 1-D wave equation, then the wave packet will travel at constant speed and the envelope of the wave will remain unchanged.

Note that "group velocity" is distinct from "phase velocity". The phase velocity is the speed at which a point of fixed phase (eg peaks) of a single mode k will travel. This is given by

$$v_{phase} = \frac{\omega}{k}.$$

To extend these ideas to more dimensions we need to introduce the idea that waves can be represented as a sum of plane waves. Each wave is therefore broken into waves of the form

$$u_{\mathbf{k}} = A_k e^{i\mathbf{k}\cdot\mathbf{x}}.$$

where the wave vector $\mathbf{k}$ gives the direction of travel of the plane wave. The PDE can then be used to find the dispersion relation between the scalar ω and $\mathbf{k}$. The group velocity can then be extended to give

$$\mathbf{v}_{group} = \vec{\nabla}_{\mathbf{k}}\, \omega$$

Similarly we have the phase velocity given by

$$\mathbf{v}_{phase} = \frac{\omega}{|k|^2}\mathbf{k}.$$

References and further readings

1 A. R. Conn, N. I. M. Gould and Ph. L. Toint, Trust-Region Methods, SIAM 2000.

2. J. Dennis and R. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear equations, (republished by) SIAM, 1996.

3. R. Fletcher, Practical Methods of Optimization, 2nd edition Wiley, 1987 (republished in 2000).

4. P. Gill, W. Murray and M. H. Wright, Practical Optimization, Academic Press, 1981.

5. N. I. M. Gould, An Introduction to Algorithms for Continuous Optimization, 2006. Available for download at http://www.numerical.rl.ac.uk/nimg/course/lectures/paper/paper.pdf.

6. J. Nocedal and S. J. Wright, Numerical Optimization, Springer Verlag, 1999 (1st edition) or 2006 (2nd edition).

7. Mathematical Modeling and Computation of Real-Time Problems: An Interdisciplinary Approach (Mathematical Engineering, Manufacturing, and Management Sciences) 1st Edition by Rakhee Kulshrestha, Chandra Shekhar, Madhu Jain, Srinivas R. Chakravarthy