GRAPHICAL METHOD OF SOLVING LPP

Graphical method to solve *Linear Programming problem* (LPP) helps to visualize the procedure explicitly. It also helps to understand the different terminologies associated with the solution of LPP. In this class, these aspects will be discussed with the help of an example. However, this visualization is possible for a maximum of two decision variables. Thus, a LPP with two decision variables is opted for discussion. However, the basic principle remains the same for more than two decision variables also, even though the visualization beyond two-dimensional case is not easily possible.

Let us consider the same LPP (general form) discussed in previous class, stated here once again for convenience.

$$
\begin{array}{lll}
\text{Maximize} & Z = 6x + 5y & \\
\text{subject to} & 2x - 3y \le 5 & (\text{C}-1) \\
 & x + 3y \le 11 & (\text{C}-2) \\
 & 4x + y \le 15 & (\text{C}-3) \\
 & x,\ y \ge 0 & (\text{C}-4)\ \&\ (\text{C}-5)
\end{array}
$$

First step to solve above LPP by graphical method, is to plot the inequality constraints one-by-one on a graph paper. Fig. 1a shows one such plotted constraint $2x - 3y \le 5$. The plotting is first done by considering this an equality constraint i.e., $2x - 3y = 5$. The hatched portion indicates the feasibility region of the inequality constraint.

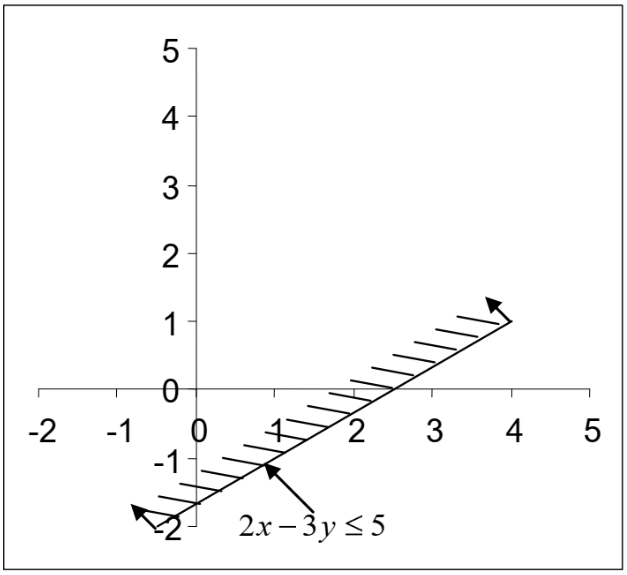

Fig. 1a Plot showing first constraint ($2x - 3y \le 5$)

Fig. 1b shows all the constraints including the nonnegativity of the decision variables (i.e., $x \geq 0$ and $y \geq 0$).

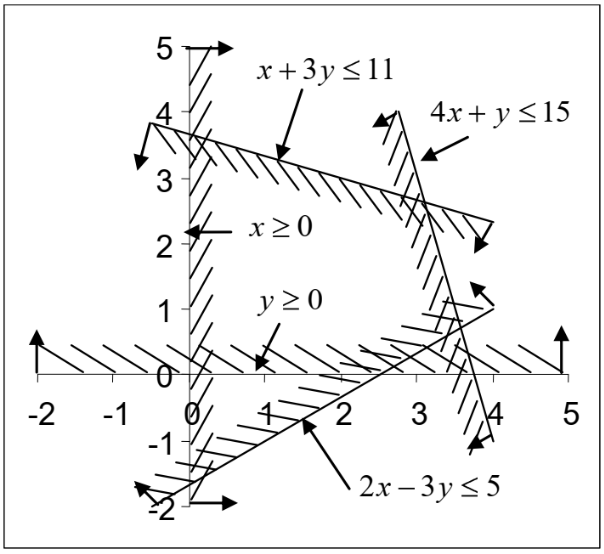

Fig. 1b Plot of all the constraints

Common region of all these constraints is known as *feasible region* (Fig. 1c). Feasible region implies that each and every point in this region satisfies all the constraints involved in the LPP.

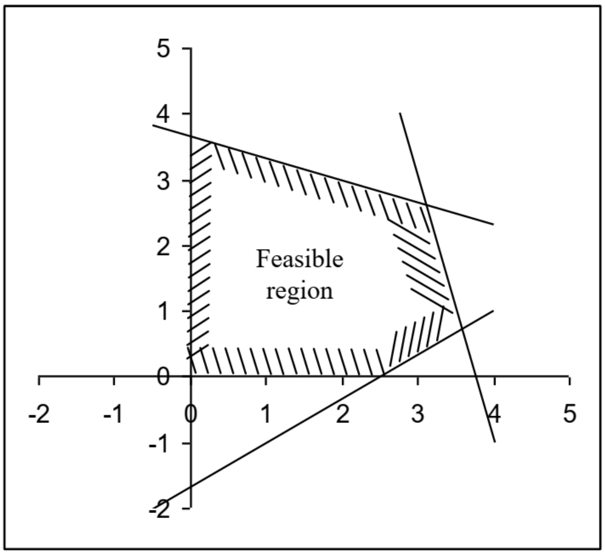

Fig. 1c Feasible region

Once the feasible region is identified, objective function ($Z = 6x + 5y$) is to be plotted on it. As the (optimum) value of Z is not known, objective function is plotted by considering any constant, k (Fig. 1d). The straight line, $6x + 5y = k$ (constant), is known as Z *line* (Fig. 1d). This line can be shifted in its perpendicular direction (as shown in the Fig. 1d) by changing the value of *k*. Note that, position of *Z line* shown in Fig. 1d, showing the intercept, *c*, on the y axis is 3. If, $6x + 5y = k \Rightarrow 5y = -6x + k \Rightarrow y = \frac{-6}{5}x + \frac{k}{5}$, i.e., $m = \frac{-6}{5}$ and $c = \frac{k}{5} = 3 \Rightarrow k = 15$.

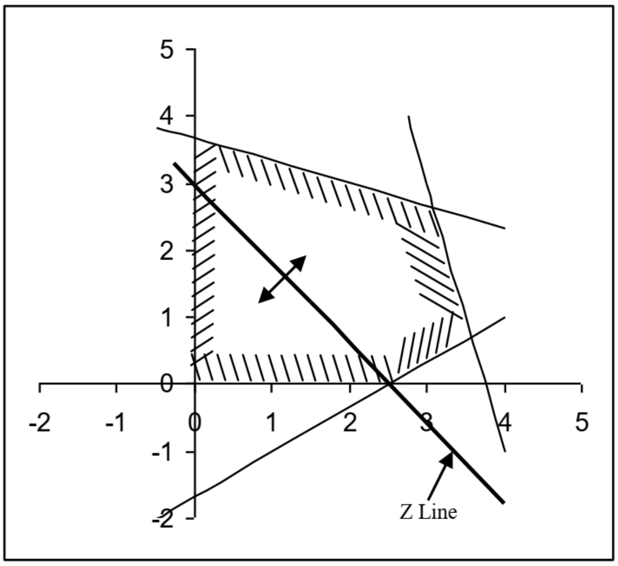

Fig. 1d Plot of *Z line* and *feasible region*

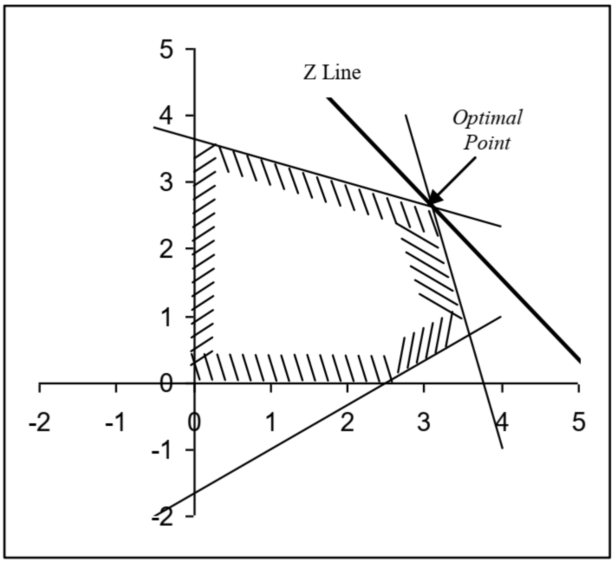

Fig. 1e Location of Optimal Point

Now it can be visually noticed that value of the objective function will be maximum when it passes through the intersection of $x + 3y = 11$ and $4x + y = 15$ (straight lines associated with the second and third inequality constraints). This is known as *optimal point* (Fig. 1e). Thus the *optimal point* of the present problem is $x^* = 3.091$ and $y^* = 2.636$. And the optimal solution is $= 6x^* + 5y^* = 31.726$

VISUAL REPRESENTATION OF DIFFERENT CASES OF SOLUTION OF LPP

A linear programming problem may have i) a unique, finite solution, ii) an unbounded solution iii) multiple (or infinite) number of optimal solutions, iv) infeasible solution and v) a unique feasible point. In the context of graphical method it is easy to visually demonstrate the different situations which may result in different types of solutions.

Unique, finite solution

The example demonstrated above is an example of LPP having a unique, finite solution. In such cases, optimum value occurs at an extreme point or vertex of the feasible region.

Unbounded solution

If the feasible region is not bounded, it is possible that the value of the objective function goes on increasing without leaving the feasible region. This is known as unbounded solution (Fig 2).

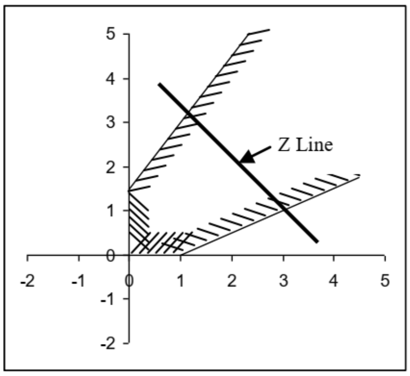

Fig. 2 Unbounded Solution

Multiple (infinite) solutions

If the *Z line* is parallel to any side of the feasible region all the points lying on that side constitute optimal solutions as shown in Fig 3.

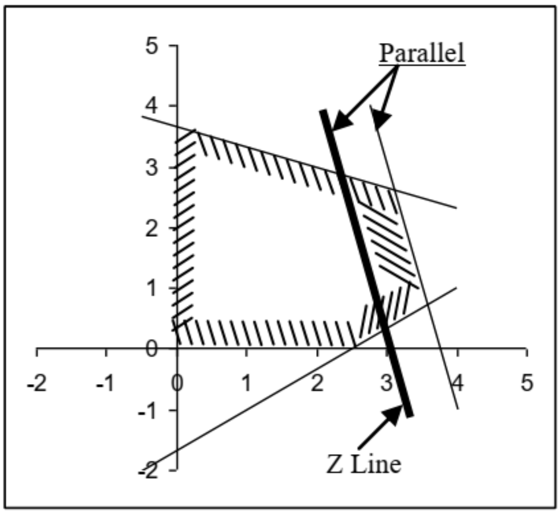

Fig. 3 Multiple (infinite) Solution

Infeasible solution

Sometimes, the set of constraints does not form a feasible region at all due to inconsistency in the constraints. In such situation the LPP is said to have infeasible solution. Fig 4 illustrates such a situation.

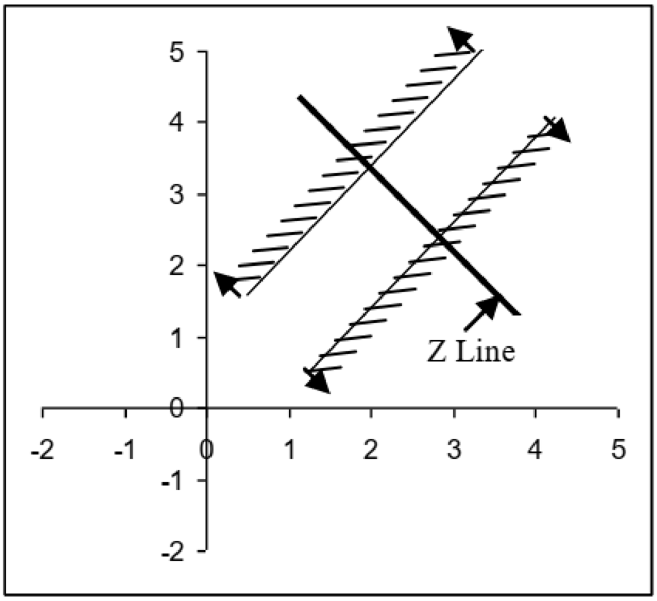

Fig. 4 Infeasible Solution

Unique feasible point

This situation arises when feasible region consist of a single point. This situation may occur only when number of constraints is at least equal to the number of decision variables. An example is shown in Fig 5. In this case, there is no need for optimization as there is only one solution.

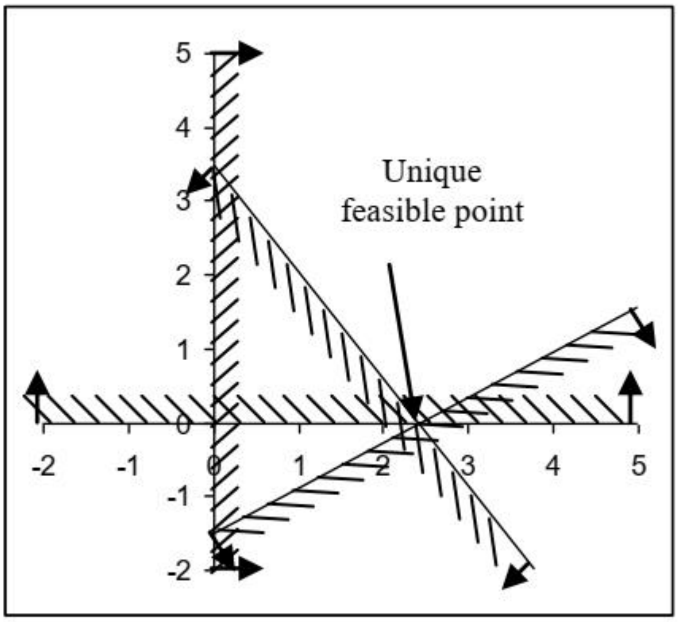

Fig. 5 Unique feasible point

OPTIMIZATION OF WATER DISTRIBUTION SYSTEMS

Simulation of distribution networks as discussed above helps to determine the hydraulic parameters such as pressure heads, tank levels etc. These models are unable to determine the optimal or minimum cost system. In addition to the cost minimization, the typical goals of water distribution systems problem in designing pipe system can be:

A) Meeting the household demands.

B) Meeting the required water pressure at all nodes of the distribution system.

C) Optimal positioning of valves.

Therefore, designing water distribution system is a multiobjective problem, which is also characterized by nonlinearity resulting from the simulation model.

Since the main purpose of a water distribution system is to supply according to the demands with adequate pressure, a typical optimization problem will be to minimize the system's cost while meeting the demands at required pressures. Hence optimization problem can be stated as:

Minimize : Total cost (Capital cost + Energy cost for pumping water throughout the system)

Subject to:

(i) Hydraulic constraints

(ii) Water demand constraints

(iii) Pressure requirements

BIBLIOGRAPHY / FURTHER READING:

1. Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, N.J, 1957.
2. Hillier F.S. and G.J. Lieberman, *Operations Research*, CBS Publishers & Distributors, New Delhi, 1987.
3. Loucks, D.P., J.R. Stedinger, and D.A. Haith, *Water Resources Systems Planning and Analysis*, Prentice-Hall, N.J., 1981.
4. Rao S.S., *Engineering Optimization – Theory and Practice*, Fourth Edition, John Wiley and Sons, 2009.
5. Taha H.A., *Operations Research – An Introduction*, 8th edition, Pearson Education India, 2008.
6. Vedula S., and P.P. Mujumdar, *Water Resources Systems: Modelling Techniques and Analysis*, Tata McGraw Hill, New Delhi, 2005.