Classical Mechanics

About these notes

These are lecture notes for the ENG 607-CLASSICAL MECHANICS course, which is a third-year option in the mathematics syllabus at the University Level. In putting together these notes, I have drawn freely from the large literature on the subject; notably from the reading list below, but also from many other books and lecture notes (notably those of Paul Tod and David Tong). Familiarity with the first year Dynamics course and second year Calculus of variations option will be assumed, although we will essentially develop the theory *ab initio*. Starred paragraphs are not examinable. There will be class assignments together with a final examination later

Reading

- H. Goldstein, C. P. Poole, J. L. Safko, *Classical Mechanics*.
- L. D. Landau, E. M. Lifshitz, *Mechanics* (*Course of Theoretical Physics, Vol. 1*).
- N. M. J. Woodhouse, *Introduction to Analytical Mechanics*.
- V. I. Arnold, *Mathematical Methods of Classical Mechanics*.

Classical Mechanics

LECTURE 1

Introduction

This course is about the Lagrangian and Hamiltonian formulations of classical mechanics. These were introduced and developed in the late 18th and 19th centuries, and recast Newton's laws in different mathematical frameworks. The reader might immediately ask what the point of this is: what mileage do you get out of rewriting Newtonian mechanics in a different language? Moreover, why is it important to study this subject?

In order to answer these questions, let's begin with the Newtonian theory itself. Newtonian mechanics is an extremely accurate theory, valid over a vast range of scales, and is applicable to many classes of dynamical problems. The axioms are also clear and simple to state. Although the later developments of special relativity, general relativity and quantum mechanics undoubtedly provide a more accurate description of the real world, these are all much more complex descriptions of Nature. Scientists, even theoretical physicists, will often try to revert to using the techniques of classical mechanics where possible. Indeed, measuring the differences from Newtonian theory usually requires very well-designed and delicate experiments.

From a computational point of view Newtonian theory, as summarized for point particles in section 1, quickly becomes cumbersome and inefficient as the systems become more complicated. In particular Newton's laws require one to work in an inertial frame in Cartesian coordinates, which is often inconvenient. The Lagrangian and Hamiltonian formalisms, on the other hand, are coordinate independent, and provide a more elegant and computationally efficient framework in which to work. For example, it much easier to solve constrained systems, as there is no need to introduce constraint forces (such as the tension in a simple pendulum, or the constraint forces for a bead moving on a wire); rigid body motion is also easier to implement. The Lagrangian formulation also introduces a new fundamental principle: the *principle of least action*, also known as *Hamilton's principle*. This gives a surprising amount of insight into classical mechanics, for example making clear the relation between symmetries and conservation laws (via Noether's theorem).

Classical Mechanics

LECTURE 1

The ideas and techniques developed in the Lagrangian and Hamiltonian formulations of classical mechanics also generalize to other areas of theoretical physics. For example, there are Lagrangian and Hamiltonian descriptions of electromagnetism and general relativity, which play an important role in formulating those theories. The ideas and principles we shall encounter were also key to the development of quantum mechanics in the 20th century. Those who have studied quantum mechanics may have noticed that the theory looks nothing like Newtonian mechanics. In fact the Hamiltonian formulation of the latter is closely related to the quantum description of a non-relativistic particle you may have already seen. The principle of least action also led to Feynman's formulation of quantum mechanics as a path integral/sum over histories, which in turn has been central to the development of particle physics in the second half of the 20th century. Finally, Lagrangian and Hamiltonian mechanics have also had a significant impact on the development of various branches of pure mathematics, particularly geometry.

Hopefully this has at least partly addressed the questions posed at the beginning of the introduction.

1 Newtonian mechanics

We begin with an overview of Newtonian mechanics for point particles. Although much of this material may already be quite familiar, it will be important to have a good grasp of these basic notions when we come to generalize the results later in the course.

1.1 Reference frames

In classical mechanics space is modelled by $\mathbb{R}^3$, equipped with the usual Euclidean metric. More precisely, a *reference frame* $\mathcal{S}$ is specified by a choice of origin O, together with a choice of Cartesian coordinate axes. With respect to this frame one then writes a position vector as $\mathbf{r} = (x, y, z)$. The trajectory of a point particle is represented by a curve $\mathbf{r} = \mathbf{r}(t)$, parametrized by time t. The *velocity* of the particle in the frame $\mathcal{S}$ is then $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$, where a dot will denote derivative with respect to time. Similarly, its *acceleration* is $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$.

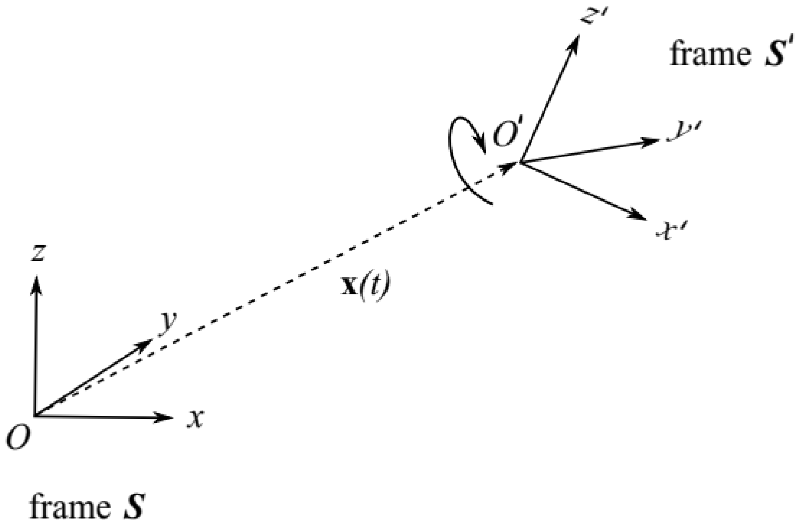

Figure 1: Relative to a choice of reference frame $\mathcal{S}$, the origin O' of another frame $\mathcal{S}'$ is at $\mathbf{x}(t)$, and the coordinate axes of $\mathcal{S}'$ may also rotate with respect to the axes of $\mathcal{S}$.

Of course the choice of frame is far from unique, and the motion of a particle viewed in two different frames can look very different. If we fix an initial choice of $\mathcal{S}$, then the origin O' of any other frame $\mathcal{S}'$ will be at some position $\mathbf{x}(t)$, measured with respect to the origin O of $\mathcal{S}$. Moreover, the coordinate axes of $\mathcal{S}'$ may rotate relative to the axes of $\mathcal{S}$ – see Figure 1. In particular if $\mathcal{S}$ and $\mathcal{S}'$ have the same origin (so $\mathbf{x}(t) = \mathbf{0}$ for all t), then the frames differ only by a rotation of the axes. A position vector $\mathbf{r}$ measured in $\mathcal{S}$ is then $\mathbf{r}' = (x', y', z') = \mathcal{R}\,\mathbf{r}$ as measured in $\mathcal{S}'$. Here $\mathcal{R} = \mathcal{R}(t)$ is a 3×3 orthogonal matrix, which in general can depend on time t. Mathematically the three-dimensional rotation group is $O(3) \subset GL(3, \mathbb{R})$, the set of real 3×3 matrices satisfying $\mathcal{R}^T\mathcal{R} = \mathcal{R}\mathcal{R}^T = \mathbb{1}$, where $\mathbb{1}$ denotes the 3×3 identity matrix. This ensures the transformation $\mathcal{R} \in O(3)$ between frames preserves Euclidean distances – all observers in Newtonian mechanics are postulated to measure the same distance between any two points. We shall describe the rotation group and rotating frames in much greater detail in section 4 when we come to discuss rigid body motion.

1.2 Newton's laws

Newton's laws of motion apply to *point particles*. These are objects whose dimensions may be neglected, to a good approximation, in describing their motion. For example, this is the case if the size of the object is small compared to the distances involved in the dynamics, *e.g.* the motion of a planet around the Sun. Of course, it is no good treating the Earth as a point particle if you want to understand the effects of its rotation. On the other hand, as we shall see in section 4 the centre of mass of an extended rigid body does behave like a point particle. Point particles have a *mass* m, and an associated *(linear) momentum* $\mathbf{p} = m\dot{\mathbf{r}}$.

The first of Newton's laws singles out a special equivalence class of reference frames, called *inertial frames*. These are defined by

> **N1**: Inertial frames exist. In such a frame an object either remains at rest or moves with constant momentum (uniform motion in a straight line), unless acted on by an external force.

Here the force that is acting is understood to have an identifiable source, *e.g.* gravity, electromagnetism, friction. A non-inertial frame $\mathcal{S}'$ is accelerating with respect to an inertial frame $\mathcal{S}$. That is, the origin O' of $\mathcal{S}'$ is accelerating with respect to O, or the axes of $\mathcal{S}'$ are rotating relative to the axes of $\mathcal{S}$. In the non-inertial frame a particle will appear to be acted on by "ficticious forces", for example the Coriolis force, or centrifugal force (more about this in section 4.2). A frame at rest on the surface of the Earth is a very good approximation to an inertial frame, ignoring the rotation of the Earth around its axis and its acceleration about the Sun.[1] Compare such a frame $\mathcal{S}$ to someone standing on a roundabout, whose frame $\mathcal{S}'$ rotates around a fixed vertical axis relative to $\mathcal{S}$: a stationary object in $\mathcal{S}$ will appear to be accelerating to the person on the roundabout.

In an inertial frame the dynamics of a particle is governed by

N2:
$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \ = \ \dot{\mathbf{p}}.$$

Assuming the mass m is constant the right hand side of Newton's second law is $\dot{\mathbf{p}} = m\ddot{\mathbf{r}}$, although one could also consider variable mass bodies, *e.g.* a rocket ship that expels the spent fuel. In this course we'll assume m is constant. The external force $\mathbf{F}$ can in general depend on the particle's position $\mathbf{r}$, its velocity $\dot{\mathbf{r}}$ (*e.g.* the drag force due to motion through a fluid), and on time t (*e.g.* a charged particle moving in a time-dependent electromagnetic field). Newton's second law is then a second order ODE for $\mathbf{r}(t)$. General theorems from the theory of differential equations guarantee

---

[1] The former does give rise to a very small measurable effect. For example, *Foucault's pendulum* detects the rotation of the Earth.

that under suitable conditions on the function $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, specifying the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ at some initial time $t = t_0$ gives a unique solution for the particle trajectory $\mathbf{r}(t)$.

Finally, if we have more than one particle, then

> **N3**: If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1.

In other words, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is often paraphrased by saying that every action has an equal and opposite reaction. There is also a *strong form* of Newton's third law:

> **N3′**: The force in **N3** above acts along the vector connecting particle 1 and particle 2.

That is, $\mathbf{F} = \mathbf{F}_{21} \propto (\mathbf{r}_1 - \mathbf{r}_2)$, where $\mathbf{r}_I$ denotes the position vector of particle I. Such forces are called *central forces*. The strong form of Newton's third law is true for the gravitational and electrostatic forces between particles:

Example (gravitational force between two point masses): According to Newton (and Hooke), the gravitational force on particle 1 due to particle 2 is given by

$$\mathbf{F}_{12} \quad = \quad -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \ . \tag{1.1}$$

Here m_I is the mass of particle I, and $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is Newton's gravitational constant. The Coulomb electrostatic force between two point charges e_1, e_2 takes a similar central inverse square law form, except unlike masses charges can be both positive and negative, leading to repulsive as well as attractive forces.

There are examples of forces that do not satisfy the strong form of Newton's third law, notably the magnetic component of the Lorentz force acting on a charged particle moving in the electromagnetic field generated by another charged particle. However, a proper discussion of this would take us too far into electromagnetic theory, and in any case requires special relativity (rather than Galilean relativity) to understand properly, so we will not discuss this further here.

1.3 Galilean transformations

Inertial frames are not unique. Rather there is an equivalence class of inertial frames, related to each other by *Galilean transformations*. Specifically, we have the following transformations of an inertial frame $\mathcal{S}$:

$$\begin{cases} \text{temporal translations, } t' = t - s, \text{ where } s \text{ is a constant,} \\ \text{spatial translations, } \mathbf{r}' = \mathbf{r} - \mathbf{x}, \text{ where } \mathbf{x} \text{ is a constant vector,} \\ \text{constant rotations, } \mathbf{r}' = \mathcal{R}\,\mathbf{r}, \text{ where } \mathcal{R} \in O(3) \text{ is a constant } 3 \times 3 \text{ orthogonal matrix,} \\ \text{Galilean boosts, } \mathbf{r}' = \mathbf{r} - \mathbf{v}t, \text{ where } \mathbf{v} \text{ is a constant velocity.} \end{cases}$$

These map uniform motion in $\mathcal{S}$ to uniform motion in $\mathcal{S}'$. Altogether they generate the 1+3+3+3 = 10-dimensional *Galilean group*.

Notice that in the first transformation we have been tacitly assuming there is a notion of *absolute time*, unique up to what we call time $t = 0$. In particular in Newtonian mechanics all observers measure the same time interval between any two events. Of course this is *not* true in special relativity (essentially it took until Einstein to realize that assuming there is an absolute notion of time *is* an assumption). Galilean boost transformations mean there is no absolute rest frame in Newtonian mechanics.

The first Galilean transformation is a statement of *homogeneity of time* (Newtonian physics is invariant under $t \to t - s$), while the second and third Galilean transformations say that space is respectively *homogeneous* and *isotropic*. These are symmetries of Galilean spacetime.

1.4 Closed systems and Galilean relativity

A *closed system* is one in which all forces are internal, acting between the constituents of the system. To be concrete let us consider a closed system of N point particles. We suppose these have constant masses m_I, $I = 1, \ldots, N$, and are at positions $\mathbf{r}_I(t)$ as measured in an inertial frame. Newton's second law **N2** for the system may be written

$$\mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t) \; = \; \dot{\mathbf{p}}_I \; = \; m_I \ddot{\mathbf{r}}_I \; , \tag{1.2}$$

where $\mathbf{F}_I$ denotes the total force on the Ith particle. For such a closed system we have

> *Galileo's principle of relativity*: Newtonian dynamics is invariant under Galilean transformations.

More precisely, if we apply the same Galilean transformation to each of the particle trajectories $\{\mathbf{r}_I(t)\}$ solving (1.2), then the resulting trajectories solve the *same* system of equations. For example, if $\{\mathbf{r}_I(t)\}$ solves (1.2) then $\{\mathbf{r}_I(t - s)\}$ must solve the same system, for all $s \in \mathbb{R}$. This invariance under temporal translations immediately implies that $\mathbf{F}_I$ must be independent of time t, so that we may write $\mathbf{F}_I = \mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$. Similarly, invariance under spatial translations and boosts means that the forces depend only on the relative positions $(\mathbf{r}_J - \mathbf{r}_K)$ and relative velocities $(\dot{\mathbf{r}}_J - \dot{\mathbf{r}}_K)$ of the particles, respectively. Finally, invariance under a rotation $\mathcal{R} \in O(3)$ means that $\mathbf{F}_I$ is further constrained to obey

$$\mathcal{R}\,\mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N) \;\; = \;\; \mathbf{F}_I(\mathcal{R}\,\mathbf{r}_1, \ldots, \mathcal{R}\,\mathbf{r}_N, \mathcal{R}\,\dot{\mathbf{r}}_1, \ldots, \mathcal{R}\,\dot{\mathbf{r}}_N) \; . \tag{1.3}$$

A closed system consisting of a *single* point particle is not very interesting: a little thought shows that the above constraints imply $\mathbf{F} = \mathbf{0}$ (a single particle cannot act on itself). Such a particle moves at constant momentum/velocity by **N1**. When treating a single point particle subject to an *external* force, we have in mind that (a) something else is responsible for producing that force, and (b) we are entirely ignoring the effect the particle has on whatever that something else is (*i.e.* we are ignoring its *back-reaction*). One can imagine such a model arising from a closed system, in which the particle has been singled out and the "external force" $\mathbf{F}$ is simply the sum of the forces on the particle from the rest of the system. Such effective descriptions of the dynamics typically won't be Galilean invariant in the way we have described.

Since this is quite a subtle point, it is worth pausing to discuss an explicit example. Consider a small body moving through a fluid with a linear drag force $\mathbf{F} = -b\,\dot{\mathbf{r}}$, where $b > 0$ is a constant. Newton's second law for our body is clearly not invariant under Galilean boosts, which take $\dot{\mathbf{r}} \to \dot{\mathbf{r}} - \mathbf{v}$. However, this is because the force law is only valid in the *rest frame* of the fluid, where the average velocity of the fluid particles is zero. The individual fluid particles will certainly be moving relative to each other, and colliding, but in the rest frame there is no net movement of the fluid as a whole. If the fluid has a net velocity $\mathbf{u}$ in our inertial frame (think of a steady flowing river), then the force law reads $\mathbf{F} = -b\,(\dot{\mathbf{r}} - \mathbf{u})$. Notice that here $\mathbf{F} = \mathbf{F}(\dot{\mathbf{r}}\,;\mathbf{u})$ also depends on the "external" parameter $\mathbf{u}$. Of course the Galilean boost also acts as $\mathbf{u} \to \mathbf{u} - \mathbf{v}$ on the fluid velocity, and with this understanding Newton's second law for the body is now Galilean invariant. The linear drag force is an *effective* force, which at a microscopic level arises due to many collisions of the body with the fluid particles. A more accurate (but completely impractical) description of the system would treat the fluid as a large number of point particles. These would all have initial velocity $\mathbf{u}$. The collisions between the fluid particles and our body, which can be treated as another particle, can then be described in a fully Galilean invariant way. The collisions will change the velocities of the fluid particles, and the linear drag force with a constant $\mathbf{u}$ is entirely ignoring this.

Of course, whether or not it is reasonable to neglect the effects of a particle on the rest of the system depends on the circumstances. It is perhaps worth pointing out though that closed systems can sometimes be rewritten/reinterpreted as non-closed systems. A good example is the *two-body problem*, of two point masses m_1, m_2 interacting via the gravitational force (1.1). This closed system may be rewritten as a problem for a *single* point particle (with mass given by the *reduced mass* $\mu = \frac{m_1 m_2}{m_1 + m_2}$) moving in an external central potential. We shall examine this example in more detail in section 2.5.