# Classical Mechanics

LECTURE 2

## 1.5 Energy and angular momentum

Consider a single point particle of mass $m$, acted on by an external force $\mathbf{F}$. The *work done* $W$ by the force $\mathbf{F}$ along a path connecting position vectors $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ is the line integral

$$W \equiv \int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \, . \tag{1.4}$$

If we now consider a trajectory of the particle $\mathbf{r}(t)$ satisfying Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$, we may compute

$$\begin{aligned} W &= \int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = m\int_{t_1}^{t_2} \ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = \frac{1}{2}m\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\,\mathrm{d}t \\ &= T(t_2) - T(t_1) \, . \end{aligned} \tag{1.5}$$

Here the particle trajectory begins at position $\mathbf{r}^{(1)} = \mathbf{r}(t_1)$ at time $t_1$, and ends at $\mathbf{r}^{(2)} = \mathbf{r}(t_2)$ at time $t_2$, and we have defined the *kinetic energy* of the particle as

$$T \equiv \frac{1}{2}m|\dot{\mathbf{r}}|^2 \, . \tag{1.6}$$

Notice that $T$ is not invariant under Galilean boosts, and so in general depends on the choice of inertial frame $\mathcal{S}$ we choose to measure it in. The equality in (1.5) is called the *work-energy theorem*: the work done by the force $\mathbf{F}$ along the particle trajectory is equal to the change in kinetic energy.

In the remainder of this course we will almost entirely focus on *conservative forces*. Recall that these arise from the gradient of a *potential function* $V = V(\mathbf{r})$ via

$$\mathbf{F} = -\nabla V(\mathbf{r}) \, . \tag{1.7}$$

In Cartesian coordinates $\mathbf{r} = (x, y, z)$ this reads $\mathbf{F} = (-\partial_x V, -\partial_y V, -\partial_z V)$. Notice in particular that $V$, and hence also $\mathbf{F}$, depends *only* on the position vector $\mathbf{r}$. Note also that $V$ is defined only up to an additive constant. It is natural to fix this freedom, where it makes sense to do so, by requiring that the potential is zero when the particle (or more generally distances between particles) is at infinity, *c.f.* (1.11) below. Conservative forces have the property that the work done is independent of the path from $\mathbf{r}^{(1)}$ to $\mathbf{r}^{(2)}$. This follows from the fundamental theorem of calculus:

$$\int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \nabla V\cdot \mathrm{d}\mathbf{r} = -V(\mathbf{r}^{(2)}) + V(\mathbf{r}^{(1)}) \, . \tag{1.8}$$

Combining with the work-energy theorem (1.5) and rearranging we thus deduce the *conservation of energy*

$$T(t_1) + V(\mathbf{r}(t_1)) \;=\; T(t_2) + V(\mathbf{r}(t_2)) \;\equiv\; E\ . \tag{1.9}$$

Thus the total energy

$$E \;\equiv\; T + V\ , \tag{1.10}$$

is constant along the trajectory.

The inverse square law force (1.1), that governs the gravitational or Coulomb interaction between point masses/charges, is conservative:

$$V(\mathbf{r}) \;=\; -\frac{\kappa}{r} \qquad \Longrightarrow \qquad \mathbf{F} \;=\; -\nabla V \;=\; -\frac{\kappa}{r^3}\mathbf{r}\ , \tag{1.11}$$

where $r = |\mathbf{r}|$ and $\kappa$ is a constant. Non-conservative forces include friction/drag forces that depend on $\dot{\mathbf{r}}$. In particular, the effective linear drag force discussed in the last subsection is not conservative. However, the *fundamental* forces in Nature seem to be conservative. For example, a body that experiences a frictional force will typically lose energy, but in practice we know what happens to this energy: it is converted into heat (average kinetic energy). Modelling this takes us outside the realm of classical mechanics and into thermodynamics. But at a fundamental level, we expect energy to be conserved.

Still focusing on the single point particle, recall that its *angular momentum* (about the origin $O$ of $\mathcal{S}$) is defined as

$$\mathbf{L} \;=\; \mathbf{r} \wedge \mathbf{p}\ . \tag{1.12}$$

We will sometimes write the cross product of two vectors using the Levi-Civita alternating symbol, discussed in the appendix at the end of these lecture notes. The angular momentum (1.12) is the *moment* of the momentum $\mathbf{p}$ about $O$. Similarly, the moment of the force about $O$ is called the *torque*

$$\boldsymbol{\tau} \;=\; \mathbf{r} \wedge \mathbf{F}\ . \tag{1.13}$$

Since $\dot{\mathbf{L}} = \dot{\mathbf{r}} \wedge \mathbf{p} + \mathbf{r} \wedge \dot{\mathbf{p}} = \mathbf{r} \wedge \dot{\mathbf{p}}$ (as $\mathbf{p} = m\dot{\mathbf{r}}$ is parallel to the velocity $\dot{\mathbf{r}}$), the moment of **N2** about the origin gives the equation

$$\boldsymbol{\tau} \;=\; \dot{\mathbf{L}}\ . \tag{1.14}$$

In particular *central forces* have $\mathbf{F} \propto \mathbf{r}$, so that the torque $\boldsymbol{\tau} = \mathbf{0}$. It then follows from (1.14) that central forces lead to *conservation of angular momentum*, $\dot{\mathbf{L}} = \mathbf{0}$.

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It is straightforward to extend these concepts to our closed system of $N$ point particles in section 1.4. If particle $J$ exerts a force $\mathbf{F}_{IJ}$ on particle $I$ for $J \neq I$, then by **N3** therefore $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$. Newton's second law for particle $I$ (1.2) then reads

$$\sum_{J \neq I} \mathbf{F}_{IJ} \;=\; \mathbf{F}_I \;=\; \dot{\mathbf{p}}_I \,. \tag{1.15}$$

We could also add an external force $\mathbf{F}_I^{\text{external}}$ to the left hand side, but the system would then not be closed. The *total momentum* of the system of particles is defined to be $\mathbf{P} = \sum_{I=1}^{N} \mathbf{p}_I$, and by summing (1.15) over all particles we deduce that

$$\dot{\mathbf{P}} \;=\; \sum_{I=1}^{N} \sum_{J \neq I} \mathbf{F}_{IJ} \;=\; \mathbf{0} \,. \tag{1.16}$$

Here the $N(N-1)$ terms in the sum of forces cancel pairwise due to **N3**: $\mathbf{F}_{IJ} = -\mathbf{F}_{JI}$. Thus for this closed system of point particles we see that the total momentum is conserved.

We may similarly define the *total angular momentum* about the origin $O$ as

$$\mathbf{L} \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \mathbf{p}_I \,. \tag{1.17}$$

As for the single point particle we compute

$$\dot{\mathbf{L}} \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \dot{\mathbf{p}}_I \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \sum_{J \neq I} \mathbf{F}_{IJ} \,. \tag{1.18}$$

In particular we have $\frac{1}{2}N(N-1)$ pairs of terms on the right hand side

$$\mathbf{r}_I \wedge \mathbf{F}_{IJ} + \mathbf{r}_J \wedge \mathbf{F}_{JI} \;=\; (\mathbf{r}_I - \mathbf{r}_J) \wedge \mathbf{F}_{IJ} \,, \tag{1.19}$$

where we have used **N3**. Imposing the *strong form* of Newton's third law **N3′** means that $\mathbf{F}_{IJ} \propto (\mathbf{r}_I - \mathbf{r}_J)$, and we see that the total angular momentum is conserved, $\dot{\mathbf{L}} = \mathbf{0}$.

One of the general results we shall derive in Lagrangian mechanics is that conservation of energy, momentum, and angular momentum for any closed system follow from the symmetries of Galilean spacetime described in section 1.3.

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## 2 Lagrangian mechanics

In this section we introduce the Lagrangian formulation of classical mechanics. The general theory is developed in sections 2.1 – 2.4, with a number of detailed examples presented in section 2.5. The reader should feel free to dip into the examples while digesting the general theory, especially on a first reading.

### 2.1 Generalized coordinates

In order to study the dynamics of a physical system, one first needs to describe its possible configurations. Any set of independent quantities that specify uniquely the position of the system at a given time are called *generalized coordinates*. These label the points of the *configuration space* $\mathcal{Q}$. The number of generalized coordinates is called the number of *degrees of freedom*.

For example, for the point particle discussed in section 1 we used Cartesian coordinates $\mathbf{r} = (x, y, z)$ with respect to an inertial frame $\mathcal{S}$. These coordinates obviously specify uniquely the position of the particle in $\mathbb{R}^3$ at a given time $t$, so $\mathcal{Q} = \mathbb{R}^3$ and the number of degrees of freedom is 3. For $N$ point particles we similarly used $N$ sets of Cartesian coordinates $\mathbf{r}_I = (x_I, y_I, z_I)$, $I = 1, \ldots, N$, giving a configuration space $\mathcal{Q} = \mathbb{R}^{3N}$ with $3N$ degrees of freedom.

For the point particle we may also consider changing to non-Cartesian coordinates. Consider the general coordinate change

$$q_i \ = \ q_i(x, y, z, t) \ , \qquad i \ = \ 1, 2, 3 \ , \tag{2.1}$$

which might depend explicitly on time $t$ (that is, the new coordinate system is moving relative to the original coordinate system). We thus replace the Cartesian coordinates $(x, y, z) \to (q_1, q_2, q_3)$. Introducing $x_1 = x$, $x_2 = y$, $x_3 = z$ we will also have the inverse coordinate transformation

$$x_i \ = \ x_i(q_1, q_2, q_3, t) \ , \qquad i \ = \ 1, 2, 3 \ . \tag{2.2}$$

This latter change of coordinates is said to be *non-singular* at a point $(q_1, q_2, q_3)$ if the *Jacobian determinant* $\det(\partial x_i/\partial q_j)$ is non-zero at that point. This condition plays a role when writing differential equations in different coordinate systems, as the Jacobian $\partial x_i/\partial q_j$ and its inverse $\partial q_i/\partial x_j$ enter the chain rule.[2]

A familiar example is *cylindrical polars*

$$x \ = \ \varrho \cos\phi \ , \qquad y \ = \ \varrho \sin\phi \ , \qquad z \ = \ z \ , \tag{2.3}$$

where $\varrho \geq 0$ and $\phi \in [0, 2\pi)$. Here $(q_1, q_2, q_3) = (\varrho, \phi, z)$, and the coordinates are adapted to rotation about the $z$-axis. Notice that $\varrho = 0$ (the $z$-axis) is a coordinate singularity, as $\det(\partial x_i/\partial q_j) = \varrho$

[2] * We may regard a coordinate transformation $(x_1, x_2, x_3) \to (q_1, q_2, q_3)$ at fixed time $t$ as a map $x : \mathbb{R}^3 \to \mathbb{R}^3$, with components $x_i = x_i(q_1, q_2, q_3)$. By the *inverse function theorem*, if the Jacobian determinant $\det(\partial x_i/\partial q_j)$ is non-zero at a point $P = (q_1, q_2, q_3)$, then there is an open set containing $P$ on which this map is invertible with differentiable inverse.

is zero at $\varrho = 0$. In practice this causes no problems provided one remembers it is there (for example it explains why the Laplacian $\partial_x^2 + \partial_y^2 = \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho + \frac{1}{\varrho^2}\partial_\phi^2$ looks singular at $\varrho = 0$).

Another natural coordinate system is *spherical polars*

$$x = r\sin\theta\cos\varphi\ , \qquad y = r\sin\theta\sin\varphi\ , \qquad z = r\cos\theta\ , \tag{2.4}$$

where $r \geq 0$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$. Here $(q_1,q_2,q_3) = (r,\theta,\varphi)$, with the coordinates adapted to rotation about the origin $r=0$. Again, there are coordinate singularities at $r=0$ and at $\theta = 0,\pi$, which are the zeros of the Jacobian determinant $\det(\partial x_i/\partial q_j) = r^2\sin\theta$.

The notion of generalized coordinates is particularly useful for *constrained* problems. You have already met many such problems in Dynamics. Perhaps the simplest example is the *simple pendulum*. Here a point mass $m$ swings on a rigid (but very light) rod of length $l$, freely pivoted at one end to the origin $O$, and constrained to move in a vertical plane containing the $z$-axis – see Figure 2. Taking the plane of motion to be the $(x,z)$ plane at $y=0$, the rod constrains the point mass coordinates to satisfy $x^2+z^2=l^2$. It is then convenient to describe the position of the system by the angle $\theta$ given by $x = l\sin\theta$, $z = -l\cos\theta$. Thus $\theta \in [-\pi,\pi)$ is a generalized coordinate for this problem, $q=\theta$, which has one degree of freedom. Since $\theta = \pi$ is identified with $\theta = -\pi$, notice that the configuration space is a circle, $\mathcal{Q} = S^1$.

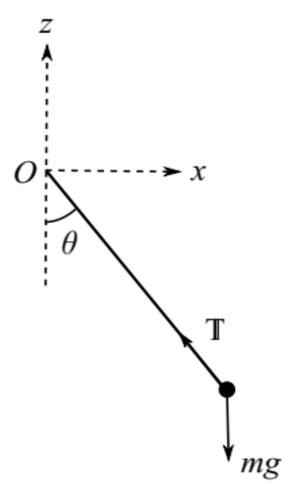


Figure 2: A simple pendulum with generalized coordinate $q = \theta$.

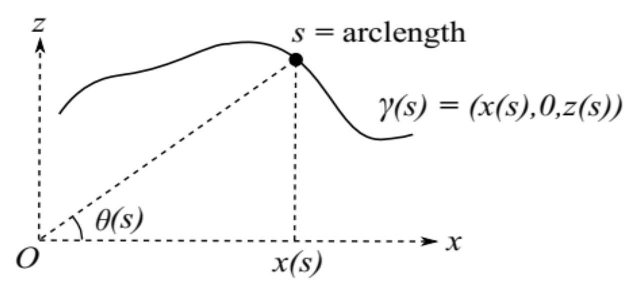


Figure 3: A bead moving on a fixed wire. We might use arclength $s$, the $x$-coordinate, or the angle $\theta$ as a generalized coordinate.

As another example consider a bead moving on a fixed wire. Here there are several natural choices of generalized coordinate. For simplicity suppose that the wire lies in the $(x,z)$ plane at $y=0$, as in Figure 3. We can specify the wire as the image of a parametrized curve $\gamma : [0,1] \to \mathbb{R}^3$ given by $\gamma(s) = (x(s),0,z(s))$. A natural generalized coordinate is then simply the arclength $s$. But there are other choices. Provided the projection onto the $x$ coordinate is injective (so that distinct points on the curve correspond to distinct values of $x$), we could also use the $x$ coordinate of the bead as a generalized coordinate. Another choice is to use the angle $\theta$ subtended from the origin, again provided the lines $\theta$ = constant through the origin intersect the curve in at most one point. For example, the latter would be a natural choice if the curve was a segment of a circle centred at the origin. This problem again has one degree of freedom, and the configuration space is an interval (assuming something stops the bead from falling off either end of the wire). Constrained motion on a surface is similar, with two degrees of freedom.

In section 4 we will consider rigid body motion. A rigid body has six degrees of freedom: three describe the position of the centre of mass, two determine the orientation in space of some axis fixed in the body, and there is one more describing rotations about that axis. For generalized coordinates it is then natural to use a mixture of Cartesian coordinates, describing the centre of mass motion, and angles, describing the orientation of the rigid body.

In general then a physical system will have $n$ degrees of freedom, described by $n$ real generalized coordinates $q_a = q_a(t)$, $a = 1, \ldots, n$. Their time derivatives $\dot{q}_a$ are called *generalized velocities*, $\ddot{q}_a$ are *generalized accelerations*, *etc.* We will use the boldface notation $\mathbf{q} = (q_1, \ldots, q_n)$, although as the examples above illustrate this is not to be confused (in general) with a vector in physical space $\mathbb{R}^3$. The dynamics of the system will trace out a curve $\mathbf{q}(t)$ in the configuration space $\mathcal{Q}$, which one could think of as a *quasi-particle* moving on $\mathcal{Q}$, whose position at any time determines the configuration of the whole system.[3] However, to avoid any possible confusion we shall generally refer to $\mathbf{q}(t)$ as the *system trajectory*, or *path*.

* The intuitive ideas of configuration space $\mathcal{Q}$ and generalized coordinates we have described can be made much more mathematically precise. A more rigorous account of a general framework for classical mechanics would take $\mathcal{Q}$ to be a *differentiable manifold*. Roughly, these are topological spaces that look locally like $\mathbb{R}^n$ (the open sets are homeomorphic to open sets in $\mathbb{R}^n$), but globally they can have much more non-trivial topology. Familiar examples are smooth curves and surfaces in $\mathbb{R}^3$ (with $n = 1$, $n = 2$ respectively). The study of geometry and calculus on these spaces is called *differential geometry*. However, expanding on these ideas would take us too far from our main subject, and in any case one doesn't need the full machinery of differential geometry to understand the fairly simple systems we shall study in these lectures. Those wishing to see more details might consult chapter 8 of *Introduction to Analytical Dynamics* by N. M. J. Woodhouse, or the book *Mathematical Methods of Classical Mechanics* by V. I. Arnold.

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[3] For example, the configuration space for two point particles moving in $\mathbb{R}^3$ is $\mathcal{Q} = \mathbb{R}^6$, which may be coordinatized by Cartesian coordinates $\mathbf{q} = (x_1, y_1, z_1, x_2, y_2, z_2)$ for the two particles. The position of the quasi-particle in $\mathcal{Q}$ then determines both physical positions of the two actual partices in $\mathbb{R}^3$.