SIMPLEX METHOD - II

INTRODUCTION

In the previous lecture the *simplex method* was discussed with required transformation of objective function and constraints. Different types of LPP solutions in the context of Simplex method will be discussed in this lecture. A discussion on minimization versus maximization will also be presented. *Duality of LP* problem is a useful property that makes the problem easier in some cases and leads to *dual simplex method*. This is also helpful in *sensitivity or post optimality analysis* of decision variables.

'Unbounded', 'Multiple' and 'Infeasible' solutions in the context of Simplex Method

As already discussed in lecture notes 2, a linear programming problem may have different type of solutions corresponding to different situations. Visual demonstration of these different types of situations was also discussed in the context of graphical method. Here, the same will be discussed in the context of Simplex method.

Unbounded solution

If at any iteration no departing variable can be found corresponding to entering variable, the value of the objective function can be increased indefinitely, i.e., the solution is unbounded.

Multiple (infinite) solutions

If in the final tableau, one of the non-basic variables has a coefficient 0 in the Z-row, it indicates that an alternative solution exists. This non-basic variable can be incorporated in the basis to obtain another optimal solution. Once two such optimal solutions are obtained, infinite number of optimal solutions can be obtained by taking a weighted sum of the two optimal solutions. Consider the problem,

$$\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \leq 6 \\ & 3x_1 + 2x_2 = 18 \\ & x_1, x_2 \geq 0 \end{array}$$

The only modification is that the coefficient of x_2 is changed from 5 to 2 in the objective function. Thus the slope of the objective function and that of third constraint are now same. It may be recalled from lecture notes 2, that if the *Z line* is parallel to any side of the feasible region (i.e., one of the constraints) all the points lying on that side constitute optimal solutions (refer fig 3 in lecture notes 2). So, reader should be able to imagine graphically that the LPP is having infinite solutions. However, for this particular set of constraints, if the objective function is made parallel (with equal slope) to either the first constraint or the second constraint, it will not lead to multiple solutions. The reason is very simple and left for the reader to find out. As a hint, plot all the constraints and the objective function on an arithmetic paper.

Now, let us see how it can be found in the simplex tableau. Coming back to our problem, final tableau is shown as follows. Full problem is left to the reader as practice.

Final tableau:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
3	Z	1	0	$\bigcirc\!\!\!\!0$	0	0	M	$1+M$	18	--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--

Coefficient of non-basic variable x_2 is zero

As there is no negative coefficient in the Z-row the optimal is reached. The solution is $Z = 18$ with $x_1 = 6$ and $x_2 = 0$. However, the coefficient of non-basic variable x_2 is zero as shown in the final simplex tableau. So, another solution is possible by incorporating x_2 in the basis. Based on the $\frac{b_r}{c_{rs}}$, x_4 will be the exiting variable. The next tableau will be as follows:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
4	Z	1	0	0	0	0	M	$1+M$	18	--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	6
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	18

Coefficient of non-basic variable x_4 is zero

Thus, another solution is obtained, which is $Z=18$ with $x_1=2$ and $x_2=6$. Again, it may be noted that, the coefficient of non-basic variable x_4 is zero as shown in the tableau. If one more similar step is performed, same simplex tableau at iteration 3 will be obtained.

Thus, we have two sets of solutions as $\begin{Bmatrix}6\\0\end{Bmatrix}$ and $\begin{Bmatrix}2\\6\end{Bmatrix}$. Other optimal solutions will be obtained as $\beta\begin{Bmatrix}6\\0\end{Bmatrix}+\ 1-\beta\ \begin{Bmatrix}2\\6\end{Bmatrix}$ where, $\beta\in\ 0,1\ $. For example, let $\beta=0.4$, corresponding solution is $\begin{Bmatrix}3.6\\3.6\end{Bmatrix}$, i.e., $x_1=3.6$ and $x_2=3.6$. Note that values of the objective function are not changed for different sets of solution; for all the cases $Z=18$.

Infeasible solution

If in the final tableau, at least one of the artificial variables still exists in the basis, the solution is indefinite.

Reader may check this situation both graphically and in the context of Simplex method by considering following problem:

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & 3x_1 + 2x_2 \ge 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Minimization versus maximization problems

As discussed earlier, standard form of LP problems consist of a maximizing objective function. Simplex method is described based on the standard form of LP problems, i.e., objective function is of maximization type. However, if the objective function is of minimization type, simplex method may still be applied with a small modification. The required modification can be done in either of following two ways.

1. The objective function is multiplied by -1 so as to keep the problem identical and 'minimization' problem becomes 'maximization'. This is because of the fact that minimizing a function is equivalent to the maximization of its negative.
2. While selecting the entering nonbasic variable, the variable having the maximum coefficient among all the cost coefficients is to be entered. In such cases, optimal solution would be determined from the tableau having all the cost coefficients as non-positive (≤ 0)

Still one difficulty remains in the minimization problem. Generally the minimization problems consist of constraints with 'greater-than-equal-to' ($\ge$) sign. For example, minimize the price (to compete in the market); however, the profit should cross a minimum threshold. Whenever the goal is to minimize some objective, lower bounded requirements play the leading role. Constraints with 'greater-than-equal-to' ($\ge$) sign are obvious in practical situations.

To deal with the constraints with 'greater-than-equal-to' ($\ge$) and $=$ sign, *Big-M* method is to be followed as explained earlier.

DUALITY OF LP PROBLEMS

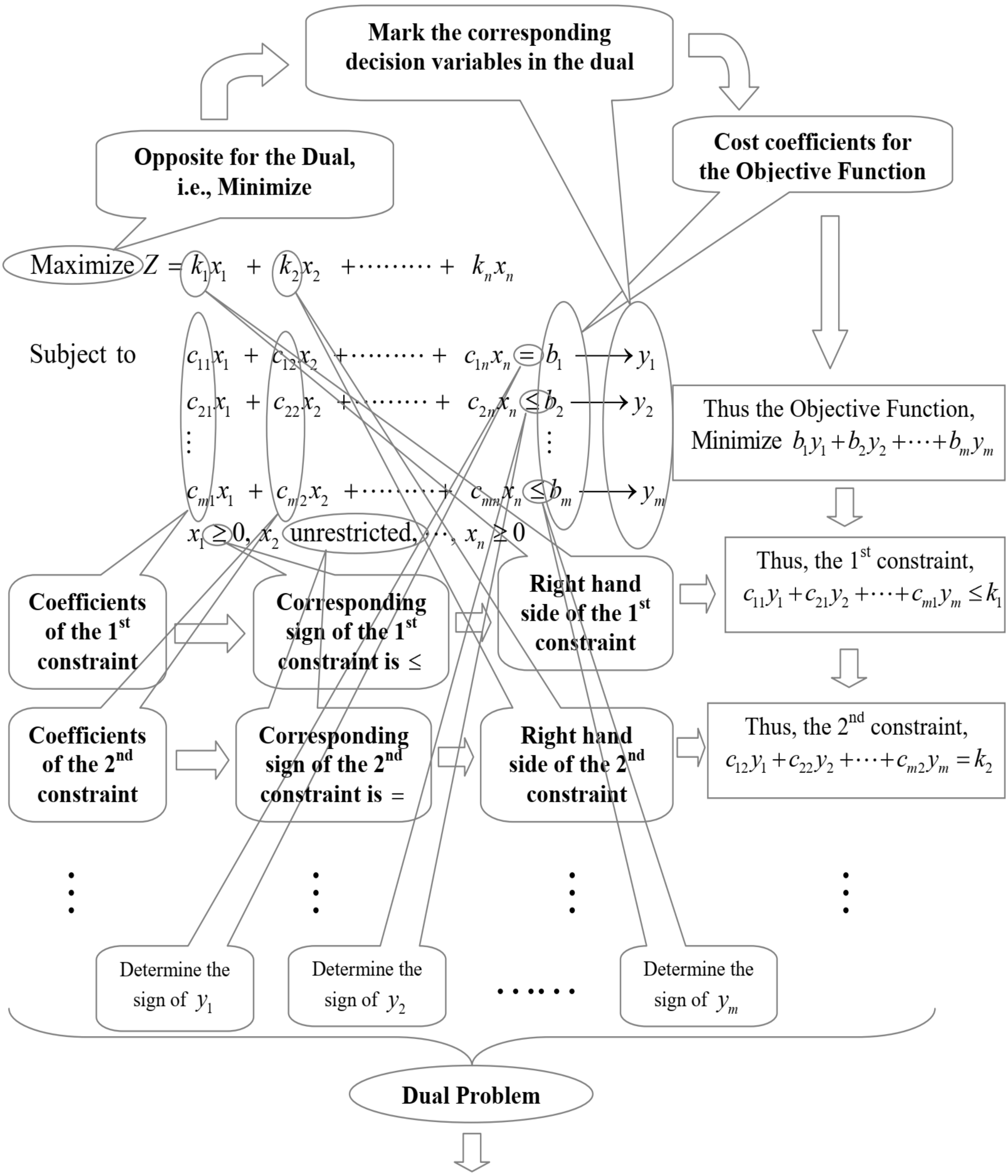
Each LP problem (called as **Primal** in this context) is associated with its counterpart known as **Dual** LP problem. Instead of primal, solving the dual LP problem is sometimes easier when a) the dual has fewer constraints than primal (time required for solving LP problems is directly affected by the number of constraints, i.e., number of iterations necessary to converge to an optimum solution which in Simplex method usually ranges from 1.5 to 3 times the number of structural constraints in the problem) and b) the dual involves maximization of an objective function (it may be possible to avoid artificial variables that otherwise would be used in a primal minimization problem).

The dual LP problem can be constructed by defining a new decision variable for each constraint in the primal problem and a new constraint for each variable in the primal. The coefficients of the j^{th} variable in the dual's objective function is the i^{th} component of the primal's requirements vector (right hand side values of the constraints in the Primal). The dual's requirements vector consists of coefficients of decision variables in the primal objective function. Coefficients of each constraint in the dual (i.e., row vectors) are the column vectors associated with each decision variable in the coefficients matrix of the primal problem. In other words, the coefficients matrix of the dual is the transpose of the primal's coefficient matrix. Finally, maximizing the primal problem is equivalent to minimizing the dual and their respective values will be exactly equal.

When a primal constraint is less than equal to type, the corresponding variable in the dual is non-negative. And equality constraint in the primal problem means that the corresponding dual variable is unrestricted in sign. Obviously, dual's dual is primal. In summary the following relationships exists between primal and dual.

Primal	**Dual**
Maximization	Minimization
Minimization	Maximization
i^{th} variable	i^{th} constraint
j^{th} constraint	j^{th} variable
$x_i \geq 0$	Inequality sign of i^{th} Constraint: $\leq$ if dual is maximization $\geq$ if dual is minimization
i^{th} variable unrestricted	i^{th} constraint with = sign
j^{th} constraint with = sign	j^{th} variable unrestricted
RHS of j^{th} constraint	Cost coefficient associated with j^{th} variable in the objective function
Cost coefficient associated with i^{th} variable in the objective function	RHS of i^{th} constraint

See the pictorial representation in the next page for better understanding and quick reference:



$$
\begin{aligned}
\text{Minimize } Z = {} & b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{Subject to } \quad & c_{11}y_1 + c_{21}y_2 + \cdots + c_{m1}y_m \le k_1 \\
& c_{12}y_1 + c_{22}y_2 + \cdots + c_{m2}y_m = k_2 \\
& \vdots \\
& c_{1n}y_1 + c_{2n}y_2 + \cdots + c_{mn}y_m \le k_n \\
& y_1 \text{ unrestricted, } y_2 \ge 0,\ \cdots,\ y_m \ge 0
\end{aligned}
$$

It may be noted that, before finding its dual, all the constraints should be transformed to 'less-than-equal-to' or 'equal-to' type for maximization problem and to 'greater-than-equal-to' or 'equal-to' type for minimization problem. It can be done by multiplying with -1 both sides of the constraints, so that inequality sign gets reversed.

An example of finding dual problem is illustrated with the following example.

Primal	**Dual**
Maximize $Z = 4x_1 + 3x_2$	Minimize $Z' = 6000\,y_1 - 2000\,y_2 + 4000\,y_3$
Subject to	Subject to
$x_1 + \frac{2}{3}x_2 \le 6000$	$y_1 - y_2 + y_3 = 4$
$x_1 - x_2 \ge 2000$	$\frac{2}{3}y_1 + y_2 \le 3$
$x_1 \le 4000$	$y_1 \ge 0$
x_1 unrestricted	$y_2 \ge 0$
$x_2 \ge 0$	$y_3 \ge 0$

It may be noted that second constraint in the primal is transformed to $-x_1 + x_2 \le -2000$ before constructing the dual.

PRIMAL-DUAL RELATIONSHIPS

Following points are important to be noted regarding primal-dual relationship:

1. If one problem (either primal or dual) has an optimal feasible solution, other problem also has an optimal feasible solution. The optimal objective function value is same for both primal and dual.

2. If one problem has no solution (infeasible), the other problem is either infeasible or unbounded.

3. If one problem is unbounded the other problem is infeasible.

DUAL SIMPLEX METHOD

Computationally, dual simplex method is same as simplex method. However, their approaches are different from each other. Simplex method starts with a nonoptimal but feasible solution where as dual simplex method starts with an optimal but infeasible solution. Simplex method maintains the feasibility during successive iterations where as dual simplex method maintains the optimality. Steps involved in the dual simplex method are:

1. All the constraints (except those with equality (=) sign) are modified to 'less-than-equal-to' ($\leq$) sign. Constraints with greater-than-equal-to' ($\geq$) sign are multiplied by -1 through out so that inequality sign gets reversed. Finally, all these constraints are transformed to equality (=) sign by introducing required slack variables.

2. Modified problem, as in step one, is expressed in the form of a simplex tableau. If all the cost coefficients are positive (i.e., optimality condition is satisfied) and one or more basic variables have negative values (i.e., non-feasible solution), then dual simplex method is applicable.

3. **Selection of exiting variable:** The basic variable with the highest negative value is the exiting variable. If there are two candidates for exiting variable, any one is selected. The row of the selected exiting variable is marked as pivotal row.

4. **Selection of entering variable:** Cost coefficients, corresponding to all the negative elements of the pivotal row, are identified. Their ratios are calculated after changing the sign of the elements of pivotal row, i.e., $ratio = \left(\frac{Cost\ \ Coefficien\,ts}{-1 \times Elements\ \ of\ \ pivotal\ \ row} \right)$.

   The column corresponding to minimum ratio is identified as the pivotal column and associated decision variable is the entering variable.

5. **Pivotal operation:** Pivotal operation is exactly same as in the case of simplex method, considering the pivotal element as the element at the intersection of pivotal row and pivotal column.

6. **Check for optimality:** If all the basic variables have nonnegative values then the optimum solution is reached. Otherwise, Steps 3 to 5 are repeated until the optimum is reached.

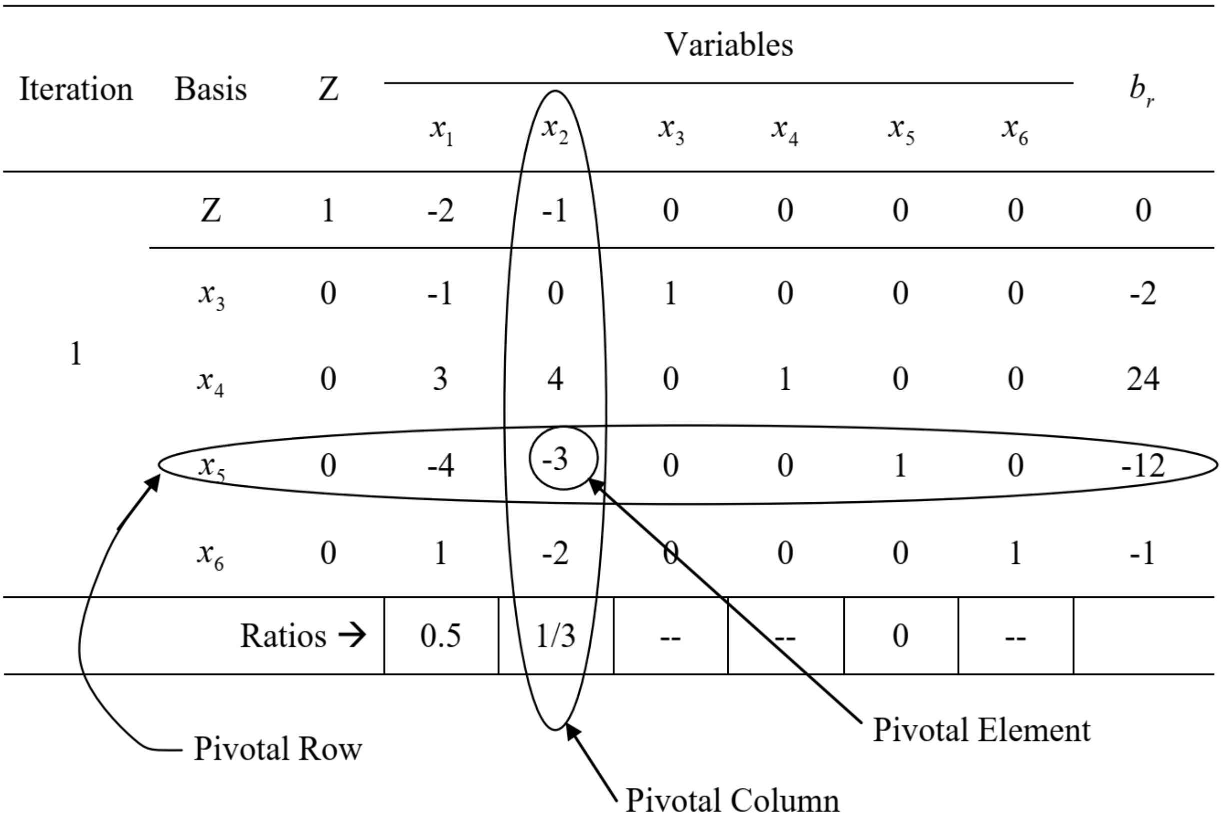
Consider the following problem:

$$\begin{array}{ll} \text{Minimize} & Z = 2x_1 + x_2 \\ \text{subject to} & x_1 \geq 2 \\ & 3x_1 + 4x_2 \leq 24 \\ & 4x_1 + 3x_2 \geq 12 \\ & -x_1 + 2x_2 \geq 1 \end{array}$$

By introducing the surplus variables, the problem is reformulated with equality constraints as follows:

$$\begin{array}{llllll} \text{Minimize} & Z = 2x_1 + x_2 & & & & \\ \text{subject to} & -x_1 & & +x_3 & = -2 \\ & 3x_1 & +4x_2 & +x_4 & = 24 \\ & -4x_1 & -3x_2 & +x_5 & = -12 \\ & x_1 & -2x_2 & +x_6 & = -1 \end{array}$$

Expressing the problem in the tableau form:



Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
1	Z	1	-2	-1	0	0	0	0	0
	x_3	0	-1	0	1	0	0	0	-2
	x_4	0	3	4	0	1	0	0	24
	x_5	0	-4	-3	0	0	1	0	-12
	x_6	0	1	-2	0	0	0	1	-1
	Ratios →		0.5	1/3	--	--	0	--	

Tableaus for successive iterations are shown below. *Pivotal Row*, *Pivotal Column* and *Pivotal Element* for each tableau are marked as usual.

Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
2	Z	1	-2/3	0	0	0	-1/3	0	4
	x_3	0	-1	0	1	0	0	0	-2
	x_4	0	-7/3	0	0	1	4/3	0	8
	x_2	0	4/3	1	0	0	-1/3	0	4
	x_6	0	11/3	0	0	0	-2/3	1	7
Ratios →			2/3	--	--	--	--	--	

Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
3	Z	1	0	0	-2/3	0	-1/3	0	16/3
	x_1	0	1	0	-1	0	0	0	2
	x_4	0	0	0	-7/3	1	4/3	0	38/3
	x_2	0	0	1	4/3	0	-1/3	0	4/3
	x_6	0	0	0	11/3	0	-2/3	1	-1/3
Ratios →			--	--	--	--	0.5	--	

Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
4	Z	1	0	0	2.5	0	0	-0.5	5.5
	x_1	0	1	0	-1	0	0	0	2
	x_4	0	0	0	5	1	0	2	12
	x_2	0	0	1	-0.5	0	0	-0.5	1.5
	x_5	0	0	0	-5.5	0	1	-1.5	0.5
Ratios →									

As all the b_r are positive, optimum solution is reached. Thus, the optimal solution is $Z = 5.5$ with $x_1 = 2$ and $x_2 = 1.5$.

Solution of Dual from Final Simplex Tableau of Primal

Primal

Maximize $Z = 4x_1 - x_2 + 2x_3$

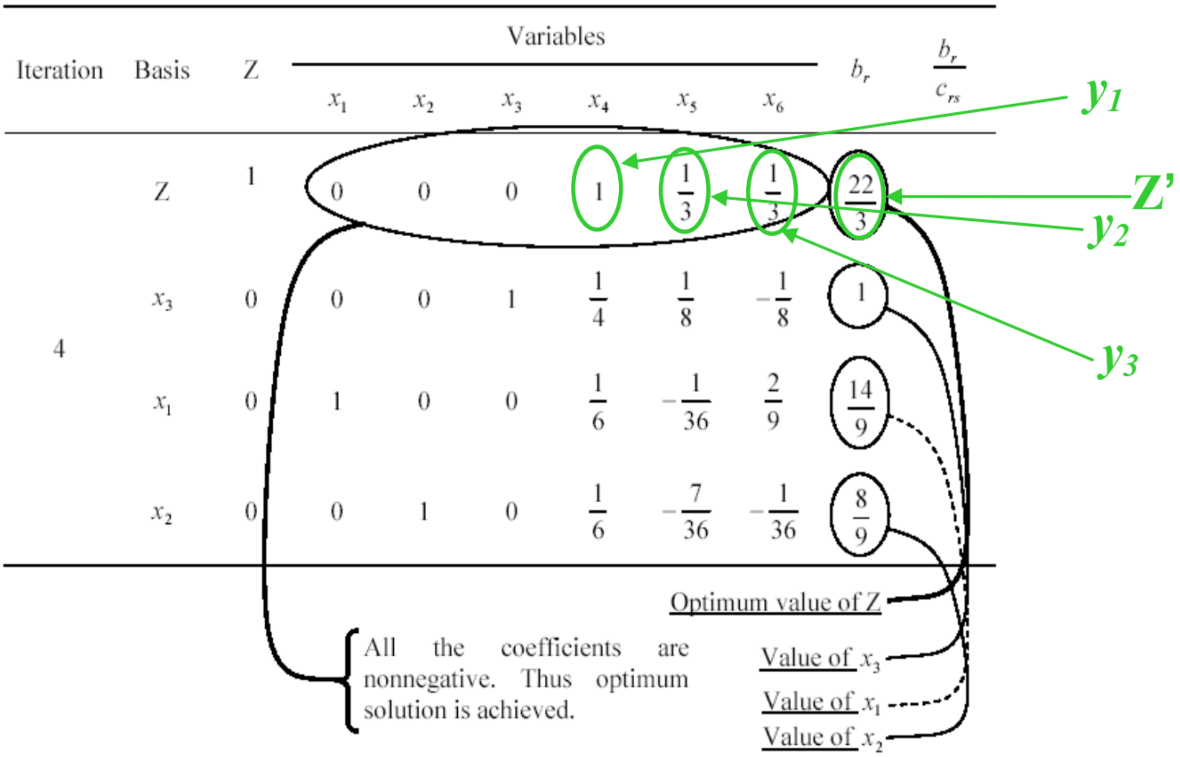
subject to
$$2x_1 + x_2 + 2x_3 \le 6$$
$$x_1 - 4x_2 + 2x_3 \le 0$$
$$5x_1 - 2x_2 - 2x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$

Dual

Minimize $Z' = 6y_1 + 0y_2 + 4y_3$

subject to
$$2y_1 + y_2 + 5y_3 \ge 4$$
$$y_1 - 4y_2 - 2y_3 \ge -1$$
$$2y_1 + 2y_2 - 2y_3 \ge 2$$
$$y_1, y_2, y_3 \ge 0$$

Final simplex tableau of primal:



Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	x_5	x_6		
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

Optimum value of Z

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of x_3

Value of x_1

Value of x_2

As illustrated above solution for the dual can be obtained corresponding to the coefficients of slack variables of respective constraints in the primal, in the Z row as, $y_1 = 1$, $y_2 = \frac{1}{3}$ and $y_3 = \frac{1}{3}$ and Z'=Z=22/3.

SENSITIVITY OR POST OPTIMALITY ANALYSIS

A dual variable, associated with a constraint, indicates a change in Z value (optimum) for a small change in RHS of that constraint. Thus,

$$\Delta Z = y_j \Delta b_i$$

where y_j is the dual variable associated with the i^{th} constraint, Δb_i is the small change in the RHS of i^{th} constraint, and ΔZ is the change in objective function owing to Δb_i.

Let, for a LP problem, i^{th} constraint be $2x_1 + x_2 \leq 50$ and the optimum value of the objective function be 250. What if the RHS of the i^{th} constraint changes to 55, i.e., i^{th} constraint changes to $2x_1 + x_2 \leq 55$? To answer this question, let, dual variable associated with the i^{th} constraint is y_j, optimum value of which is 2.5 (say). Thus, $\Delta b_i = 55 - 50 = 5$ and $y_j = 2.5$. So, $\Delta Z = y_j \Delta b_i = 2.5 \times 5 = 12.5$ and revised optimum value of the objective function is $250 + 12.5 \;= 262.5$.

It may be noted that Δb_i should be so chosen that it will not cause a change in the optimal basis.

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