PRELIMINARIES: CANONICAL FORM OF LINEAR PROGRAMMING PROBLEM – LPP

INTRODUCTION

Linear Programming (LP) is the most useful optimization technique used for the solution of engineering problems. The term 'linear' implies that the objective function and constraints are 'linear' functions of 'nonnegative' decision variables. Thus, the conditions of LP problems (LPP) are

1. Objective function must be a linear function of decision variables
2. Constraints should be linear function of decision variables
3. All the decision variables must be nonnegative

For example,

$$
\begin{array}{lll}
\text{Maximize} & Z = 6x + 5y & \prec \text{Objective Function} \\
\text{subject to} & 2x - 3y \le 5 & \prec \text{1st Constraint} \\
 & x + 3y \le 11 & \prec \text{2nd Constraint} \\
 & 4x + y \le 15 & \prec \text{3rd Constraint} \\
 & x,\ y \ge 0 & \prec \text{Nonnegativ ity Condition}
\end{array}
$$

is an example of LP problem. However, example shown above is in "general" form.

STANDARD FORM OF LPP

Standard form of LPP must have following three characteristics:

1. Objective function should be of maximization type
2. All the constraints should be of equality type
3. All the decision variables should be nonnegative

The procedure to transform a general form of a LPP to its standard form is discussed below. Let us consider the following example.

$$
\begin{array}{ll}
\text{Minimize} & Z = -3x_1 - 5x_2 \\
\text{subject to} & 2x_1 - 3x_2 \le 15 \\
 & x_1 + x_2 \le 3 \\
 & 4x_1 + x_2 \ge 2 \\
 & x_1 \ \ge 0 \\
 & x_2 \ \text{unrestricted}
\end{array}
$$

The above LPP is violating the following criteria of standard form:

1. Objective function is of minimization type
2. Constraints are of inequality type
3. Decision variable x_2 is unrestricted, i.e., it can take negative values also, thus violating the non-negativity criterion.

However, a standard form for this LPP can be obtained by transforming it as follows: Objective function can be rewritten as

$$\text{Maximize} \qquad Z' = -Z = 3x_1 + 5x_2$$

The first constraint can be rewritten as: $2x_1 - 3x_2 + x_3 = 15$. Note that, a new nonnegative variable x_3 is added to the left-hand-side (LHS) to make both sides equal. Similarly, the second constraint can be rewritten as: $x_1 + x_2 + x_4 = 3$. The variables x_3 and x_4 are known as *slack variables*. The third constraint can be rewritten as: $4x_1 + x_2 - x_5 = 2$. Again, note that a new nonnegative variable x_5 is subtracted form the LHS to make both sides equal. The variable x_5 is known as *surplus variable*.

Decision variable x_2 can expressed by introducing two extra nonnegative variables as

$$x_2 = x_2' - x_2''$$

Thus, x_2 can be negative if $x_2' < x_2''$ and positive if $x_2' > x_2''$ depending on the values of x_2' and x_2''. x_2 can be zero also if $x_2' = x_2''$.

Thus, the standard form of above LPP is as follows:

$$\begin{aligned} \text{Maximize} \qquad & Z' = -Z = 3x_1 + 5(x_2' - x_2'') \\ \text{subject to} \qquad & 2x_1 - 3(x_2' - x_2'') + x_3 = 15 \\ & x_1 + (x_2' - x_2'') + x_4 = 3 \\ & 4x_1 + (x_2' - x_2'') - x_5 = 2 \\ & x_1,\ x_2',\ x_2'',\ x_3,\ x_4,\ x_5 \geq 0 \end{aligned}$$

After obtaining solution for x_2' and x_2'', solution for x_2 can be obtained as, $x_2 = x_2' - x_2''$.

CANONICAL FORM OF LPP

Canonical form of standard LPP is a set of equations consisting of the 'objective function' and all the 'equality constraints' (standard form of LPP) expressed in *canonical form*. Understanding the canonical form of LPP is necessary for studying *simplex method*, the most popular method of solving LPP. *Simplex method* will be discussed in some other class. In this class, *canonical form* of a set of linear equations will be discussed first. Canonical form of LPP will be discussed next.

Canonical form of a set of linear equations

Let us consider a set of three equations with three variables for ease of discussion. Later, the method will be generalized.

Let us consider the following set of equations,

$$3x + 2y + z = 10 \qquad (A_0)$$

$$x - 2y + 3z = 6 \qquad (B_0)$$

$$2x + y - z = 1 \qquad (C_0)$$

The system of equations can be transformed in such a way that a new set of three different equations are obtained, each having only one variable with nonzero coefficient. This can be achieved by some *elementary operations*.

The following operations are known as *elementary operations*.

1. Any equation E_r can be replaced by kE_r, where k is a nonzero constant.
2. Any equation E_r can be replaced by $E_r + kE_s$, where E_s is another equation of the system and k is as defined above.

Note that the transformed set of equations through *elementary operations* is equivalent to the original set of equations. Thus, solution of the transformed set of equations will be the solution of the original set of equations too.

Now, let us transform the above set of equation (A_0, B_0 and C_0) through *elementary operations* (shown inside bracket in the right side).

$$x + \frac{2}{3}y + \frac{1}{3}z = \frac{10}{3} \qquad (A_1 = \frac{1}{3}A_0)$$

$$0 - \frac{8}{3}y + \frac{8}{3}z = \frac{8}{3} \qquad (B_1 = B_0 - A_1)$$

$$0 - \frac{1}{3}y - \frac{5}{3}z = -\frac{17}{3} \qquad (C_1 = C_0 - 2A_1)$$

Note that variable x is eliminated from equations B_0 and C_0 to obtain B_1 and C_1 respectively. Equation A_0 in the previous set is known as *pivotal equation*.

Following similar procedure, y is eliminated from A_1 and C_1 as follows, considering B_1 as pivotal equation.

$$x + 0 + z = 4 \qquad (A_2 = A_1 - \frac{2}{3}B_2)$$

$$0 + y - z = -1 \qquad (B_2 = -\frac{3}{8}B_1)$$

$$0 + 0 - 2z = -6 \qquad (C_2 = C_1 + \frac{1}{3}B_2)$$

Finally, z is eliminated form A_2 and B_2 as follows, considering C_2 as pivotal equation.

$$x + 0 + 0 = 1 \qquad (A_3 = A_2 - C_3)$$

$$0 + y + 0 = 2 \qquad (B_3 = B_2 + C_3)$$

$$0 + 0 + z = 3 \qquad (C_3 = -\frac{1}{2}C_2)$$

Thus we end up with another set of equations which is equivalent to the original set having one variable in each equation. Transformed set of equations, (A_3, B_3 and C_3), thus obtained are said to be in *canonical form*. Operation at each step to eliminate one variable at a time, from all equations except one, is known as *pivotal operation*. It is obvious that the number of *pivotal operations* is the same as the number of variables in the set of equations. Thus we did three *pivotal operations* to obtain the canonical form of the set of equations having three variables each.

It may be noted that, at each *pivotal operation*, the pivotal equation is transformed first and using the transformed pivotal equation, other equations in the system are transformed. For example, while transforming, A_1, B_1 and C_1 to A_2, B_2 and C_2, considering B_1 as *pivotal equation*, B_2 is obtained first. A_2 and C_2 are then obtained using B_2. Transformation can be obtained by some other *elementary operations* also but will end up in the same canonical form. The procedure explained above is used in *simplex algorithm* which will be discussed later. The *elementary operations* involved in *pivotal operations*, as explained above, will help the reader to follow the analogy while understanding the *simplex algorithm*.

To generalize the procedure explained above, let us consider the following system of n equations with n variables.

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \qquad (E_1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2 \qquad (E_2)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + a_{nn}x_n = b_n \qquad (E_n)$$

Canonical form of above system of equations can be obtained by performing n pivotal operations through elementary operations. In general, variable $x_i \quad i = 1, 2, \cdots, n$ is eliminated from all equations except j^{th} equation for which a_{ji} is nonzero.

General procedure for one pivotal operation consists of following two steps,

1. Divide j^{th} equation by a_{ji}. Let us designate it as (E'_j), i.e., $E'_j = \dfrac{E_j}{a_{ji}}$

2. Subtract a_{ki} times of equation (E'_j) from k^{th} equation $(K = 1, 2, \cdots j-1, j+1, \cdots, n)$, i.e., $E_k - a_{ki}E'_j$

Above steps are repeated for all the variables in the system of equations to obtain the canonical form. Finally the canonical form will be as follows:

$$1x_1 + 0x_2 + \cdots\cdots\cdots + 0x_n = b_1'' \qquad (E_1^c)$$

$$0x_1 + 1x_2 + \cdots\cdots\cdots + 0x_n = b_2'' \qquad (E_2^c)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$0x_1 + 0x_2 + \cdots\cdots\cdots + 1x_n = b_n'' \qquad (E_n^c)$$

It is obvious that solution of the system of equations can be easily obtained from *canonical form*, such as:

$$x_i = b_i''$$

which is the solution of the original set of equations too as the *canonical form* is obtained through *elementary operations*.

Now let us consider more general case for which the system of equations has m equations with n variables ($n \geq m$). It is possible to transform the set of equations to an equivalent *canonical form* from which at least one solution can be easily deduced.

Let us consider the following general set of equations.

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \qquad (E_1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2 \qquad (E_2)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots + a_{mn}x_n = b_m \qquad (E_m)$$

By performing *n pivotal operations* (described earlier) for any m variables (say, $x_1, x_2, \cdots x_m$, called *pivotal variables*), the system of equations reduced to *canonical form* will be as follows:

$$1x_1 + 0x_2 + \cdots\cdots\cdots + 0x_m + a''_{1,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{1n}x_n = b''_1 \qquad (E_1^c)$$

$$0x_1 + 1x_2 + \cdots\cdots\cdots + 0x_m + a''_{2,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{2n}x_n = b''_2 \qquad (E_2^c)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$0x_1 + 0x_2 + \cdots\cdots\cdots + 1x_m + a''_{m,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{mn}x_n = b''_m \qquad (E_m^c)$$

Variables, $x_{m+1}, \cdots, x_n$, of above set of equations are known as *nonpivotal variables* or independent variables. One solution that can be obtained from the above set of equations is $x_i = b''_i$ for $i = 1, 2, \cdots, m$ and $x_i = 0$ for $i = m+1, \cdots, n$. This solution is known as *basic solution*. *Pivotal variables*, $x_1, x_2, \cdots x_m$, are also known as *basic variables*. *Nonpivotal variables*, $x_{m+1}, \cdots, x_n$, are known as *nonbasic variables*.

Canonical form of a set of LPP

Similar procedure can be followed in the case of a standard form of LPP. Objective function and all constraints for such standard form of LPP constitute a linear set of equations. In general this linear set will have m equations with n variables ($n \geq m$). The set of canonical form obtained from this set of equations is known as canonical form of LPP.

If the *basic solution* satisfies all the constraints as well as non-negativity criterion for all the variables, such *basic solution* is also known as *basic feasible solution*. It is obvious that, there can be ${}^n c_m$ numbers of different *canonical forms* and corresponding *basic feasible solutions*. Thus, if there are 10 equations with 15 variables there exist ${}^{15}c_{10}$= 3003 solutions, a huge number to be inspected one by one to find out the optimal solution. This is the reason which motivates for an efficient algorithm for solution of the LPP. *Simplex method* is one such popular method, which will be discussed after *graphical method*.

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