River flow; porous-medium flow; two-phase flow.

Nonlinear waves

Traffic flow, river flow, two-phase flow

A relative simple model of flow of traffic on a single lane road describes the density of traffic as a continuous variable and hence the movement as being like a fluid. Such models are rather over simplified since stochastic behaviour of drivers is a large element in the dynamics, however we present it here as it is easy to visualise. Taking the density to be ρ (cars/km) and the speed to be u (km/s) we exploit "conservation of cars" on a 1-D road to give

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

The simplest model is then completed by including a constitutive equation which, for example assumes that drivers travel at a speed determined by the density of traffic on the road. Such a constitutive relation might assume that cars have a maximum speed limit and they go slower the higher the traffic density and might take the form

$$u = u_{max}(1 - \rho/\rho_{max})$$

where, u_{max} is the maximum speed and ρ_{max} is a constant representing the maximum density cars will fit on the road (and at that density the traffic is stationary). The resulting equation for the traffic density is the kinematic equation

$$\frac{\partial \rho}{\partial t} + u_{max}\frac{\partial(\rho - \rho^2/\rho_{max})}{\partial x} = 0.$$

For flow in an estuary or a river we can return to the ideas of "Stokes waves" given earlier in the course. The main difference here is that the tidal variations on an estuary (the height variations) can get to be as large as the depth of the water. Hence a model should account for this rather than consider the very small waves that were analysed in the section on Stokes waves. The basic model assumes that in the inviscid fluid the velocity is approximately uniform with depth. Hence if h is the water depth, u is the average velocity, then the mass conservation equation can be integrated in the vertical direction between the estuary bed at $z = 0$ to the water surface $z = y$ to get

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} = 0$$

while conservation of momentum with just gravity acting on the inviscid fluid, again integrated from $z = 0$ to $z = h$, is

$$\frac{\partial(hu)}{\partial t} + \frac{\partial(hu^2)}{\partial x} + \frac{\partial(gh^2)}{\partial x} = 0$$

This system is typically referred to as the "shallow water equations". The previous two equations are in conservation form but quite often they are presented in a form found by subtracting u times the first equation from the second to give the system

$$\frac{\partial h}{\partial t} + +u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

In many oil fields, once the oil has finished coming out of wells under its own pressure, a second phase is reached as water is pumped into the oil reservoir to push a further significant fraction of the oil out (there are interesting problems to do with how to extract even more oil once the water pumping has ceased to be effective). To model water pushing oil out of rock we start by assuming both fluids are incompressible but that at any point there is a saturation, S, which is the volume fraction of water in the rock pores (and hence the volume fraction of oil in the pores is $(1-S)$). Conservation of mass for each fluid therefore gives

$$\frac{\partial(S\rho_{water})}{\partial t} + +\frac{\partial(S\rho_{water}\mathbf{q}_{water})}{\partial x} = 0, \qquad \frac{\partial((1-S)\rho_{oil}}{\partial t} + +\frac{\partial((1-S)\rho_{oil}\mathbf{q}_{oil})}{\partial x} = 0$$

where $\mathbf{q}_{water}$ and $\mathbf{q}_{oil}$ are the flow of the respective fluid phases (water or oil). For momentum balance we use "Darcy flow" to describe the flow of each phase.

$$\mathbf{q}_{oil} = \frac{-K_{oil}}{\mu_{oil}}\frac{\partial p_{oil}}{\partial x}, \qquad \mathbf{q}_{water} = \frac{-K_{ware}}{\mu_{water}}\frac{\partial p_{water}}{\partial x}$$

where K_{oil} and K_{water} are the "permeabilities" of the fluid and are strong function of S at any point (eg $K_{water} \to 0$ as $S \to 0$, $K_{oil} \to 0$ as $S \to 1$). The μ are the respective fluid viscosities and the p are the pressures in the fluids. If the capillary pressure between oil and water is small (not usually true) then we assume that

$$p_{oil} = p_{water}.$$

From this we find the following equations hold

$$S_t + (SK_{water}p_x)x = 0, \qquad ((SK_{water} + (1-S)K_{oil})p_x)_x = 0.$$

If we now integrate the final equation and impose a fixed total flux of fluid, Q at some point x, then

$$p_x = \frac{Q}{(SK_{water} + (1-S)K_{oil})}$$

and the saturation is governed by a single equation

$$S_t + \left(\frac{QSK_{water}}{(SK_{water} + (1-S)K_{oil})}\right)_x = 0.$$

This is called the Buckley-Leverett model of 2-phase flow.

Classification of systems of PDEs

Throughout this course we have been looking at a number of different PDE problems. We now turn to discuss how to classify the different types of PDEs and to discuss the consequences in terms of the initial and boundary data that should be imposed to make the problem wellposed. When considering classifying PDEs the two approaches are to look at semi-linear second order equations or systems of quasi-linear first order equations. Here we adopt the latter method as it allows us to consider large systems and the former method can be put into this same framework. We shall simplify the discussion by working with just two independent variables x, y with a dependent vector $\mathbf{z}(x, y)$ of length n and make a few comments about how some of the ideas relate to systems in higher dimensions.

We consider the quasi-linear system

$$\mathbf{A}(x, y, \mathbf{z})\frac{\partial \mathbf{z}}{\partial x} + \mathbf{B}(x, y, \mathbf{z})\frac{\partial \mathbf{z}}{\partial y} = \mathbf{C}(x, y, \mathbf{z})$$

where $\mathbf{A}$, $\mathbf{B}$ are matrices of size $(n \times n)$ and $\mathbf{C}$ is a vector of length n.

Many second order linear equations can be written in this form but care must be taken to ensure the resulting system is just 2×2. There are many physical systems where the governing laws are derived using a number of first order equations although the result in often presented as a second order equation. (for example Newton's second law: $\dot{x} = v$, $\dot{v} = \text{Force}/m$).

Simple examples are:

Laplace's equation $z_{1_{xx}} + z_{1_{yy}} = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The wave equation $z_{1_{xx}} = z_{1_{yy}}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the heat equation $z_{1_{xx}} - z_{1_y} = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_y = \begin{pmatrix} z_2 \\ 0 \end{pmatrix}$$

The classification of such equations is made by locally freezing the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ and examining the characteristics of the resulting linearised system.

Method of characteristics

For the simplest case of a single quasi-linear PDE the characteristics are relatively easy to find. Given

$$a(x, y, z)z_x + b(x, y, z)z_y = c(x, y, z)$$

we seek to change coordinates from x, y to s, r such that the PDE reduces to a PDE. Hence using the chain rule

$$x_s z_x + y_s z_y = z_s$$

we see this equivalent to the PDE if we choose

$$\frac{\partial x}{\partial s} = a(x, y, z) \qquad \frac{\partial y}{\partial s} = b(x, y, z) \qquad \frac{\partial z}{\partial s} = c(x, y, z).$$

We can then observe that the solution can be obtained by solving this system of ODES (the variable r is only a parameter in them). The system will require data at some point $s = 0$ and this is provided by the initial data where z is given on a curve $f(x, y) = 0$ which we can parametrises by the variable r.

The resulting curves (x, y), where r is constant, are the "projected characteristics" of the system and indicate the path along which the "information" from the initial data travels (formally the "characteristics" are the curves (x, y, u) where r is constant, but this distinction is seldom made) . The local slope of these projected characteristics is given by

$$\frac{dy}{dx} = \frac{y_s}{x_s} = \frac{b}{a} = \lambda$$

or

$$\lambda a - b = 0.$$

(note a simple check on what slope *lambda* represents is to note the operators are $\mathbf{A}\partial/\partial x$ and $\mathbf{B}\partial/\partial y$ so that the dimensions are $\mathbf{A}/dx$ and $\mathbf{B}/dy$ Hence dimensionally the equation for the slope must be $|\mathbf{A}dy/dx - \mathbf{B}| = 0$.)

This approach also demonstrated that the initial data must not be given along a line that is a characteristics (ie initial data must not a curve with a tangent vector $x_r = a$, $y_r = b$ as, unless we are very lucky, the data for z will be inconsistent with $z_r = c$.

One final note is that this method parametrises the solution by s, r and it is not obvious that we will have a 1-1 mapping of s, r to x, y. we revisit this problem later.

Solve the PDE

$$u_t + u\,u_x = 1$$

for $u(x,t)$ in $t > 0$, subject to the initial condition $u = x$ on $t = 0$. The characteristics are given by

$$\frac{dt}{dr} = 1, \qquad \frac{dx}{dr} = u, \qquad \frac{du}{dr} = 1$$

and the initial data may be parametrised by

$$t = 0, \qquad x = s, \qquad u = s \quad \text{at } r = 0.$$

Solving for t first, we see that $t = r + A(s)$ and the initial data then makes $t = r$ and thus we may replace r by t henceforth. The initial-value problem for u has the solution

$$u = s + t$$

so that the problem for x becomes

$$\frac{dx}{dt} = s + t \qquad \text{with } x = s \text{ when } t = 0$$

whose solution is

$$x = s + st + \frac{1}{2}t^2$$

Now we can now find s in terms of u, t, x and use it to obtain the solution in explicit form

$$u = \frac{x + t + \frac{1}{2}t^2}{1 + t}$$

Returning to the classification problem for a system of first order PDEs. we now perform a procedure similar to that for a single first order PDE. Note that if we had n independent PDEs then there would be n independent characteristics at each point in (x, y). Our approach is simply to find the slope of these characteristics at each point (x, y) and we denote this slope locally by $\lambda = dy/dx$. This slope must satisfy the condition

$$\|\mathbf{B} - \lambda\mathbf{A}\| = 0.$$

This is a polynomial for λ and the roots dependent of the coefficients and hence possibly on x, y and $\mathbf{z}$. We classify the local behaviour of the PDE based on these roots.

Hyperbolic

If all the roots are real and distinct (no repeated roots) then we can imagine that information is travelling in various directions at any point. Formally the "information" is the set of left eigenvectors. Such a system is *hyperbolic* and exhibits wavelike behaviour with the waves travelling along the characteristics. Hence if data is specified on some curve it will only affect the solution in a limited region of x, y due to the wave behaviour. Typically we would have one of the variable being "time" and the other representing space. Note that a 'time-like" variable need not be actual time but can be a space coordinate.

Parabolic

Note if a degenerate case occurs and roots are equal then the system is *parabolic* if the eigenspace is also degenerate and hyperbolic otherwise.

Elliptic

If all the roots are complex (non-zero imaginary part) then there is no concept of propagating information. Such a system is *elliptic* and if data is specified on some curve then it will affect the solution everywhere in (x, y). This global influence of data makes such models very difficult to interpret as having a time-like variable since data affect both the future and the past. Such equations typically appear when considering steady state problems.

Mixed systems and other

There are of course a huge variety of other possible behaviours of the roots. A general rule is that a problem where one of independent variables is actually time then all the roots λ should be real, although not necessarily distinct, everywhere in (x, y) and $\mathbf{z}$.

Some general approximate rules for boundary conditions can be extracted from characteristics. If the system is hyperbolic then one variable is time-like and hence we want the problem to be posed so that events are not influenced by the future. This implies that as the time-like variable increases we want information to propagate along the characteristics from the past to future. At boundaries of the region this means that as time increases any characteristic entering the region from the boundary must be given data in the form of a boundary or initial condition.
(General rule: the number of conditions needed at a boundary equals the number of characteristics entering the solution region at that point as time increases)
Note: there is a further complication that the information must include some aspect of the eigenvector related to the particular characteristic. Furthermore there the conditions at the boundary cannot specify values of the information of characteristics leaving the region. This tells us that all variables should be specified initially and at boundaries the number of variables will change as the number of characteristics entering or leaving the region changes. Hyperbolic systems, due to their time-like variable, are typically posed on finite regions in the space variable and semi-infinite regions in the time-like variable.

If the system is elliptic then the slope of the characteristics, λ, all appear in complex conjugate pairs (note this requires that there are an even number of variables and assumes all parameters in the equation are real). For these systems it is necessary to impose conditions equal to half the number variables at each point on the boundary. Such problems are typically posed on finite regions of (x, y) although this is not always necessary.

References and further readings

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