Classical Mechanics

LECTURE 6

3 Small oscillations

Having set up appropriate generalized coordinates, derived the Lagrangian, and hence Lagrange equations of motion, the next task in analysing any physical system is to identify its *equilibrium configurations*. These are constant solutions $\mathbf{q}(t) = \mathbf{q}^{(0)}$ to the equations of motion. Of course there may be no such solutions, but sometimes there can be many, and having found an equilibrium point the next question is whether or not it is *stable*. That is, does a small perturbation of the system tend to return it to the equilibrium point, or drive it away? For stable equilibria we will show that small oscillations are described by a set of decoupled harmonic oscillators.

3.1 Equilibria and quadratic Lagrangians

Consider a general Lagrangian for a system with n degrees of freedom of the form

$$L \;=\; T - V \;=\; \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{q})\dot{q}_a\dot{q}_b - V(\mathbf{q}) \;. \tag{3.1}$$

Here without loss of generality we take $T_{ab}(\mathbf{q}) = T_{ba}(\mathbf{q})$ to be symmetric – compare to the discussion around (2.46). The Lagrange equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{b=1}^{n} T_{ab}\,\dot{q}_b\right) - \frac{1}{2}\sum_{b,c=1}^{n}\frac{\partial T_{bc}}{\partial q_a}\dot{q}_b\dot{q}_c \;=\; -\frac{\partial V}{\partial q_a} \;, \qquad a = 1,\ldots,n \;. \tag{3.2}$$

By definition an equilibrium point has $\mathbf{q}(t) = \mathbf{q}^{(0)} \in \mathcal{Q}$ independent of time t, so that $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = \mathbf{0}$ throughout the motion. It follows that the left hand side of (3.2) is zero at an equilibrium point, and hence defining the *generalized force* as

$$\mathbf{F} \;\equiv\; -\frac{\partial V}{\partial \mathbf{q}} \;, \tag{3.3}$$

we see this is precisely a *critical point* of the potential V, or equivalently a point at which the generalized force is zero.

In order to analyse the dynamics near to such a point we make a Taylor expansion of the Lagrangian (3.1) around $\mathbf{q}^{(0)}$. Without loss of generality we choose coordinates so that $\mathbf{q}^{(0)} = \mathbf{0}$, so as to simplify the following formulae. We then expand

$$\begin{aligned} T \;&=\; \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{0})\dot{q}_a\dot{q}_b + O(q^3) \;, \\ V \;&=\; V(\mathbf{0}) + \frac{1}{2}\sum_{a,b=1}^{n}\frac{\partial^2 V}{\partial q_a \partial q_b}(\mathbf{0})q_a q_b + O(q^3) \;, \end{aligned} \tag{3.4}$$

in a hopefully obvious notation. In particular we assume that both $\mathbf{q}$ and $\dot{\mathbf{q}}$ are small, and have neglected all terms of order 3 and higher. Notice that the linear term in the expansion of V is zero precisely because we are at a critical point. We may effectively drop the constant potential energy

term $V(\mathbf{0})$ as it does not contribute to the Lagrange equations of motion. Thus the dynamics near to the equilibrium point is governed by the *quadratic Lagrangian*

$$L_{\text{quadratic}} \;=\; \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{T}_{ab}\,\dot{q}_a\dot{q}_b - \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{V}_{ab}\,q_a q_b \;, \tag{3.5}$$

where $\mathcal{T}_{ab} = T_{ab}(\mathbf{0})$, $\mathcal{V}_{ab} = \frac{\partial^2 V}{\partial q_a \partial q_b}(\mathbf{0})$ are the components of constant $n \times n$ symmetric matrices. We write these matrices as $\mathcal{T} = (\mathcal{T}_{ab})$, $\mathcal{V} = (\mathcal{V}_{ab})$. We shall study the solutions to (3.5) in the next subsection, but we should bear in mind that these solutions will only describe the approximate dynamics near to equilibrium if they remain small. Otherwise the higher order terms we have neglected will become relevant.

3.2 Normal frequencies and normal modes

The Lagrange equations of motion for the quadratic Lagrangian (3.5) are

$$\sum_{b=1}^{n} \mathcal{T}_{ab}\,\ddot{q}_b \;=\; -\sum_{b=1}^{n} \mathcal{V}_{ab}\,q_b \;, \qquad a \;=\; 1,\ldots,n \;, \tag{3.6}$$

which are a set of n linear homogeneous second order ODEs in $\mathbf{q}(t)$ with constant coefficients. We may write these in matrix form

$$\mathcal{T}\ddot{\mathbf{q}} \;=\; -\mathcal{V}\,\mathbf{q} \;. \tag{3.7}$$

Notice that in this notation the kinetic energy is $T = \frac{1}{2}\dot{\mathbf{q}}^T\mathcal{T}\dot{\mathbf{q}}$, while the potential energy is $V = \frac{1}{2}\mathbf{q}^T\mathcal{V}\,\mathbf{q}$. Provided $\mathcal{T}$ is invertible we may solve (3.7) in terms of $\ddot{\mathbf{q}}$ as

$$\ddot{\mathbf{q}} \;=\; -\mathcal{T}^{-1}\mathcal{V}\,\mathbf{q} \;. \tag{3.8}$$

In physical situations $\mathcal{T}$ will be positive definite, and hence invertible, because the kinetic energy T is positive definite. We thus henceforth assume that $\mathcal{T}$ is positive definite.

We may now solve (3.7) using generalized eigenvectors and eigenvalues, as follows. We seek solutions of the form

$$\mathbf{q}(t) \;=\; f(t)\boldsymbol{\alpha} \;, \tag{3.9}$$

where $\boldsymbol{\alpha}$ is a constant (non-zero) vector. With this ansatz the equation of motion (3.7) becomes

$$\ddot{f}(t)\mathcal{T}\boldsymbol{\alpha} \;=\; -f(t)\mathcal{V}\,\boldsymbol{\alpha} \;. \tag{3.10}$$

Since $\mathcal{T}$ is invertible, so $\mathcal{T}\boldsymbol{\alpha}$ is non-zero, this implies that

$$\ddot{f} \;=\; -\lambda f \;, \qquad \text{where} \qquad (\lambda\mathcal{T} - \mathcal{V})\boldsymbol{\alpha} \;=\; \mathbf{0} \;, \tag{3.11}$$

and λ is a constant. It follows that $\boldsymbol{\alpha}$ lies in the kernel of $\lambda\mathcal{T} - \mathcal{V}$, and hence the determinant of this matrix must be zero.

This motivates defining the *characteristic equation* of the system to be

$$\det(\lambda\mathcal{T} - \mathcal{V}) \;=\; 0 \;. \tag{3.12}$$

Since this is a polynomial in λ of degree n, we will have n solutions $\lambda_1, \ldots, \lambda_n$.[7] These are precisely the eigenvalues of $\mathcal{T}^{-1}\mathcal{V}$. The corresponding eigenvectors are $\boldsymbol{\alpha}_a$, $a = 1, \ldots, n$, satisfying

$$(\lambda_a\mathcal{T} - \mathcal{V})\,\boldsymbol{\alpha}_a \;=\; \mathbf{0} \;, \tag{3.13}$$

and the equation of motion (3.7) is solved by $\mathbf{q}(t) = f_a(t)\boldsymbol{\alpha}_a$, where

$$\ddot{f}_a \;=\; -\lambda_a f_a \;. \tag{3.14}$$

Next we note that the matrix $\mathcal{T}^{-1}\mathcal{V}$ is diagonalizable. This follows since it is similar to the matrix $\mathcal{T}^{-1/2}\mathcal{V}\mathcal{T}^{-1/2}$, which is symmetric as the square root $\mathcal{T}^{-1/2}$ is symmetric[8], and real symmetric matrices are diagonalizable. There is hence a basis of $\mathbb{R}^n$ consisting of eigenvectors $\{\boldsymbol{\alpha}_a\}$ of $\mathcal{T}^{-1}\mathcal{V}$. Hence $\mathbf{q}_a(t) = f_a(t)\boldsymbol{\alpha}_a$ are n linearly independent solutions to (3.7), and the general solution is a linear combination of these.[9]

An important physical fact is that the eigenvalues we just defined are *real*. This is a simple consequence of the fact that $\mathcal{T}$ and $\mathcal{V}$ are real and symmetric, and $\mathcal{T}$ is positive definite. Indeed, taking the dot product of (3.13) with the complex conjugate $\bar{\boldsymbol{\alpha}}_a$ of $\boldsymbol{\alpha}_a$ we have $\bar{\boldsymbol{\alpha}}_a^T\mathcal{V}\,\boldsymbol{\alpha}_a = \lambda_a\bar{\boldsymbol{\alpha}}_a^T\mathcal{T}\boldsymbol{\alpha}_a$. But for any real symmetric matrix $\mathcal{S}$ the quantity $\bar{\boldsymbol{\alpha}}_a^T\mathcal{S}\,\boldsymbol{\alpha}_a$ is real, as follows simply by showing it is equal to its complex conjugate. Thus $(\bar{\lambda}_a - \lambda_a)\bar{\boldsymbol{\alpha}}_a^T\mathcal{T}\boldsymbol{\alpha}_a = 0$, and since $\mathcal{T}$ is positive definite we deduce that $\bar{\lambda}_a = \lambda_a$.

If $\lambda > 0$ then the solution to $\ddot{f} = -\lambda f$ is

$$f(t) \;=\; A\cos(\omega t + \beta) \;, \tag{3.15}$$

where we have written $\lambda = \omega^2$, A, β are constants of integration, and without loss of generality we take $\omega > 0$, $A > 0$. Thus the corresponding eigenmode executes simple harmonic oscillations, with *amplitude* A and *angular frequency* ω. In fact we shall simply refer to ω as the *frequency* of the mode. The modulus of the solution (3.15) is bounded by A for all times t.

On the other hand if $\lambda = 0$ then one obtains a linearly growing solution $f(t) = A + Bt$, while $\lambda < 0$ leads to exponential growth $f(t) = A\exp(\sqrt{-\lambda}t) + B\exp(-\sqrt{-\lambda}t)$. In particular in the latter case the mode quickly becomes large and the quadratic approximations we made in section 3.1 break down. If the system possesses an eigenvalue $\lambda < 0$ we hence say that it has a

[7] Notice that $\det(\lambda\mathbb{1} - \mathcal{T}^{-1}\mathcal{V})$ is the *characteristic polynomial* of the matrix $\mathcal{T}^{-1}\mathcal{V}$ you defined in first year linear algebra, and $\lambda_1, \ldots, \lambda_n$ are its roots.

[8] * For completeness: since $\mathcal{T}$ is positive definite and symmetric, so is $\mathcal{T}^{-1}$. It is hence diagonalizable, $\mathcal{T}^{-1} = \mathcal{P}D\mathcal{P}^T$, with $\mathcal{P} \in O(n)$ orthogonal and $D = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ diagonal with positive eigenvalues $\mu_a > 0$, $a = 1, \ldots, n$. Then $\mathcal{T}^{-1/2} = \mathcal{P}D^{1/2}\mathcal{P}^T$, where $D^{1/2} = \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$, satisfies $(\mathcal{T}^{-1/2})^2 = \mathcal{T}^{-1}$ and is clearly symmetric.

[9] If you recall that $\mathbf{q}(t) = (q_1(t), \ldots, q_n(t))$ then $\mathbf{q}_a(t)$ will have two subscripts when written out in components: $\mathbf{q}_a(t) = (q_{a,1}(t), \ldots, q_{a,n}(t)$. Don't get confused over the different meanings of these subscripts! One labels the generalized coordinates, while the other labels different solutions to the equations of motion.

linear instability. If $\lambda = 0$ then this may be a translational mode (often called a *flat direction* of the potential, meaning its derivative is identically zero in that direction), or else one needs to analyse the third order, or higher order, terms in the expansions (3.4) of section 3.1 to determine whether or not it is stable.

In particular if *all* the eigenvalues $\lambda_a > 0$, $a = 1, \ldots, n$, then we have a point of *stable equilibrium*. The frequencies $\omega_a = \sqrt{\lambda_a} > 0$ are called the *normal frequencies*, with the corresponding eigenvectors $\boldsymbol{\alpha}_a$ called *normal modes*. Since $\{\boldsymbol{\alpha}_a\}$ form a basis for $\mathbb{R}^n$, we may write any $\mathbf{q} = \sum_{a=1}^n f_a \, \boldsymbol{\alpha}_a$, and the $(f_1, \ldots, f_n)$ may then be used as a new set of generalized coordinates, called *normal coordinates*. Since each solution $f_a(t)$ to the equations of motion satisfies

$$\ddot{f}_a + \omega_a^2 f_a \;\; = \;\; 0 \;, \tag{3.16}$$

we see that the coordinate transformation $(q_1, \ldots, q_n) \to (f_1, \ldots, f_n)$ essentially diagonalizes the original quadratic Lagrangian (3.5) into a set of n decoupled harmonic oscillators of frequencies ω_a.

3.3 Examples

Having developed the general theory, we now apply it to some examples.

Mass-spring system

Consider three springs of natural length a and spring constant k lying in a line. One end of the first spring is fixed at a point A, while the other end is attached to a particle of mass m. This mass is in turn attached to one end of the second spring, with the other end attached to a second particle of mass m. Finally this second mass is attached to one end of the third spring, with the other end fixed at a point B. The distance between A and B is $3a$. We denote the horizontal displacement of the first mass from its equilibrium position by x, and similarly the horizontal displacement of the second mass by y – see Figure 9. The kinetic energy of the system is then

$$T \;\; = \;\; \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \;, \tag{3.17}$$

while the potential energy is

$$V \;\; = \;\; \frac{1}{2} k \left[x^2 + (y - x)^2 + y^2 \right] \;. \tag{3.18}$$

Here we have used the fact that if a spring with spring constant k is displaced by an amount Δ from its natural length, then the corresponding potential energy is $\frac{1}{2} k \Delta^2$.

Notice that the equilibrium point $\partial_x V = \partial_y V = 0$ is at $x = y = 0$ (as one would expect), and moreover the Lagrangian $L = T - V$ is *already* quadratic in x, y. We may then write the kinetic

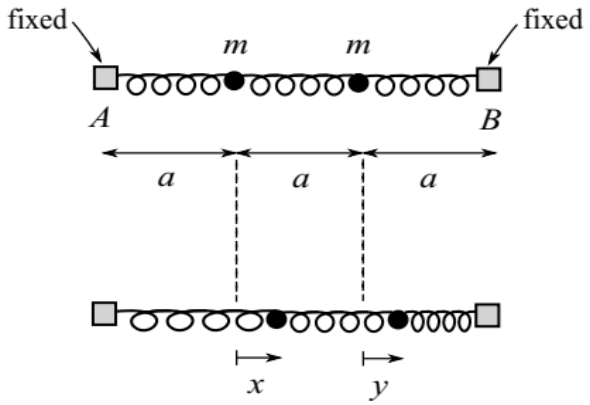

Figure 9: The spring-mass system. The upper diagram shows the equilibrium configuration. In the lower diagram we have shown the horizontal displacements x and y of the two masses from their equilibrium positions.

and potential energy terms in matrix form as

$$\begin{aligned} T &= \frac{1}{2}\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} , \\ V &= \frac{1}{2}\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} 2k & -k \\ -k & 2k\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} . \end{aligned} \tag{3.19}$$

Thus

$$\lambda\mathcal{T} - \mathcal{V} = \begin{pmatrix} m\lambda - 2k & k \\ k & m\lambda - 2k\end{pmatrix} , \tag{3.20}$$

The solutions to the characteristic equation

$$0 = \det(\lambda\mathcal{T} - \mathcal{V}) = (m\lambda - k)(m\lambda - 3k) , \tag{3.21}$$

are hence $\lambda_1 = k/m$, $\lambda_2 = 3k/m$, giving normal frequencies

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}} , \qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3k}{m}} . \tag{3.22}$$

The corresponding eigenvectors are

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 1 \\ 1\end{pmatrix} , \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} 1 \\ -1\end{pmatrix} , \tag{3.23}$$

which give the normal modes. In particular the lower frequency mode ω_1 corresponds to the two masses oscillating in phase, while the higher frequency mode ω_2 corresponds to the two masses oscillating out of phase. Of course this is what one might have expected.

Double pendulum

A *double pendulum* consists of a simple pendulum of mass m_1 and length l_1 pivoted at the origin, together with another simple pendulum of mass m_2 and length l_2, pivoted at the mass m_1. The whole system moves freely in a vertical plane under gravity. If θ_1 and θ_2 denote the angles each pendulum makes with the vertical, then the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\ &\quad + m_1 g l_1\cos\theta_1 + m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1)\ . \end{aligned} \tag{3.24}$$

Deriving this is an exercise on Problem Sheet 1.

For simplicity we shall analyse the normal modes in the case where both pendula have the same parameters, *i.e.* $l_1 = l_2 = l$ and $m_1 = m_2 = m$. In particular the potential energy is then

$$V = -mgl(2\cos\theta_1 + \cos\theta_2)\ . \tag{3.25}$$

As one would expect $\theta_1 = \theta_2 = 0$ is a critical point of V, and is a point of stable equilibrium. To analyse this we Taylor expand V around $\theta_1 = \theta_2 = 0$:

$$V = -mgl\left[2(1-\tfrac{1}{2}\theta_1^2) + (1-\tfrac{1}{2}\theta_2^2) + \text{fourth order}\right]\ . \tag{3.26}$$

Dropping the constant term we thus obtain the quadratic potential

$$\begin{aligned} V_{\text{quadratic}} &= \frac{1}{2}mgl(2\theta_1^2 + \theta_2^2) \\ &= \frac{1}{2}\begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\begin{pmatrix}2mgl & 0\\ 0 & mgl\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\ . \end{aligned} \tag{3.27}$$

Similarly expanding the kinetic term T and dropping cubic and higher order terms we find

$$\begin{aligned} T_{\text{quadratic}} &= \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\ , \\ &= \frac{1}{2}\begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{pmatrix}\begin{pmatrix}2ml^2 & ml^2\\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}\ . \end{aligned} \tag{3.28}$$

Thus

$$\lambda\mathcal{T} - \mathcal{V} = \begin{pmatrix}2ml^2\lambda - 2mgl & ml^2\lambda\\ ml^2\lambda & ml^2\lambda - mgl\end{pmatrix}\ , \tag{3.29}$$

The solutions to the characteristic equation

$$0 = \det(\lambda\mathcal{T} - \mathcal{V}) = m^2l^4\left[2\left(\lambda - \frac{g}{l}\right)^2 - \lambda^2\right]\ , \tag{3.30}$$

are then $\lambda_1 = (2-\sqrt{2})\frac{g}{l}$, $\lambda_2 = (2+\sqrt{2})\frac{g}{l}$. Both are positive, so the equilibrium is stable, as expected. The normal frequencies are hence

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{(2-\sqrt{2})\frac{g}{l}}\ , \qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{(2+\sqrt{2})\frac{g}{l}}\ , \tag{3.31}$$

with corresponding normal modes

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}. \tag{3.32}$$

Again the lower frequency normal mode has the pendula swinging in phase, while for the higher frequency mode they are out of phase.