# FINAL EXAMINATION

*Time: 3 Hours*

*Instruction to students: Attempt all questions*

*All these are examples and exercises we discussed in class. Good luck!*

## Question one (20 Marks)

Consider the function

$$f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2, \quad x = (x_1 \;\; x_2)^T \in \mathbb{R}^2.$$

(a) Compute the gradient vector and the Hessian matrix of $f$ at (any) $x \in \mathbb{R}^2$. Find all stationary points of $f$. Show that $x^* = (1 \;\; 1)^T$ is the unique global minimizer of $f$ and that the Hessian of $f$ at $x^*$ is positive definite.

(b) Show that the Hessian matrix $\nabla^2 f(x)$ of $f$ is singular if and only if $x$ satisfies the condition

$$x_2 - x_1^2 = 0.05.$$

Hence show that $\nabla^2 f(x)$ is positive definite for all $x$ such that $f(x) < 0.025$.

(c) Show that $f$ is not a convex function.

## Question two (20 Marks)

Show that the function

$$f(x) = (x_2 - x_1^2)^2 + x_1^5$$

has only one stationary point which is neither a local maximum nor a local minimum.

## Question three (20 Marks)

Apply Newton's method (without linesearch) to minimizing the univariate function

$$f(x) = \frac{11}{546}x^6 - \frac{38}{364}x^4 + \frac{1}{2}x^2, \quad x \in \Re,$$

starting from $x^0 = 1.01$ and let $\{x^k\}$ be the generated sequence of iterates.

Show that the Hessian matrix $\nabla^2 f(x)$ is positive definite for all $x$, and that the sequence $\{f(x^k)\}$ is monotonically decreasing. Prove that the limit points of the sequence of iterates $\{x^k\}$ are $+1$ and $-1$ as $k \to \infty$, and that $\nabla f(\pm 1) \neq 0$. (*hint*: think of the Newton iterate $x^{k+1}$ as a continuous function of $x^k$; or use numerical results carefully.)

## Question four (20 Marks)

Newton's method (without linesearch) is used to minimize the function of one variable

$$f(x) = x^6 - 14x^4 + 49x^2 - 36, \quad x \in \Re.$$

What is the order of convergence in the two cases when (a) the starting point $x^0$ is close to $\sqrt{7}$ and (b) $x^0$ is close to 0. Find a nonzero value of $x^0$ such that $x^{k+1} = -x^k$ on every iteration.

## Question five (20 Marks)

Determine if solutions to the following PDE problems are continuously dependent on their initial conditions:

$$(i)\quad \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \qquad (ii)\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (iii)\quad \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}$$

$$(iv)\quad \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad (v)\quad \frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} \qquad (vi)\quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$(vii)\quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \qquad (viii)\quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$