Classical Mechanics

LECTURE 8

4.4 Euler's equations

In this section we consider a *free* rigid body, meaning that no external forces act on it. From Newton's laws we thus have that the total momentum $\mathbf{P}$ given by (4.22) is constant, and hence $\dot{\mathbf{P}} = M\dot{\mathbf{v}}_G = \mathbf{0}$. One can deduce this from Newton's laws for point particles by considering the body as made up of many particles of small masses m_I at positions $\mathbf{r}_I$, with $M = \sum_{I=1}^{N} m_I$ being the total mass. The only forces acting are the constraint forces keeping the masses at fixed distances apart, $|\mathbf{r}_I - \mathbf{r}_J| =$ constant, and summing over the whole body these forces cancel, as in equation (1.16). We thus deduce that the centre of mass velocity $\mathbf{v}_G$ is constant. Via a Galilean boost we may take the inertial frame $\hat{\mathcal{S}}$ to be such that the centre of mass is at rest at the origin, $\hat{O} = G$, which we henceforth do.

The remaining dynamics is then entirely in the rotation of the basis $\{\mathbf{e}_i\}$ for the rest frame $\mathcal{S}$ of the body. Recall from section 4.1 that

$$\mathbf{e}_i(t) \;=\; \sum_{j=1}^{3} \mathcal{R}_{ij}(t)\,\hat{\mathbf{e}}_j \;, \tag{4.42}$$

where now $\{\hat{\mathbf{e}}_i\}$ is a time-independent basis for the inertial frame $\hat{\mathcal{S}}$. The Coriolis formula (4.11) applied to $\mathbf{r} = \mathbf{e}_i$ immediately gives

$$\dot{\mathbf{e}}_i \;=\; \boldsymbol{\omega} \wedge \mathbf{e}_i \;, \tag{4.43}$$

although we can also easily check this from (4.42): differentiating the latter with respect to time t we have

$$\dot{\mathbf{e}}_i \;=\; \sum_{j=1}^{3} \dot{\mathcal{R}}_{ij}\hat{\mathbf{e}}_j \;=\; \sum_{k=1}^{3} (\dot{\mathcal{R}}\mathcal{R}^T)_{ik}\mathbf{e}_k \;=\; -\sum_{k=1}^{3} \Omega_{ik}\mathbf{e}_k \;. \tag{4.44}$$

Using (4.6) we may then write this as

$$\dot{\mathbf{e}}_i \;=\; \sum_{j,k=1}^{3} \epsilon_{ikj}\omega_j\mathbf{e}_k \;=\; \sum_{j=1}^{3} \mathbf{e}_j \wedge \mathbf{e}_i\,\omega_j \;=\; \boldsymbol{\omega} \wedge \mathbf{e}_i \;, \tag{4.45}$$

where recall $\boldsymbol{\omega} = \sum_{j=1}^{3} \omega_j \mathbf{e}_j$. In the second equality in (4.45) we have used that the frame is right-handed, so $\mathbf{e}_i \wedge \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\mathbf{e}_k$.

Choosing the basis $\{\mathbf{e}_i\}$ to be aligned with the principal axes of the body, we may write the total angular momentum as

$$\mathbf{L} = \sum_{i=1}^{3} I_i \omega_i \mathbf{e}_i , \tag{4.46}$$

where recall that I_i are the principal moments of inertia. This is likewise conserved, $\dot{\mathbf{L}} = \mathbf{0}$, as with no external forces acting there is no external torque (again, compare to the system of point particles in section 1.5). Using (4.46) we thus compute

$$\mathbf{0} = \dot{\mathbf{L}} = \sum_{i=1}^{3} I_i (\dot{\omega}_i \mathbf{e}_i + \omega_i \dot{\mathbf{e}}_i) , \tag{4.47}$$

which combining with (4.45) leads to the three equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 , \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 , \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 . \end{aligned} \tag{4.48}$$

This is a set of coupled first order ODEs for the three functions $\omega_i(t)$, $i = 1, 2, 3$, which determine how the rigid body rotates. The system also depends on the constants I_i, and collectively the equations are known as *Euler's equations.* If we include an external torque $\boldsymbol{\tau}$ then the components τ_i simply appear on the right hand side of (4.48) via $\dot{\mathbf{L}} = \boldsymbol{\tau}$.

Let us examine (4.48) further. If we multiply the i^{th} equation by ω_i and then sum over i we find the resulting equation is equivalent to

$$2T \equiv \sum_{i=1}^{3} I_i \omega_i^2 \tag{4.49}$$

being constant. Of course we immediately recognize T as the rotational kinetic energy (4.41), which we expect to be conserved for this closed system. Similarly multiplying the i^{th} equation by $I_i\omega_i$ and summing we find

$$\mathbf{L} \cdot \mathbf{L} = L^2 = \sum_{i=1}^{3} I_i^2 \omega_i^2 \tag{4.50}$$

is constant. Here $\mathbf{L} = \sum_{i=1}^{3} I_i\omega_i\mathbf{e}_i$ is the total angular momentum (4.24). Again, on general grounds we expect this to be conserved.

We have thus found two conserved quantities. A little algebra allows us to rearrange the equations as

$$\begin{aligned} 2I_3T - L^2 &= I_1(I_3 - I_1)\omega_1^2 + I_2(I_3 - I_2)\omega_2^2 \ , \\ 2I_2T - L^2 &= I_1(I_2 - I_1)\omega_1^2 + I_3(I_2 - I_3)\omega_3^2 \ , \\ I_1^2\dot{\omega}_1^2 &= (I_2 - I_3)^2\omega_2^2\omega_3^2 \ . \end{aligned} \tag{4.51}$$

We may thus algebraically eliminate ω_2, ω_3 using the first two equations to obtain a first order ODE for $\omega_1(t)$ of the form

$$\dot{\omega}_1^2 = F(\omega_1; I_i, T, L^2) \ , \tag{4.52}$$

where F is quadratic in ω_1^2. Thus the problem is *completely integrable*, since we may solve $t = \int \mathrm{d}\omega_1/\sqrt{F}$, and then substitute the solution for $\omega_1(t)$ into the remaining two equations in (4.51) to obtain $\omega_2(t)$, $\omega_3(t)$ algebraically. In general the integral that arises here is called an *elliptic integral of the first kind*, and cannot be written in terms of more elementary functions. However, one can gain some *qualitative* understanding of the general solutions to these equations using geometric methods. In particular notice that equations (4.49) and (4.50), when viewed as constraints on the angular momentum $L_i = I_i\omega_i$, say that $\mathbf{L} = (L_1, L_2, L_3)$ lies on an ellipsoid with semiaxes $\sqrt{2TI_1}$, $\sqrt{2TI_2}$, $\sqrt{2TI_3}$, and on a sphere of radius L, respectively. For futher discussion see Example 5.7 in the book by Woodhouse, or section 37 of the book by Landau and Lifschitz. We shall instead examine some special cases:

Example (axisymmetric body): Consider an axisymmetric body (also known as a *symmetric top*) with $I_1 = I_2$. Then the third Euler equation in (4.48) implies that $\dot{\omega}_3 = 0$, *i.e.* $\omega_3 =$ constant. We may then combine the first two equations into the single *complex* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1 + \mathrm{i}\omega_2) = \mathrm{i}\nu(\omega_1 + \mathrm{i}\omega_2) \ , \qquad \text{where} \qquad \nu \equiv \frac{(I_3 - I_1)}{I_1}\omega_3 \ . \tag{4.53}$$

The solution is hence $\omega_1 + \mathrm{i}\omega_2 = A\exp(\mathrm{i}\nu t)$, where the constant amplitude A may be taken to be real by choosing a suitable origin for the time t. Thus

$$\boldsymbol{\omega}(t) = (A\cos\nu t, A\sin\nu t, \omega_3) \ . \tag{4.54}$$

Since the component ω_3 along the axis of symmetry $\mathbf{e}_3$ is constant, this shows that the vector $\boldsymbol{\omega}$ rotates uniformly with angular speed ν about the axis of symmetry, with constant modulus $|\boldsymbol{\omega}| = \sqrt{A^2 + \omega_3^2}$ and making a constant angle $\cos^{-1}(\omega_3/\sqrt{A^2 + \omega_3^2})$. Since the components of the angular momentum $\mathbf{L}$ in this basis are $L_i = I_i\omega_i$, the angular momentum vector $\mathbf{L}$ has a similar motion. That is, $\mathbf{L}$ rotates uniformly about the axis of symmetry $\mathbf{e}_3$, making a fixed angle $\theta = \cos^{-1}(I_3\omega_3/|\mathbf{L}|)$ with it.

Of course this description is as viewed from the rest frame $\mathcal{S}$ of the body. Viewed from the inertial frame $\hat{\mathcal{S}}$ we know that the angular momentum vector is constant. Viewed in this frame it is the axis of symmetry $\mathbf{e}_3$ that rotates uniformly about the constant direction of $\mathbf{L}$. We shall describe this in more detail in section 4.5.

Example: Suppose $I_1 < I_2$, $I_3 = I_1 + I_2$ and the body is set in rotation with $\omega_2 = 0$ and $\omega_3 = \sqrt{(I_2 - I_1)/(I_2 + I_1)}\,\omega_1$. Then the solution to the Euler equations is $\omega_1(t) = c_1 \mathrm{sech}(c_2 t)$, where $c_1 = \sqrt{2T/I_2}$ and $c_2 = \sqrt{(I_2 - I_1)/(I_2 + I_1)}\,c_1$. Verifying the details is left to Problem Sheet 3.

Stability

Euler's equations (4.48) have three special solutions where $(\omega_1, \omega_2, \omega_3) = (\omega, 0, 0)$, $(0, \omega, 0)$, or $(0, 0, \omega)$, respectively, corresponding to rotation around each of the three principal axes with constant angular speed ω. When I_1, I_2, I_3 are all different, so that without loss of generality we may assume $I_1 < I_2 < I_3$, then the first and last of these solutions are stable, while the second is unstable.

To see this consider the first solution, and let us perturb it by writing $\boldsymbol{\omega} = (\omega + \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Then ignoring quadratic order terms in ε_i, as in section 3, the Euler equations (4.48) read

$$\begin{aligned} I_1 \dot{\varepsilon}_1 &= 0 \, , \\ I_2 \dot{\varepsilon}_2 &= (I_3 - I_1)\omega\varepsilon_3 \, , \\ I_3 \dot{\varepsilon}_3 &= (I_1 - I_2)\omega\varepsilon_2 \, . \end{aligned} \qquad (4.55)$$

In particular the last two equations imply that

$$I_2 \ddot{\varepsilon}_2 = (I_3 - I_1)\omega\dot{\varepsilon}_3 = \frac{(I_3 - I_1)(I_1 - I_2)}{I_3}\omega^2\varepsilon_2 \, . \qquad (4.56)$$

Since $I_1 < I_2 < I_3$ this takes the form $\ddot{\varepsilon}_2 = -\lambda_2\varepsilon_2$ where $\lambda_2 = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\omega^2 > 0$. We thus find simple harmonic motion in the ε_2 and ε_3 directions, and the solution is stable.

The computation for the solution $\boldsymbol{\omega} = (\varepsilon_1, \omega + \varepsilon_2, \varepsilon_3)$ is very similar, only we now find that

$$I_1 \ddot{\varepsilon}_1 = \frac{(I_2 - I_3)(I_1 - I_2)}{I_3}\omega^2\varepsilon_1 \, , \qquad (4.57)$$

This takes the form $\ddot{\varepsilon}_1 = -\lambda_1\varepsilon_1$, where now $\lambda_1 = -\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\omega^2 < 0$ and the exponential solution means this is *unstable*.

We leave it as an exercise to check that the final case where $\boldsymbol{\omega} = (0, 0, \omega)$ is stable. One can demonstrate this very effectively by taking any rigid body with well-separated principal moments of inertia $I_1 \ll I_2 \ll I_3$, and spinning it about each principal axis. For example, we showed earlier that a rectangular cuboid of mass M and side lengths $2a$, $2b$, $2c$ has $I_1 = \frac{1}{3}M(b^2 + c^2)$,

$I_2 = \frac{1}{3}M(c^2 + a^2)$, $I_3 = \frac{1}{3}M(a^2 + b^2)$. Then $a \gg b \gg c$ implies $I_1 \ll I_2 \ll I_3$, and rotation about the middle length axis is the unstable direction. Try it – preferably with something that will not break if you drop it!

4.5 Euler angles and $SO(3)$

As we've already mentioned, the motion of a rigid body may be described by three coordinates for its centre of mass G, together with three angles which determine the orientation of the axes $\{\mathbf{e}_i\}$ fixed in the body relative to the axes $\{\hat{\mathbf{e}}_i\}$ of the inertial frame $\hat{\mathcal{S}}$. A natural choice for these latter three generalized coordinates is *Euler angles.*

We might as well suppose that the origins O, $\hat{O}$ of $\mathcal{S}$, $\hat{\mathcal{S}}$ coincide, as we are only interested in the angles between coordinate axes. The axes of the two frames are then related as in (4.1), where $\mathcal{R} \in SO(3)$. We then have

Proposition: There exist angles $\theta \in [0, \pi]$, $\varphi, \psi \in [0, 2\pi)$ such that

$$\mathcal{R} = \begin{pmatrix} \cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi & \cos\theta\sin\varphi\cos\psi + \cos\varphi\sin\psi & -\sin\theta\cos\psi \\ -\cos\theta\cos\varphi\sin\psi - \sin\varphi\cos\psi & -\cos\theta\sin\varphi\sin\psi + \cos\varphi\cos\psi & \sin\theta\sin\psi \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (4.58)$$

Moreover, θ, φ, ψ are determined uniquely by $\mathcal{R}$, provided that $|\mathcal{R}_{33}| \neq 1$.

N.B. There are different conventions for Euler angles. We're using the so-called "y-convention", in which the second rotation we describe below is about the "y-axis" (more precisely, the $\tilde{\mathbf{e}}_2$ axis). The other commonly used convention is called the "x-convention", where one instead rotates about the x-axis. This amounts to the simple replacement $\varphi \to \varphi - \frac{\pi}{2}$, $\psi \to \psi + \frac{\pi}{2}$.

Proof: There are essentially two approaches to deriving (4.58). One can either proceed algebraically, which is the method used on pages 9 and 10 of the book by Woodhouse, or else one can proceed geometrically. We shall follow the geometric route, as it gives a better intuition for the Euler angles.

Consider the two axes $\mathbf{e}_3$ and $\hat{\mathbf{e}}_3$. Each of these defines an orthogonal plane, namely the $(\mathbf{e}_1, \mathbf{e}_2)$ plane and the $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ plane, respectively. If these planes coincide then $\mathbf{e}_3 = \pm\hat{\mathbf{e}}_3$, which is the special case $|\mathcal{R}_{33}| = 1$ mentioned above. In this special case the two frames simply differ by an $O(2)$ rotation of this plane. Otherwise the intersection of the two planes defines a unique line though the origin, called the *line of nodes.* We specify its direction by defining the unit vector

$$\mathbf{n} = \hat{\mathbf{e}}_3 \wedge \mathbf{e}_3 . \qquad (4.59)$$

Since $\mathbf{n}$ is a unit vector lying in both the $(\mathbf{e}_1, \mathbf{e}_2)$ plane and the $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ plane we may write

$$\mathbf{n} = \sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2 , \qquad \text{and} \qquad \mathbf{n} = -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 . \tag{4.60}$$

This defines uniquely the two angles φ, $\psi \in [0, 2\pi)$. The angle $\theta \in [0, \pi]$ is simply the angle between $\mathbf{e}_3$ and $\hat{\mathbf{e}}_3$. This is probably the clearest description of the Euler angles – see Figure 12.

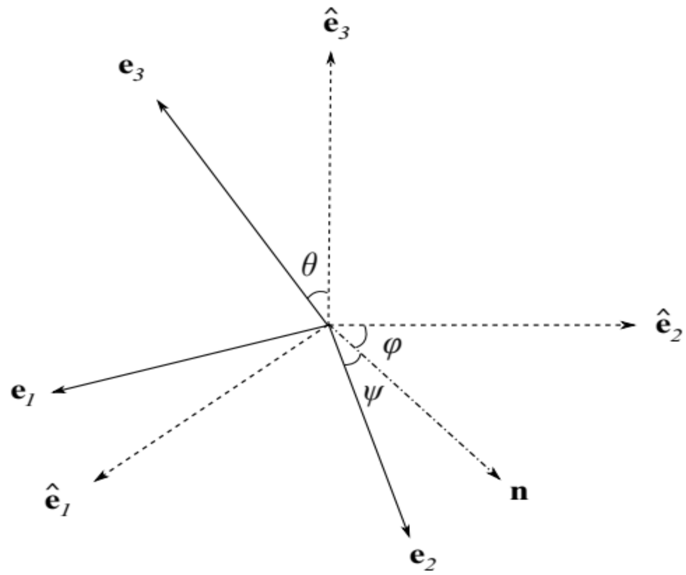

Figure 12: The Euler angles θ, φ, ψ between the two orthonormal bases $\{\hat{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$. The angle between $\hat{\mathbf{e}}_3$ and $\mathbf{e}_3$ is θ, while the vector $\mathbf{n} = \hat{\mathbf{e}}_3 \wedge \mathbf{e}_3$ along the line of nodes is orthogonal to both of these vectors. The angle between $\hat{\mathbf{e}}_2$ and $\mathbf{n}$ is denoted φ, while the angle between $\mathbf{e}_2$ and $\mathbf{n}$ is ψ.

Having given a geometric description of the angles, as suggested by the second line of (4.58) we may construct the rotation matrix $\mathcal{R}$ between the two frames as a sequence of three 2×2 rotations. We start with the $\{\hat{\mathbf{e}}_i\}$ basis. Clearly we'd like to rotate $\hat{\mathbf{e}}_3$ onto $\mathbf{e}_3$, but to do this we need to rotate about an axis perpendicular to both vectors, *i.e.* the line of nodes $\mathbf{n}$. Thus we first rotate about the $\hat{\mathbf{e}}_3$ axis by an angle φ, which brings the $\hat{\mathbf{e}}_2$ vector into line with $\mathbf{n}$. Concretely the new orthonormal basis is

$$\tilde{\mathbf{e}}_1 = \cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2 , \qquad \tilde{\mathbf{e}}_2 = \mathbf{n} = -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 , \qquad \tilde{\mathbf{e}}_3 = \hat{\mathbf{e}}_3 . \tag{4.61}$$

The line of nodes is the new basis vector $\tilde{\mathbf{e}}_2 = \mathbf{n}$, and rotating about this through an angle θ brings $\tilde{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$ into line with $\mathbf{e}_3$. The new orthonormal basis is

$$\begin{aligned} \mathbf{e}'_1 &= \cos\theta\,(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) - \sin\theta\,\hat{\mathbf{e}}_3 , \qquad \mathbf{e}'_2 = \mathbf{n} = -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 , \\ \mathbf{e}'_3 &= \sin\theta\,(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) + \cos\theta\,\hat{\mathbf{e}}_3 . \end{aligned} \tag{4.62}$$

Now that we have rotated $\hat{\mathbf{e}}_3$ onto $\mathbf{e}_3$, the last step is to ensure that in the final basis $\{\mathbf{e}_i\}$ we have $\mathbf{n} = \sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2$, which was the definition of ψ in (4.60). Since currently $\mathbf{n} = \mathbf{e}'_2$, we achieve this by rotating about the $\mathbf{e}'_3 = \mathbf{e}_3$ direction by an angle ψ. This hence explains the sequence of three rotations on the second line of (4.58), of course read right to left as we are mapping $\hat{\mathbf{e}}_i \to \mathbf{e}_i = \sum_{j=1}^3 \mathcal{R}_{ij}\hat{\mathbf{e}}_j$. The case where $|\mathcal{R}_{33}| = 1$ (equivalently $\sin\theta = 0$) is a *coordinate singularity*. If one wants to analyse the behaviour near here one should really change coordinates (much like near the origin $\varrho = 0$ in plane polar coordinates, where the angle ϕ is not defined and one should change to Cartesian coordinates $x = \varrho\cos\phi$, $y = \varrho\sin\phi$).

Since the Euler angles are generalized coordinates for the rotation group $SO(3)$, we must be able to write the angular velocity $\boldsymbol{\omega}$ of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ in terms of these variables. Of course here $\theta = \theta(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$. The brute force method would be to take the rather ugly 3×3 matrix on the first line of (4.58) and substitute it into the definition of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in (4.5). This works, but is a little tedious by hand.[11]

There is a much nicer way to get to the result though, again using the decomposition on the second line of (4.58). Let us denote the various frames in the proof above as $\hat{\mathcal{S}}$, $\tilde{\mathcal{S}}$, $\mathcal{S}'$ and $\mathcal{S}$, respectively. Then the rotation matrices from $\hat{\mathcal{S}} \to \tilde{\mathcal{S}}$, $\tilde{\mathcal{S}} \to \mathcal{S}'$ and $\mathcal{S}' \to \mathcal{S}$ are precisely the three 2×2 rotation matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.63}$$

respectively. Comparing to the example in (4.12), we thus see that the angular velocity of $\mathcal{S}$ relative to $\mathcal{S}'$ is $\dot{\psi}\,\mathbf{e}_3$, the angular velocity of $\mathcal{S}'$ relative to $\tilde{\mathcal{S}}$ is $\dot{\theta}\,\mathbf{n}$, and the angular velocity of $\tilde{\mathcal{S}}$ relative to $\hat{\mathcal{S}}$ is $\dot{\varphi}\,\hat{\mathbf{e}}_3$. Using the comment at the end of section 4.1, the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is then simply given by the sum of these:

$$\begin{aligned} \boldsymbol{\omega} &= \dot{\psi}\,\mathbf{e}_3 + \dot{\theta}\,\mathbf{n} + \dot{\varphi}\,\hat{\mathbf{e}}_3 \qquad (4.64) \\ &= \dot{\psi}\,\mathbf{e}_3 + \dot{\theta}\,(\sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2) + \dot{\varphi}\,(-\sin\theta\cos\psi\,\mathbf{e}_1 + \sin\theta\sin\psi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3)\ . \end{aligned}$$

Thus writing $\boldsymbol{\omega} = \sum_{i=1}^3 \omega_i\mathbf{e}_i$, the components ω_i in the frame $\mathcal{S}$ are

$$\begin{aligned} \omega_1 &= \dot{\theta}\,\sin\psi - \dot{\varphi}\,\sin\theta\cos\psi\ , \\ \omega_2 &= \dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi\ , \\ \omega_3 &= \dot{\psi} + \dot{\varphi}\,\cos\theta\ . \end{aligned} \tag{4.65}$$

Example (axisymmetric body): As an application of these results, let's return to the free axisymmetric body discussed in the previous subsection. We showed there that in the *body frame* the angular momentum vector $\mathbf{L}$ rotated about the $\mathbf{e}_3$ axis at a fixed angle θ, with angular speed $\nu = (I_3 - I_1)\omega_3/I_1$. In the inertial frame $\hat{\mathcal{S}}$ the angular momentum $\mathbf{L}$ is constant, and for simplicity

[11] With a computer algebra package it takes less than a minute to type in this matrix and get the formula (4.65) for $\boldsymbol{\omega}$.

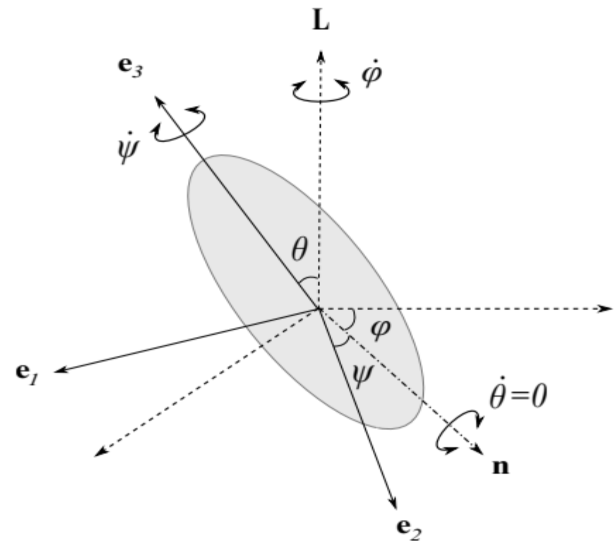

Figure 13: The precession of an axisymmetric body. The dotted axes are those of the fixed inertial frame $\hat{\mathcal{S}}$. We take the conserved angular momentum $\mathbf{L}$ to lie along the third axis $\hat{\mathbf{e}}_3$. The axis of symmetry $\mathbf{e}_3$ of the body makes a constant angle θ with $\mathbf{L}$, so that $\dot\theta = 0$, and precesses around $\mathbf{L}$ with rate $\dot\varphi$.

we take it to lie along the $\hat{\mathbf{e}}_3$ direction. From Figure 13 we thus see that $\dot\theta = 0$ (the angle between the axis of symmetry $\mathbf{e}_3$ and $\mathbf{L}$ is constant) while comparing (4.65) to the solution (4.54) we see that $\dot\psi = -\nu$ and $\omega_3 = \dot\psi + \dot\varphi\cos\theta$. Notice here that $\dot\psi = -\nu$ is the rate of rotation of the body about its symmetry axis $\mathbf{e}_3$, which since $\mathbf{L}$ is fixed in $\hat{\mathcal{S}}$ explains why $\mathbf{L}$ appears to rotate about $\mathbf{e}_3$ with angular speed $+\nu$ in the body frame. We may also deduce that

$$\dot\varphi = \frac{\omega_3 + \nu}{\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta} . \qquad (4.66)$$

This is the constant rate of *precession* of the $\mathbf{e}_3$ axis around $\mathbf{L}$, as viewed in the inertial frame $\hat{\mathcal{S}}$.

4.6 The Lagrange top

Euler's equations (4.48) determine the time-dependence of the angular velocity $\boldsymbol{\omega}$, but in general these are not a useful description of the dynamical evolution, especially when we consider rigid bodies moving under gravity (in (4.48) there are no external forces acting).

Consider a rigid body with rest frame $\mathcal{S}$, with the origin $O = G$ at the centre of mass and the axes aligned with the principal axes. We may use Cartesian coordinates (x, y, z) for the position of O with respect to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. The orientation of the body may then be specified by giving the Euler angles θ, φ, ψ of the frame $\mathcal{S}$ relative to the fixed inertial frame $\hat{\mathcal{S}}$. Altogether $x, y, z, \theta, \varphi, \psi$ are six generalized coordinates for the configuration space. The kinetic energy of the body is given by (4.40), where the centre of mass velocity is $\mathbf{v}_G = (\dot x, \dot y, \dot z)$. We may then use (4.65) to express the rotational kinetic energy (4.41) in terms of Euler angles. Altogether the kinetic energy of the body relative to $\hat{\mathcal{S}}$ is then

$$\begin{aligned} T &= \frac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{1}{2}I_1(\dot{\theta}\,\sin\psi-\dot{\varphi}\,\sin\theta\cos\psi)^2 \\ &\quad +\frac{1}{2}I_2(\dot{\theta}\,\cos\psi+\dot{\varphi}\,\sin\theta\sin\psi)^2+\frac{1}{2}I_3(\dot{\psi}+\dot{\varphi}\,\cos\theta)^2 \ . \end{aligned} \tag{4.67}$$

The main application we will address in this section is the dynamics of a rigid body moving under gravity. An important fact is that if a body is placed in a uniform gravitational field, say of strength g in the downward z-direction, then the body acts as if all the mass was located at the centre of mass G. This follows since the total gravitational potential energy of the body is

$$V = \int_R \rho(\mathbf{r})\, gz\, \mathrm{d}V = Mg\, z_G \ , \tag{4.68}$$

where z_G is the z-coordinate of the centre of mass $\mathbf{r}_G$ in (4.20). In particular, the Lagrangian for a rigid body moving under gravity is $L=T-V$, where T is given by (4.67) and V is given by (4.68). The form of this Lagrangian immediately implies that the centre of mass motion $(x(t),y(t),z(t))$, behaves like a point particle of mass M at the centre of mass G, while the rotational motion about the centre of mass is entirely unaffected by the gravitational field.

A different situation has the body instead rotating about a general point P fixed both in $\hat{\mathcal{S}}$ *and* fixed in the body. In this case we may take the frame $\mathcal{S}$ to be centred at $O=P$, with axes aligned with the principal axes of $\mathcal{I}^{(P)}$, so that

$$T = \frac{1}{2}I_1(\dot{\theta}\,\sin\psi-\dot{\varphi}\,\sin\theta\cos\psi)^2+\frac{1}{2}I_2(\dot{\theta}\,\cos\psi+\dot{\varphi}\,\sin\theta\sin\psi)^2+\frac{1}{2}I_3(\dot{\psi}+\dot{\varphi}\,\cos\theta)^2 \ . \tag{4.69}$$

Here I_i are now the principal moments of inertia about the point P. This follows from the original expression (4.37) for the kinetic energy, where the middle term in the expression on the right hand side is now zero since $\mathbf{v}_O=\mathbf{0}$ (rather than because $O=G$). The form (4.69) of the kinetic energy is relevant for problems where the rigid body is suspended from a particular point P, but may move freely about this point.

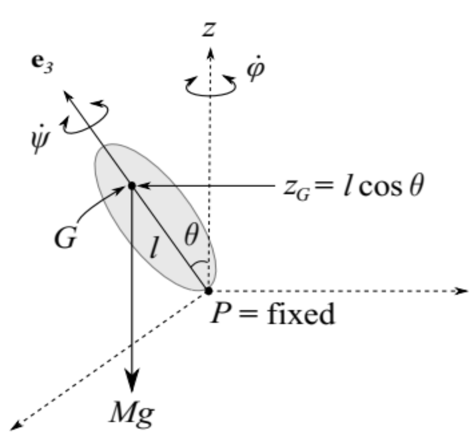

Figure 14: A Lagrange top.

In general this problem is not integrable, but certain special cases are. We will study the *Lagrange top*, which is a *constrained problem* in which an axisymmetric body is suspended from a point P lying on the axis of symmetry $\mathbf{e}_3$. If we denote by l the fixed distance between P and G along this axis, then from (4.68) the potential energy is

$$V = Mgl\cos\theta \ . \tag{4.70}$$

In particular notice that θ and φ are spherical polar angles for the axis of symmetry $\mathbf{e}_3$. That is, $\mathbf{e}_3$ makes an angle θ with the upward vertical z-direction, while $\dot{\varphi}$ is the angular speed with which the axis $\mathbf{e}_3$ rotates about this vertical. The Lagrangian for the Lagrange top is hence given by

$$L = T - V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - Mgl\cos\theta \ . \tag{4.71}$$

We stress that as in (4.69) I_i are the principal moments of inertia *about* P. We immediately see that both ψ and φ are ignorable coordinates, with conserved momenta

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial\dot{\psi}} = I_3(\dot{\psi} + \dot{\varphi}\cos\theta) = I_3\omega_3 \ , \\ p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} = I_1\sin^2\theta\,\dot{\varphi} + p_\psi\cos\theta \ . \end{aligned} \tag{4.72}$$

Notice that p_ψ is simply the angular momentum of the top about its axis of symmetry, where the corresponding constant angular velocity ω_3 is called the *spin*. Since $\partial L/\partial t = 0$ the energy E is conserved, where

$$2E = I_1\dot{\theta}^2 + I_1\dot{\varphi}^2\sin^2\theta + I_3\omega_3^2 + 2Mgl\cos\theta \ . \tag{4.73}$$

It is then useful to introduce $u = \cos\theta$ and rearrange the equations as

$$\begin{aligned} \dot{\varphi} &= \frac{p_\varphi - I_3\omega_3 u}{I_1(1-u^2)} \ , \\ \dot{\psi} &= \omega_3 - \frac{(p_\varphi - I_3\omega_3 u)u}{I_1(1-u^2)} \ , \\ I_1\dot{u}^2 &= F(u) \ , \end{aligned} \tag{4.74}$$

where we have introduced

$$F(u) \equiv (2E - I_3\omega_3^2 - 2Mglu)(1-u^2) - \frac{(p_\varphi - I_3\omega_3 u)^2}{I_1} \ . \tag{4.75}$$

We have thus reduced the problem to quadratures, and as for the free Euler equations the u-integral is an elliptic integral. However, one can understand the *qualitative* behaviour of the dynamics without solving the equations directly.

Suppose we set the top in motion with starting value $\theta = \cos^{-1}(u_2)$ and $\dot{\theta} = 0$. We shall fix the angular speed $\omega_3 > 0$ and p_φ, and suppose that $u_c \equiv p_\varphi/I_3\omega_3$ satisfies $0 < u_c < 1$. This latter condition will lead to various interesting behaviour, as we shall see. Next consider the differential equation $I_1\dot{u}^2 = F(u)$, where $F(u)$ is given by (4.75). The constant E in this expression is fixed by the initial condition

$$0 = F(u_2) = (2E - I_3\omega_3^2 - 2Mglu_2)(1-u_2^2) - \frac{(p_\varphi - I_3\omega_3 u_2)^2}{I_1} . \tag{4.76}$$

The function $F(u)$ is a cubic with positive coefficient of u^3, and we also note that $F(\pm 1) = -(p_\varphi \mp I_3\omega_3)^2/I_1 < 0$. The physical range of interest is where $-1 \leq u = \cos\theta \leq 1$ and $F(u) \geq 0$. In Figure 15 we have sketched the graph of $F(u)$ as we vary u_2, with all other parameters kept fixed. Notice that $F(u)$ has two roots $u_1 \leq u_2 \in (-1, 1)$ in the physical range.

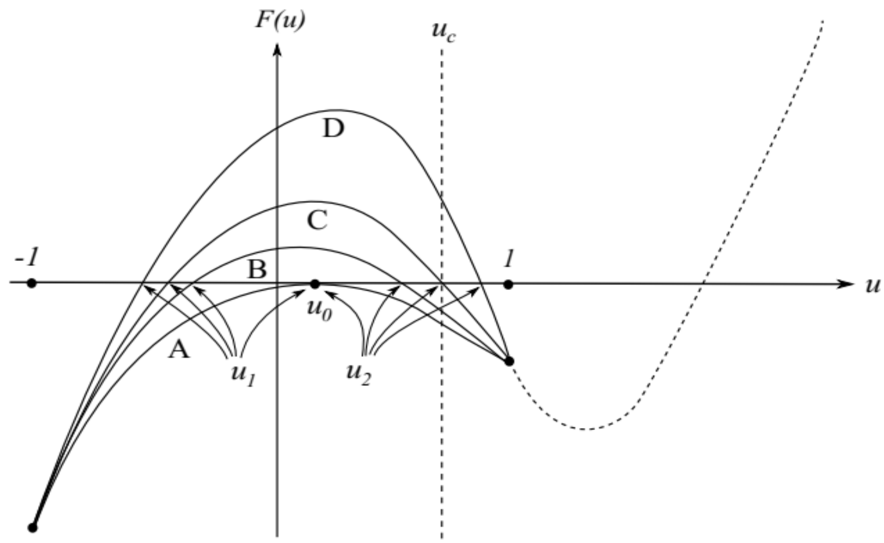

Figure 15: Sketch of the cubic function $F(u)$ for various values of initial condition u_2 (where $F(u_2) = 0$). The physical region is where $u = \cos\theta \in [-1, 1]$ and $F(u) \geq 0$. The motion of $u(t)$ is constrained to lie between two roots $u_1 \leq u_2$ of $F(u)$. As we vary u_2 the endpoints $F(\pm 1) = -(p_\varphi \mp I_3\omega_3)^2/I_1 < 0$ of the curves are fixed. We have also noted the critical value $u_c \equiv p_\varphi/I_3\omega_3 \in (0, 1)$. There are then 4 distinct behaviours, corresponding to the 4 curves A, B, C and D.

Notice that to the left of the line $u = u_c$, meaning $-1 < u < u_c$, we see from (4.74) that $\dot{\varphi} > 0$ is positive. On the other hand to the right of $u = u_c$, meaning $u_c < u < 1$, we instead have $\dot{\varphi} < 0$ is negative. This leads to 4 distinct behaviours for the motion of the top. Notice that in this motion the axis of symmetry $\mathbf{e}_3$ traces out a curve on the unit sphere, with spherical polar coordinates (θ, φ), centred at P:

(A) Here there is a critical value $u_0 = u_1 = u_2$ which is a root of both $F(u)$ and $F'(u)$. The graph of $F(u)$ hence touches the u-axis at $u = u_0$, and the trajectory is simply $u = u_0 =$ constant. The axis thus steadily precesses about the vertical with θ and $\dot{\varphi} > 0$ both constant. This traces out a horizontal circle on the sphere with $\theta = \theta_0 = \cos^{-1}(u_0)$.

(B) Here u_2 satisfies $u_0 < u_2 < u_c$, and $u(t) \in [u_1, u_2]$ oscillates between the two roots $u_1 < u_2$ of $F(u)$ on either side of u_0. But in this motion we always have $\dot{\varphi} > 0$. The axis of symmetry hence rotates anticlockwise about the vertical (viewed from above the north pole $\theta = 0$), but at the same time its angle θ with the vertical oscillates between the two values corresponding to the roots of F. This oscillation in θ is called *nutation*.

(C) There is a limiting case where $u_2 = u_c$. Here $\dot{\varphi} = 0$ when θ reaches its minimum, corresponding to $u = u_c$. While this case might look fine-tuned, it's also exactly what happens when you simply let go of the rigid body from rest, *i.e.* both $\dot{\theta} = 0$ and $\dot{\varphi} = 0$ initially (of course it can still spin about its axis, $\dot{\psi} = \omega_3$ in this case).

(D) Perhaps most interesting is the final case when $u_c < u_2 < 1$. In this case $\dot{\varphi}$ is *negative* for appropriately small range of θ, and the curve traced out on the sphere loops back on itself.

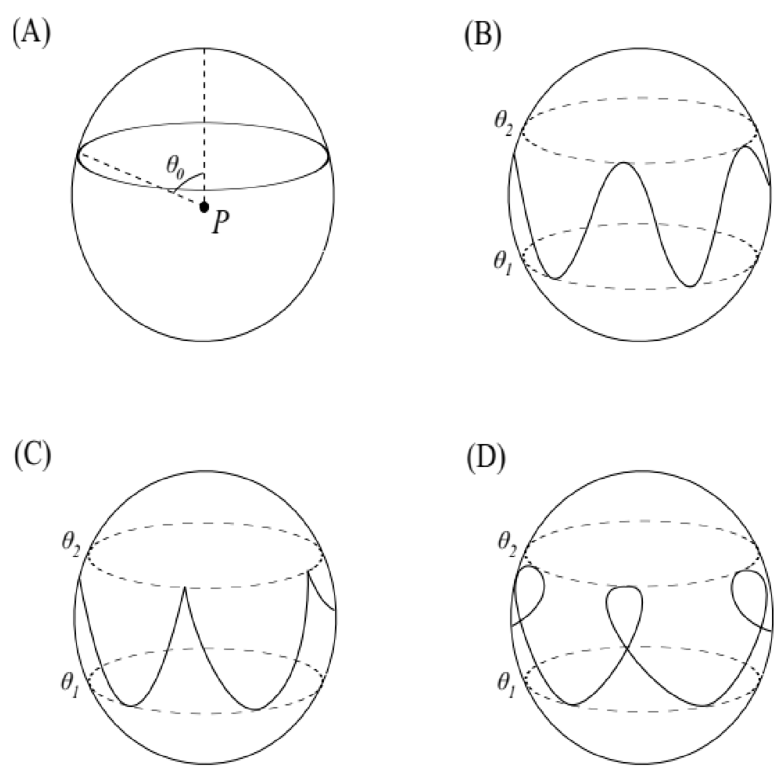

Figure 16: The distinct behaviours of the Lagrange top corresponding to the curves A, B, C and D in Figure 15. In each case we've shown the curve traced out by the axis of symmetry on the unit sphere centred at P.