Eigenvalue problems; resonance and high-frequency asymptotics

Eigenvalue problems and resonance

As we have seen, the propagation of sound or light waves in two spatial dimensions is governed by the wave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$

where ψ is some state variable such as pressure or a scalar electric field and c is the wave speed (note that in general electromagnetic waves are governed by vector wave equations, rather than a scalar wave equation and there are situations where this creates behaviour that the scalar equation cannot mimic.

We look for time-periodic (or "monochromatic") solutions with constant frequency ω by setting

$$\psi(x, y, t) = \phi(x, y)\mathrm{e}^{-\mathrm{i}\omega t}.$$

Then ϕ satisfies the *Helmholtz equation*

$$\nabla^2\phi + k^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \tag{2.5}$$

where $k = \omega/c$ is the wavenumber (*i.e.* 2π divided by the wavelength).

One question we can then ask is what are the possible vibration modes within a region. Depending on the type of boundary we might take the boundary to be governed by simple Dirichlet conditions $\phi = 0$ and then the modes of vibration are the eigensolutions corresponding to the eigenvalues, k_n of the problem. For a simple region (eg a rectangle) these modes can be determined by separation of variables.

Green's functions

Green's Functions for Poisson and Helmholtz Equations

Introduction

The typical problem we want to consider is that of finding solutions $\phi(\mathbf{x})$ of

$$\mathcal{L}[\phi] = f(\mathbf{x})$$

where $\mathcal{L}$ is the Laplacian ∇ or the Helmholtz operator $\nabla + k^2$ and $f(\mathbf{x})$ is a given function.

The nature of the solution depends on the boundary conditions and in this section we start by assuming that solutions are required in unbounded space. We assume that $f(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ so that there are no 'sources at infinity'. We will derive the Green's function for this problem in order to allow solutions to be found for any $f(\mathbf{x})$.

Three dimensional δ-function

We start by defining the three dimensional delta function by

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3),$$

where the RHS has the conventional one-dimensional delta functions, or alternatively by the properties

$$\delta(\mathbf{x}) = 0 \text{ for } \mathbf{x} \neq \mathbf{0}, \quad \text{and} \quad \int \delta(\mathbf{x})\psi(\mathbf{x})\, d^3\mathbf{x} = \psi(\mathbf{0})$$

for all test functions[1] $\psi(\mathbf{x})$.

In spherical polar co-ordinates (with $r^2 = x_1^2 + x_2^2 + x_3^2$) we have

$$\delta(\mathbf{x}) = \frac{1}{4\pi}\frac{\delta(r)}{r^2}. \tag{2.6}$$

Note: Although we have said that

$$g(x)\delta(x) = g(0)\delta(x)$$

this only applies to functions $g(x)$ which are continuous at $x = 0$. An expression of the form $g(x)\delta(x)$ when $g(x)$ is not continuous at $x = 0$ must be interpreted as a generalised function in its own right. Thus $\delta(r)/r^2$ is a generalised function and it is most certainly not equal to $\delta(r)/0^2$.

Free space Green's function for the Poisson equation

Here we wish to solve the problem

$$\nabla\phi = f(\mathbf{x})$$

with $\phi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$; that is, the effect of sources falls off far away from these sources. We can do this in much the same way as used when finding Green's functions for ODEs.

[1] In n dimensions, a test function is any infinitely differentiable function which vanishes together with all its partial derivatives of all orders as $|\mathbf{x}| \to \infty$.

Both Helmholtz and Poisson equations are self-adjoint so suppose we have a Green's function, $G(\mathbf{x},\mathbf{y})$, that satisfies

$$\nabla_x G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \quad G(\mathbf{x},\mathbf{y}) \to 0 \text{ as } |\mathbf{x}-\mathbf{y}| \to \infty$$

where ∇_x indicates differentiation with respect to $\mathbf{x} = (x_1, x_2, x_3)$ and not with respect to $\mathbf{y} = (y_1, y_2, y_3)$;

Then we will put

$$\phi(\mathbf{x}) = \int G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d^3\mathbf{y}$$

Note, since the integral is with respect to $\mathbf{y}$ and the derivatives are with respect to $\mathbf{x}$, if we want to put this into our PDE we will have

$$\nabla_x \phi = \int \nabla_x G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d^3\mathbf{y} = \int \delta(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, d^3\mathbf{y} = f(\mathbf{x}).$$

Hence if we can find a Green's function for the problem we can solve for general $f(\mathbf{x})$.

Now, we return to the question of finding G and observe that the Laplace operator ∇_x is independent of the origin of the $\mathbf{x}$-coordinates, so we can move these to $\mathbf{y}$. That is, if we put $\mathbf{x}' = \mathbf{x} - \mathbf{y}$ then the problem for G becomes

$$\nabla_{x'} G = \delta(\mathbf{x}') \tag{2.7}$$

Since the problem is posed over an infinite region and we only want decay at infinity we expect G to be radially symmetric, so we look for a function

$$G(\mathbf{x}') = G(r)$$

in which case (2.7) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) = \frac{1}{4\pi}\frac{\delta(r)}{r^2}$$

For $r > 0$, $\delta(r) = 0$, so we find that

$$G = -\frac{A}{r} + B \quad (\text{for } r > 0)$$

where A and B are constants. Since $G \to 0$ as $r \to \infty$, $B = 0$.

To find A we argue as follows. Since $\nabla_{x'} G = \delta(\mathbf{x}')$ we must have

$$\int_V \nabla_{x'} G\, d^3\mathbf{x}' = 1$$

for any volume V that includes the origin. By the divergence theorem we also have

$$\int_S \frac{\partial G}{\partial n}\, d^2\mathbf{x}' = \int_V \nabla_{\mathbf{x}'} G\, d^3\mathbf{x}' = 1$$

where S is the surface of V.

Now, choose V to be a sphere of radius R. The outward normal to the surface is just the unit vector pointing from the origin to the point on the surface and so

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} = \frac{A}{R^2}$$

on the surface $r = R$ of the sphere. Also, on the surface $r = R$, the unit of area $d^2\mathbf{x}'$ is given by $R^2 \sin\theta\, d\theta d\varphi$, so

$$\int_0^{\pi} \int_0^{2\pi} \left. \frac{\partial G}{\partial r} \right|_{r=R} R^2 \sin\theta\, d\theta d\varphi = 1$$

Thus

$$A \int_0^{\pi} \int_0^{2\pi} \sin\theta\, d\theta d\varphi = 1$$

and so

$$A = \frac{1}{4\pi}.$$

Thus

$$G = -\frac{1}{4\pi\, r} = -\frac{1}{4\pi} \frac{1}{|\mathbf{x}'|} = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

or

$$G = -\frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}$$

and the solution of the problem

$$\nabla\phi = f(\mathbf{x}), \quad \phi \to 0 \text{ as } |\mathbf{x}| \to \infty$$

is

$$\phi(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{f(\mathbf{y})\, d^3\mathbf{y}}{|\mathbf{x} - \mathbf{y}|}$$

or

$$\phi(x_1, x_2, x_3) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y_1, y_2, y_3)\, dy_1\, dy_2\, dy_3}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}.$$

Note that, strictly speaking, $G(\mathbf{x},\mathbf{y})$ is not defined at $\mathbf{x}=\mathbf{y}$. We should interpret $G(\mathbf{x},\mathbf{y})$ as some sort of generalised function. One useful interpretation uses the Heaviside function H with

$$G(\mathbf{x},\mathbf{y}) = \lim_{\epsilon\to 0} -\frac{1}{4\pi}\frac{H(|\mathbf{x}-\mathbf{y}|-\epsilon)}{|\mathbf{x}-\mathbf{y}|}.$$

The Helmholtz equation

Consider the wave equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2\nabla\psi.$$

with wave speed c.

If we look for time harmonic standing waves of frequency ω;

$$\psi(\mathbf{x},t) = e^{-i\omega t}\phi(\mathbf{x})$$

we find that $\phi(\mathbf{x})$ satisfies the Helmholtz equation:

$$(\nabla + k^2)\phi = 0.$$

where $k = \omega/c$ is the wave number. The solutions of the Helmholtz equation represent (the spatial part of) standing wave solutions of the wave equation.

If there is a harmonic momentum source (i.e., a harmonic disturbance $f(\mathbf{x})e^{-i\omega t}$ which is producing the waves) then it appears on the right-hand-side of the Helmholtz equation,

$$(\nabla + k^2)\phi = f(\mathbf{x}).$$

We think of $f(\mathbf{x})$ as a wave source, see figure 2.1.

Physically we expect waves to propagate away from the disturbance generating them and not towards it. This gives us a *radiation condition* which replaces the condition that $\phi\to 0$ as $|\mathbf{x}|\to\infty$ used for Poisson's equation. (We shall describe this radiation condition shortly.)

The Green's function for the Helmholtz equation satisfies

$$(\nabla_x + k^2)G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}).$$

subject a to suitable *radiation condition.* Then

$$\phi(\mathbf{x}) = \int G(\mathbf{x},\mathbf{y})f(\mathbf{y})\,d^3\mathbf{y}$$

is the solution of

$$(\nabla + k^2)\phi = f(\mathbf{x})$$

(subject to the same radiation condition as the Green's function).

As before, it is convenient to introduce $\mathbf{x}' = \mathbf{x} - \mathbf{y}$ in which case the problem becomes

$$(\nabla_{x'} + k^2)G = \delta(\mathbf{x}')$$

which clearly has spherical symmetry. So, we look for a solution with $G(\mathbf{x}') = G(r)$, and the problem is then

$$\frac{1}{r}\left(\frac{\partial^2}{\partial r^2}(rG) + k^2(rG)\right) = \frac{\delta(r)}{4\pi r^2}$$

since

$$\nabla \mapsto \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ ).$$

So for $r > 0$ we have

$$\frac{\partial^2}{\partial r^2}(rG) + k^2(rG) = 0$$

which implies that $rG = \alpha e^{ikr} + \beta e^{-ikr}$ or

$$G = \frac{A}{4\pi r}e^{ikr} + \frac{B}{4\pi r}e^{-ikr}$$

If we now consider Ge^{-jkt}, which is a solution of the wave equation, we have

$$Ge^{-ikt} = \frac{A}{4\pi r}e^{ik(r-t)} + \frac{B}{4\pi r}e^{-ik(r+t)}.$$

Now any function $f(r-t)$ represents a wave moving away from $r = 0$ towards $r \to \infty$ as t increases (i.e., outward radiation). On the other hand a function $g(r+t)$ represents a wave moving in towards $r = 0$ from $r \to \infty$ (i.e., inward radiation).

The δ-function in the problem for G represents a disturbance at the origin; physically we expect waves to propagate outward away from this disturbance and not inward from infinity. This gives us our *radiation condition*—there should only be waves moving away from the disturbance at the origin. Thus we must take $B = 0$.

Hence we have

$$G = \frac{A}{4\pi r}e^{ikr}, \quad r > 0.$$

We can extend this to all values of r by defining G to be the generalised function

$$G = \lim_{\epsilon \to 0} \left(\frac{AH(r-\epsilon)}{4\pi r} e^{ikr} \right).$$

We compute the Laplacian of G to find

$$\nabla G = \frac{-Ak^2 e^{ikr}}{4\pi r} - A\delta(\mathbf{x}')$$

so that

$$(\nabla + k^2)G = -A\delta(\mathbf{x}')$$

and hence we take $A = -1$:

$$G(\mathbf{x}') = -\frac{1}{4\pi r} e^{ikr} = -\frac{1}{4\pi |\mathbf{x}'|} e^{ik|\mathbf{x}'|}$$

Finally, recall that $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}')$ so

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} e^{ik|\mathbf{x}-\mathbf{y}|}$$

or

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi} \frac{\exp\left(ik\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}\right)}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}}$$

Note that as $k \to 0$ we recover the Green's function for the Poisson equation.

Inhomogeneous Helmholtz equation

The solution of the inhomogeneous Helmholtz problem

$$(\nabla + k^2)\phi = f(\mathbf{x})$$

(where we assume $f(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$) which satisfies the outward radiation condition is given by

$$\phi(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{ik|\mathbf{x}-\mathbf{y}|} \, d^3\mathbf{y}$$

or

$$\phi(x_1, x_2, x_3) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2, y_3) \times$$

$$\times \frac{\exp\left(ik\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}\right)}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}} dy_1 \, dy_2 \, dy_3.$$

This represents the (spatial part of) an outgoing train of waves caused by a disturbance in the region where $f(\mathbf{x}) \neq 0$.

Green's functions for Bounded Regions

In this section, we deal with the Helmholtz equation

$$(\nabla + k^2)\phi = f(\mathbf{x})$$

and determine Green's functions relevant to problems where the region of interest is bounded. Results for the Poisson equation follow by taking the limit $k \to 0$.

Interior problems for Helmholtz and Poisson equations

In the previous section we considered the free space problems of the form

$$(\nabla_x + k^2)\phi = f(\mathbf{x}), \tag{2.8}$$

and

$$(\nabla_x + k^2)G = \delta(\mathbf{x} - \mathbf{y}) \tag{2.9}$$

and expressed the solution of (2.8) in the form

$$\phi(\mathbf{x}) = \int f(\mathbf{y})G(\mathbf{x}, \mathbf{y})\, d^3\mathbf{y}. \tag{2.10}$$

To deal with problems where boundaries are present, it is convenient to adopt a more general point of view. Suppose we regard the point $\mathbf{x}$ as fixed and $\mathbf{y}$ as the variable in the equation for the Green's function. That is, suppose the Green's function satisfies

$$(\nabla_y + k^2)G(\mathbf{x}, \mathbf{y}) = \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} + k^2\right) G = \delta(\mathbf{x} - \mathbf{y}).$$

Now, write equations (2.8) and (2.9) as

$$(\nabla_y + k^2)\phi = f(\mathbf{y}) \tag{2.11}$$

and

$$(\nabla_y + k^2)G = \delta(\mathbf{x} - \mathbf{y}) \tag{2.12}$$

but regard $\mathbf{x}$ as the point in a bounded region V in $\mathbf{y}$-space where we want the solution ϕ.

Suppose that $G(\mathbf{x}, \mathbf{y})$ is any solution of (2.11)); there are infinitely many solutions of (2.11) and specific forms for G are obtained only when we impose boundary conditions on the surface S of V.[2] Multiplying (2.11) by $G(\mathbf{x}, \mathbf{y})$, multiplying (2.12) by $\phi(\mathbf{y})$ and subtracting these gives:

$$G\nabla_y\phi(\mathbf{y}) - \phi(\mathbf{y})\nabla_y G = f(\mathbf{y})G - \phi(\mathbf{y})\delta(\mathbf{x} - \mathbf{y}).$$

Integrate this with respect to $\mathbf{y}$ over V then gives

$$\int_V \Big(G(\mathbf{x},\mathbf{y})\nabla_y\phi(\mathbf{y}) - \phi(\mathbf{y})\nabla_y G(\mathbf{x},\mathbf{y}) \Big)\, d^3\mathbf{y} \\ = \int_V f(\mathbf{y})G(\mathbf{x},\mathbf{y})\, d^3\mathbf{y} - \int_V \phi(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\, d^3\mathbf{y}.$$

Using Green's theorem and the properties of $\delta(\mathbf{x}-\mathbf{y})$ we then find

$$\phi(\mathbf{x}) = \int_V f(\mathbf{y})G(\mathbf{x},\mathbf{y})\, d^3\mathbf{y} \\ + \int_S \left(\phi(\mathbf{y})\frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\frac{\partial \phi}{\partial n_y}(\mathbf{y}) \right) d^2\mathbf{y} \qquad (2.13)$$

where $\partial/\partial n_y$ denotes normal derivative with respect to the $\mathbf{y}$ variables, i.e.,

$$\frac{\partial G}{\partial n_y} = (\nabla_y G)\cdot \mathbf{n}_y.$$

[2]This is because to any solution of (2.12) we can add a solution ψ of (2.11) and then $(\nabla + k^2)(G+\psi) = (\nabla + k^2)G + (\nabla + k^2)\psi = \delta(\mathbf{x}-\mathbf{y})$.

The representation (2.13) is called the Kirchhoff–Helmholtz representation. It gives the value of $\phi(\mathbf{x})$ inside the region V in terms of the source distribution $f(\mathbf{x})$ in V and the values of ϕ and $\partial\phi/\partial n$ on the surface S. It is true for *any* $G(\mathbf{x},\mathbf{y})$ that satisfies (2.12).

When attempting to solve (2.8) analytically, we attempt to choose G so that we minimise the amount of information we need to know about ϕ and $\partial\phi/\partial n$ on the boundary. For example, if we are given a Dirichlet problem, so ϕ is prescribed on the boundary, then we try to find G so that $G(\mathbf{x},\mathbf{y}) = 0$ when $\mathbf{y}$ is on the boundary. This eliminates the unknown $\partial\phi/\partial n$ and allows us to calculate ϕ in terms of known quantities. Similarly, for the Neumann problem where we know $\partial\phi/\partial n$ on the boundary we try to find G so that $(\partial G/\partial n)(\mathbf{x},\mathbf{y}) = 0$ when $\mathbf{y}$ is on the boundary. This eliminates the unknown ϕ from the integral over the surface.

Note that (2.13) can also be used numerically. We choose a simple G, say a free space Green's function, and this gives us an integral equation to solve numerically. For example if we are given a Neumann problem with $\partial\phi/\partial n$ specified on the boundary (but we do not know ϕ on the boundary) then by choosing $\mathbf{x}$ to be a point on the boundary, (2.8) becomes an integral equation for the unknown $\phi(\mathbf{x})$ on the boundary. This is solved numerically, and once we know $\phi(\mathbf{x})$ on the boundary, then (2.8) tells us the value of $\phi(\mathbf{x})$ at all points inside the boundary. This is the essence of "boundary integral methods". Note that the integral equation is two dimensional whereas the original problem is three dimensional. This reduction in dimensionality is why boundary integral methods are powerful for efficiently solving problems.

The Dirichlet Problem This is the problem of finding ϕ in V given that

$$(\nabla + k^2)\phi = f(\mathbf{x}) \quad \text{in } V$$
$$\phi = g(\mathbf{x}) \quad \text{on } S.$$

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown $\partial\phi/\partial n$ from the integral, that is, we attempt to find a Green's function such that

$$(\nabla_y + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \quad \text{in } V$$
$$G(\mathbf{x}, \mathbf{y}) = 0 \quad \text{when } \mathbf{y} \text{ on } S.$$

In practice it may be difficult to find such a G, but assuming G is known the solution is then, from the Kirchhoff-Helmholtz representation,

$$\phi(\mathbf{x}) = \int_V f(\mathbf{y})G(\mathbf{x}, \mathbf{y})d^3\mathbf{y} + \int_S g(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y})\, d^2\mathbf{y}.$$

The Neumann problem

This is the problem of finding ϕ in V given that

$$(\nabla + k^2)\phi = f(\mathbf{x}) \quad \text{in } V$$
$$\frac{\partial\phi}{\partial n} = g(\mathbf{x}) \quad \text{on } S.$$

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown ϕ from the integral, that is, we attempt to find a Green's function such that

$$(\nabla_y + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \quad \text{in } V$$
$$\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{when } \mathbf{y} \text{ on } S.$$

Assuming G can be found, the solution is then

$$\phi(\mathbf{x}) = \int_V f(\mathbf{y})G(\mathbf{x}, \mathbf{y})d^3\mathbf{x} - \int_S g(\mathbf{y})G(\mathbf{x}, \mathbf{y})\, d^2\mathbf{y}.$$

Robin boundary conditions

This is the problem of finding ϕ in V given that

$$(\nabla + k^2)\phi = f(\mathbf{x}) \quad \text{in } V$$
$$\frac{\partial\phi}{\partial n}(\mathbf{x}) + \lambda(\mathbf{x})\phi(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } S$$

where $f(\mathbf{x})$, $g(\mathbf{x})$ and $\lambda(\mathbf{x})$ are all given functions.

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown $\partial\phi/\partial n$ from the problem using the fact that

$$\frac{\partial \phi}{\partial n}(\mathbf{x}) = g(\mathbf{x}) - \lambda(\mathbf{x})\phi(\mathbf{x}) \quad \text{on } S.$$

so the Kirchhoff-Helmholtz representation (2.13) becomes

$$\begin{aligned}\phi(\mathbf{x}) = &\int_V f(\mathbf{y})G(\mathbf{x},\mathbf{y})\, d^3\mathbf{y} \\ &+ \int_S \phi(\mathbf{y}) \left(\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) + \lambda(\mathbf{y})G(\mathbf{x},\mathbf{y}) \right) d^2\mathbf{y} \\ &- \int_S g(\mathbf{y})G(\mathbf{x},\mathbf{y})\, d^2\mathbf{y}\end{aligned}$$

Then, as we do not know ϕ on the surface S, we choose G so that this term is eliminated. That is, we choose G to be a solution of

$$\begin{aligned}(\nabla_y + k^2)G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \quad &\text{in } V \\ \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) + \lambda(\mathbf{y})G(\mathbf{x},\mathbf{y}) = 0 \quad &\text{when } \mathbf{y} \text{ on } S.\end{aligned}$$

Assuming G can be found, the solution is then

$$\phi(\mathbf{x}) = \int_V f(\mathbf{y})G(\mathbf{x},\mathbf{y}) d^3\mathbf{x} - \int_S g(\mathbf{y})G(\mathbf{x},\mathbf{y})\, d^2\mathbf{y}.$$

The Reciprocal Theorem

In all of the problems in this section, the Green's function is symmetric, that is

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}).$$

This is why the general method in which we find G by solving a partial differential equation in $\mathbf{y}$ variables (used in this section) is equivalent to the more naive method of the previous chapter where we found G by solving a partial differential equation in $\mathbf{x}$ variables.

We illustrate this symmetry for the Dirichlet problem for the Poisson equation:

Reciprocal Theorem

It is worth noting a mathematical property of the Green's function that may be exploited in certain situations. If $G(\mathbf{x},\mathbf{y})$ is the solution of

$$\begin{aligned}\nabla G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \quad &\text{in } V \\ G(\mathbf{x},\mathbf{y}) = 0 \quad &\text{for } \mathbf{x} \text{ on } S\end{aligned}$$

then

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}).$$

Hence physically the effect at a point $\mathbf{x}$ of imposing a force at point $\mathbf{y}$ is the same at the effect at a point $\mathbf{y}$ of imposing a force at point $\mathbf{x}$.

Exterior problems

The general Kirchhoff-Helmholtz solution (2.13) can be shown to be valid for exterior problems provided physical conditions ensure that ϕ decays at large distances (i.e, for $k = 0$ $\phi \to 0$ as $|\mathbf{x}| \to \infty$) or has outgoing wave behaviour (i.e, for $k \neq 0$ satisfies the radiation condition).

High-frequency asymptotics

The study of waves has produced some very innovative mathematical ideas. One of the most commonly encountered of these relates to studying situations where the wave lengths of interest are much shorter than the variations in size of the

region of interest. Hence gravity waves in rivers. estuaries; electromagnetic waves from radar, illuminations; noise in rooms and from aircraft. Mathematically these situations can be explored by examining behaviour of a monochromatic wave using Helmholtz equations (either scalar or vector) in the limit of $k \to \infty$. Below we examine some of these ideas under the heading of geometrical optics.

Geometrical optics

The theory of *geometrical optics* arises from considering the limit $k \to \infty$ to analyse the behaviour of (2.5). Note that a naive approach to taking $k \to \infty$ simply neglects all the derivatives in the problem and gets nowhere.

The central idea to taking this limit is to recognise that the solution ϕ will vary on a very short length scale $(1/k)$ due to the wave structure. However we anticipate that although the waves will have this high frequency their amplitude may varying over a much longer scale. The approach we use it the so-called *WKBJ method*, which involves writing ϕ in the form

$$\phi(x,y) = A(x,y)\mathrm{e}^{\mathrm{i}ku(x,y)}, \tag{3.11}$$

where A and u represent the *amplitude* and *phase* respectively of the solution. Then (2.5) becomes

$$\nabla^2 A + \mathrm{i}k\left(A\nabla^2 u + 2\boldsymbol{\nabla} A \cdot \boldsymbol{\nabla} u\right) + k^2 A\left(1 - |\boldsymbol{\nabla} u|^2\right) = 0. \tag{3.12}$$

Having had this WKBJ assumption we now seek solutions in which A is asymptotic expansions of the form

$$A \sim A_0 + \frac{A_1}{k} + \frac{A_2}{k^2} + \dots.$$

Note that we can simply allow u to be an order one function independent of ϵ as, if we did assume an asymptotic expansion for u, there would be ambiguity since such behaviour is already accounted for in the expansion of A.

Using these assumptions and considering the leading order problem for, (3.12) implies that u_0 satisfies the *Eikonal equation*

$$|\boldsymbol{\nabla} u|^2 = 1. \tag{3.13}$$

Then the successive terms in the amplitude expansion satisfy the *transport equations*

$$\begin{aligned} 2\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} A_0 + A_0 \nabla^2 u &= 0, \\ 2\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} A_n + A_n \nabla^2 u &= \mathrm{i}\nabla^2 A_{n-1}, \quad n \geq 1. \end{aligned}$$

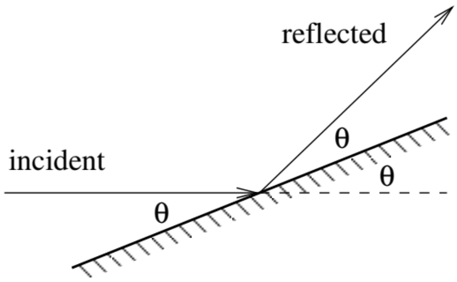

Figure 3.4: Illustration of Snell's law.

The Eikonal equation (3.13) may be solved exactly as in Example 1. The rays correspond to light rays and all the familiar properties of geometrical optics, for example that light travels in straight lines, follow from the solution of Charpit's equations.

Example 3. Reflecting plane waves

One obvious solution of (3.13) is $u = x$, which corresponds to a plane wave moving in the x-direction. Now we examine what happens if such a wave impinges on a reflecting wall given by a curve Γ in the (x, y)-plane. We decompose the state variable ϕ into an *incident* wave ϕ_I, namely a plane wave with constant amplitude a, and a *reflected* wave ϕ_R:

$$\phi = \phi_I + \phi_R, \quad \phi_I = a\mathrm{e}^{\mathrm{i}kx}.$$

Now we apply the WKBJ ansatz to ϕ_R:

$$\phi_R = A\mathrm{e}^{\mathrm{i}ku(x,y)}.$$

The boundary conditions depend on the physical situation being modelled and exactly what the state variable ϕ represents. The simplest case is to impose the Dirichlet condition $\phi = 0$ on Γ, which leads to

$$u = x, \quad A_0 = -a \quad \text{on } \Gamma. \tag{3.14}$$

Other possibilities are $\partial\phi/\partial n = 0$ or the "Robin condition" $\partial\phi/\partial n + \lambda\phi = 0$, but it is readily verified that the leading-order boundary conditions (3.14) are unchanged in either case.

The solution of Charpit's equations for the Eikonal equation was already obtained in Example 1:

$$p = p_0(s), \quad q = q_0(s), \quad x = x_0(s) + p_0(s)\tau, \quad y = y_0(s) + q_0(s)\tau, \quad u = u_0(s) + \tau.$$

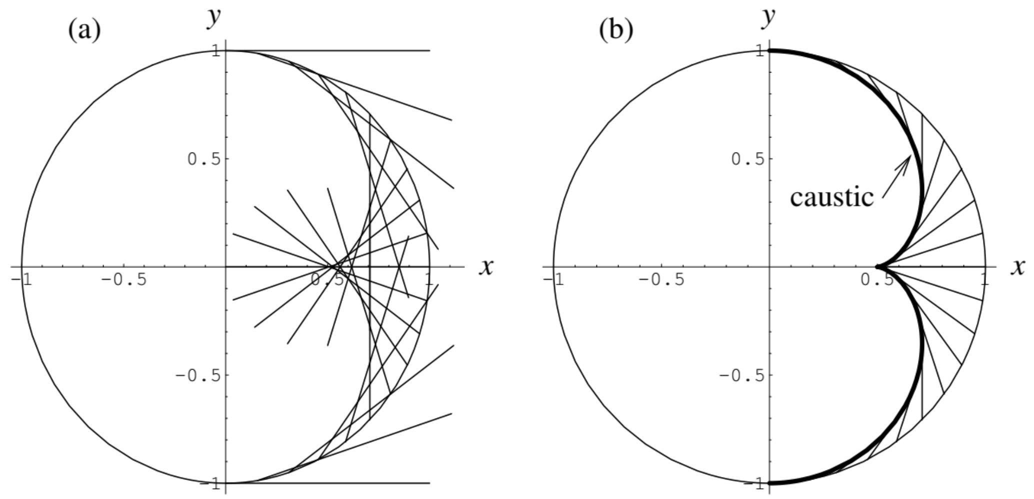

Figure 3.5: (a) Reflected rays for Example 4. (b) Rays truncated by a caustic on which $J = 0$.

For simplicity we suppose that s parametrises arc-length along Γ so we can write $x_0' = \cos\theta$, $y_0' = \sin\theta$ where θ is the angle between Γ and the x-axis. The boundary condition (3.14) implies that $u_0(s) = x_0(s)$, and then p_0 and q_0 are obtained from

$$x_0' p_0 + y_0' q_0 = x_0', \quad p_0^2 + q_0^2 = 1.$$

This system has two solutions, one of which is $p_0 = 1$, $q_0 = 0$, corresponding to the incident wave. The reflected wave is given by the other solution

$$p_0 = 1 - 2(y_0')^2 = \cos(2\theta), \quad q_0 = 2x_0' y_0' = \sin(2\theta).$$

Hence the reflected ray makes an angle of 2θ with the x-axis. This is *Snell's law*: as illustrated in Figure 3.4, it implies that the angle of incidence to the wall equals the angle of reflection from it.

Note that in analysing this problem we have not had to consider any shocks created by interaction of the waves. This is because the underlying physical problem is modelled by Helmholtz equation, which because it is linear, we could break the solution into an incident wave ϕ_I and a reflected wave π_R and the fact that these waves might simultaneously occur at the same position and time is physically acceptable.

Example 4. The caustic in a teacup
As a special case of Example 3, we now consider the case where Γ is the unit circle, parametrised (say) by $x_0(s) = \cos(s)$, $y_0(s) = \sin(s)$. The rays reflected from the

circle are given parametrically by

$$x = \cos(s) - \tau\cos(2s), \quad y = \sin(s) - \tau\sin(2s).$$

and the phase is then given by $u = \cos(s) + t$.

As shown in Figure 3.5(a), the reflected waves start to cross a finite distance from the circle. The envelope of the rays is where the Jacobian

$$J = \frac{\partial x}{\partial s}\frac{\partial F}{\partial q} - \frac{\partial y}{\partial s}\frac{\partial F}{\partial p} = 0,$$

which may be solved to give

$$\tau = \tfrac{1}{2}\cos(s).$$

Thus $J = 0$ on the curve given parametrically by

$$x = \cos(s)\left(1 - \tfrac{1}{2}\cos(2s)\right), \quad y = \sin(s) - \tfrac{1}{2}\cos(s)\sin(2s) = \sin^3(s), \qquad (3.15)$$

and such a curve is called a *caustic*. A single-valued solution may be obtained by truncating the rays when they reach the caustic, as shown in Figure 3.5(b). A caustic similar to this may often be observed if a bright light is shone on a cup of coffee.

An alternative way to find the caustic is to note that the rays are given by

$$F(x, y; s) = x\sin(2s) - y\cos(2s) - \sin(s) = 0.$$

Their envelope may therefore be found by solving

$$F = \frac{\partial F}{\partial \lambda} = 0,$$

that is

$$x\sin(2s) - y\cos(2s) - \sin(s) = 2x\cos(2s) + 2y\sin(2s) - \cos(s) = 0,$$

whose simultaneous solution reproduces (3.15).

Example 4 illustrates that a single-valued ray solution may be obtained by truncating rays at any caustic where the Jacobian is zero. It may be shown that the asymptotic ansatz (3.11) breaks down, with $A \to \infty$ as the caustic is approached. The method of matched asymptotic expansions yields the appropriate correction in the neighbourhood of a caustic and allows the behaviour in the dark zone beyond the caustic (corresponding to *complex rays*) to be found.

References and further readings

1 A. R. Conn, N. I. M. Gould and Ph. L. Toint, Trust-Region Methods, SIAM 2000.

2. J. Dennis and R. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear equations, (republished by) SIAM, 1996.

3. R. Fletcher, Practical Methods of Optimization, 2nd edition Wiley, 1987 (republished in 2000).

4. P. Gill, W. Murray and M. H. Wright, Practical Optimization, Academic Press, 1981.

5. N. I. M. Gould, An Introduction to Algorithms for Continuous Optimization, 2006. Available for download at http://www.numerical.rl.ac.uk/nimg/course/lectures/paper/paper.pdf.

6. J. Nocedal and S. J. Wright, Numerical Optimization, Springer Verlag, 1999 (1st edition) or 2006 (2nd edition).

7. Mathematical Modeling and Computation of Real-Time Problems: An Interdisciplinary Approach (Mathematical Engineering, Manufacturing, and Management Sciences) 1st Edition by Rakhee Kulshrestha, Chandra Shekhar, Madhu Jain, Srinivas R. Chakravarthy