LINEAR PROGRAMMING APPLICATIONS

INTRODUCTION

In the previous lectures, we discussed about the standard form of a LP and the commonly used methods of solving LPP. LP finds many applications in the field of water resources and structural design which include many types like planning of urban water distribution, reservoir operation, crop water allocation etc. In this lecture, applications of LP in deciding the optimal irrigation allocation and water quality management are discussed.

EXAMPLE – WATER RESOURCES

(1) Consider two crops 1 and 2. One unit of crop 1 brings four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let x be the amount of water required for A units of crop 1 and y be the same for B units of crop 2. The linear relations between the amounts of crop produced (i.e., demands A and B) and the available water (i.e., x and y) for two crops are shown below.

$$A = 0.5(x - 2) + 2$$

$$B = 0.6(y - 3) + 3$$

Minimum amount of water that must be provided to 1 and 2 to meet their demand is two and three units respectively. Maximum availability of water is ten units. Find out the optimum pattern of irrigation.

Solution:

The objective is to maximize the profit from crop 1 and 2, which can be represented as

$$Maximize\ f = 4A + 5B;$$

Expressing as a function of the amount of water,

$$\text{Maximize} \quad f = 4[0.5(x - 2) + 2] + 5[0.6(y - 3) + 3] = 2x + 3y + 10$$

subject to

$x + y \leq 10$; *Maximum availability of water*

$x \geq 2$; *Minimum amount of water required for crop 1*

$y \geq 3$; *Minimum amount of water required for crop 2*

The above problem is same as maximizing $f' = 2x + 3y$ subject to same constraints.

Changing the problem into standard form by introducing slack variables S_1, S_2, S_3

$$\text{Maximize} \; f' = 2x + 3y$$

subject to

$$x + y + S_1 = 10$$

$$-x + S_2 = -2$$

$$-y + S_3 = -3$$

This problem is solved by forming the simplex table as below

Starting Solution:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	-2	-3	0	0	0	0	
S_1	1	1	1	0	0	10	10
S_2	-1	0	0	1	0	-2	-
S_3	0	-1	0	0	1	-3	3

Iteration 1:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	-2	0	0	0	-3	9	-
S_1	1	0	1	0	1	7	7
S_2	-1	0	0	1	0	-2	2
y	0	1	0	0	-1	3	-

Iteration 2:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	0	0	0	-2	-3	13	-
S_1	0	0	1	1	1	5	5
x	1	0	0	-1	0	2	-
y	0	1	0	0	-1	3	-3

Iteration 3:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	0	0	3	1	0	28	-
S_3	0	0	1	1	1	5	-
x	1	0	0	-1	0	2	-
y	0	1	1	1	0	8	-

Hence the solution is

$x = 2; y = 8; \; f' = 28$

Therefore, $f = 28+10 = 38$

Thus, water allocated to crop A is 2 units and to crop B is 8 units and total profit yielded is 38 units.

EXAMPLE – WATER QUALITY MANAGEMENT

Waste load allocation for water quality management in a river system can be defined as determination of optimal treatment level of waste, which is discharged into a river; such that the water quality standards set by Pollution Control Agency (PCA) are maintained through out the river. Conventional waste load allocation involves minimization of treatment cost subject to the constraint that the water quality standards are not violated.

Consider a simple problem, where, there are M dischargers, who discharge waste into the river, and I checkpoints, where the water quality is measured by PCA. Let x_j be the treatment level and a_j be the unit treatment cost for j^{th} discharger $(j=1,2,\ldots,M)$ and c_i be the dissolved oxygen (DO) concentration at checkpoint i $(i=1,2,\ldots,I)$, which is to be controlled. Therefore the decision variables for the waste load allocation model are x_j $(j=1,2,\ldots,M)$.

Thus, the objective function can be expressed as

$$Minimize \quad f = \sum_{j=1}^{M} a_j x_j$$

The relationship between the water quality indicator, c_i (DO) at a checkpoint and the treatment level upstream to that checkpoint is linear (based on Streeter-Phelps Equation) when all other parameters involved in water quality simulations are constant. Let *g(x)* denotes the linear relationship between c_i and x_j. Then,

$$c_i = g(x_j) \qquad \forall i, j$$

Let c_P be the permissible DO level set by PCA, which is to be maintained through out the river. Therefore,

$$c_i \geq c_P \qquad \forall i$$

Solution of the optimization model using simplex algorithm gives the optimal fractional removal levels required to maintain the water quality of the river.

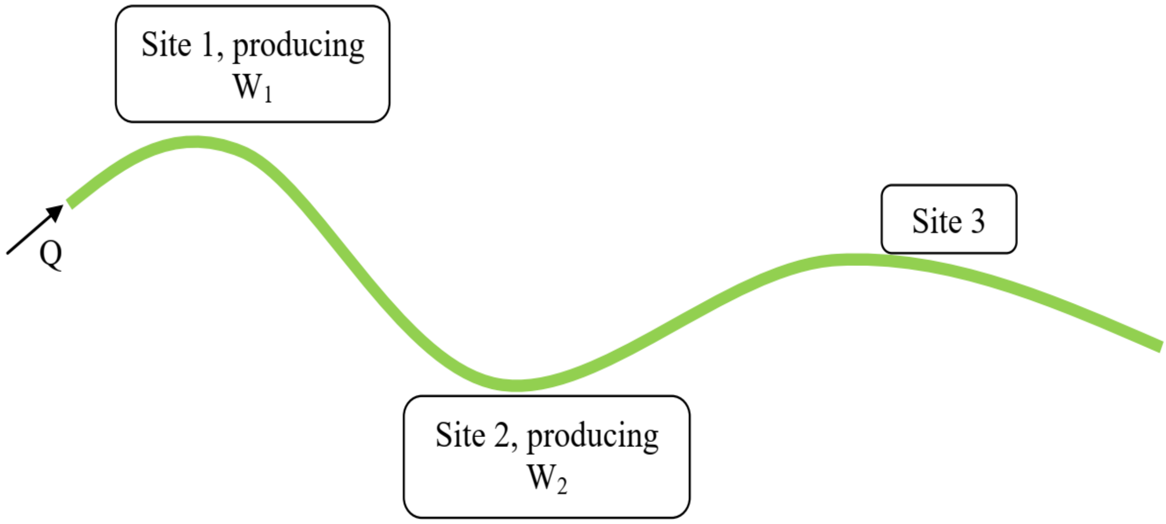
<u>Exercise problem:</u>

The stream shown below receives wastewater effluent from two point sources at Sites 1 and 2. There are 2 dischargers and 3 check points. The concentration of pollutant at the sites 2 and 3 should not exceed the maximum desired concentration. The values of the parameters are given below:

Parameters	Unit	Value
Flow upstream of site 1, Q_1 Flow upstream of site 2, Q_2 Flow upstream of site 3, Q_3	m^3/s	12 15 17
Pollutant produced at site 1, W_1 Pollutant produced at site 2, W_2	(mg/l) (m^3/s)	1500 750
Pollutant concentration u/s of site 1, P_1 Maximum pollutant concentration allowed at	mg/l	28 18

site 2, P_2 Maximum pollutant concentration allowed at site 3, P_3		18
Unit treatment cost for 1st discharger, a_1 Unit treatment cost for 2nd discharger, a_2		100 70
Fraction of pollutant from site 1 transferred to site 2, c_{12} Fraction of pollutant from site 1 transferred to site 3, c_{13} Fraction of pollutant from site 2 transferred to site 3, c_{23}		0.30 0.15 0.50

Find the level of wastewater treatment (waste removed) at sites 1 and 2 to achieve the desired concentrations at sites 2 and 3 at a minimum total cost.



LINEAR PROGRAMMING SOFTWARES

MMO Software

This is an MS-Dos based software to solve various types of problems. Opening screen can be seen as shown in Fig. 2. Press any key to see Main menu screen of MMO as shown in Fig. 3. Use arrow keys from keyboard to select different models.

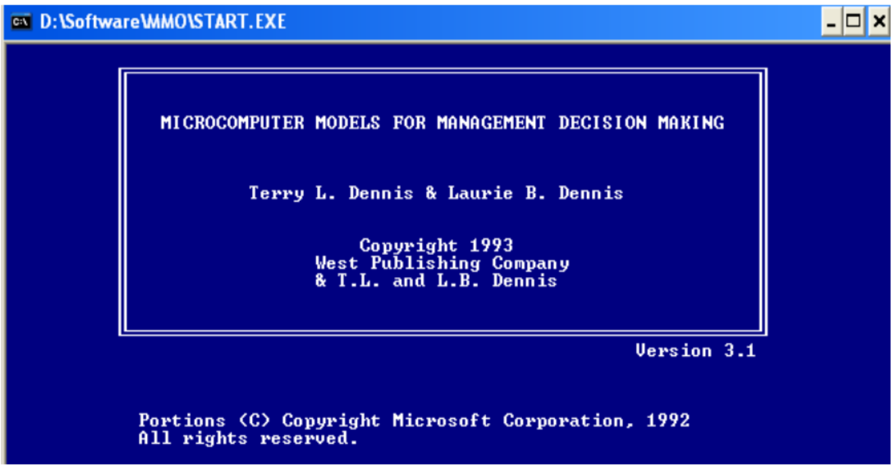


Fig. 2. Opening Screen of MMO

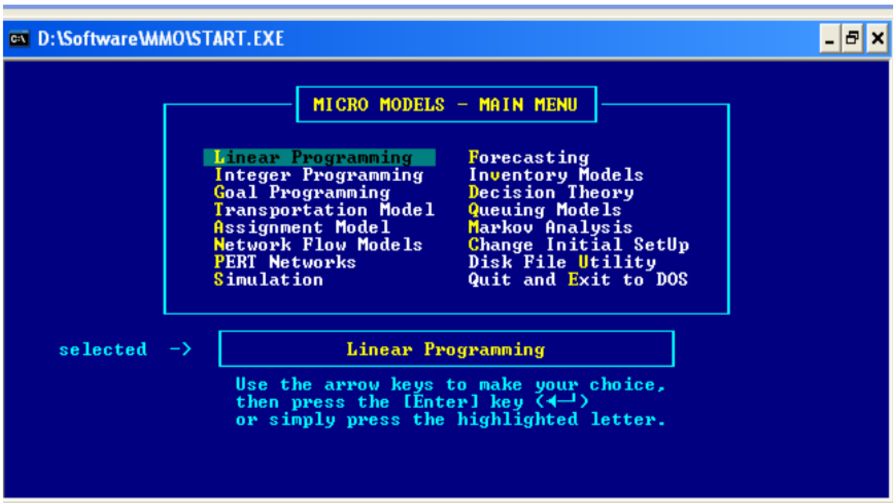


Fig. 3 Main Menu Screen of MMO

Select "Linear Programming" and press enter. Two options will appear as follows:

SOLUTION METHOD: GRAPHIC/ SIMPLEX

<u>SIMPLEX Method using MMO</u>

Select SIMPLEX in Linear Programming option of MMO software. Screen for "data entry method" will appear (Fig. 4).

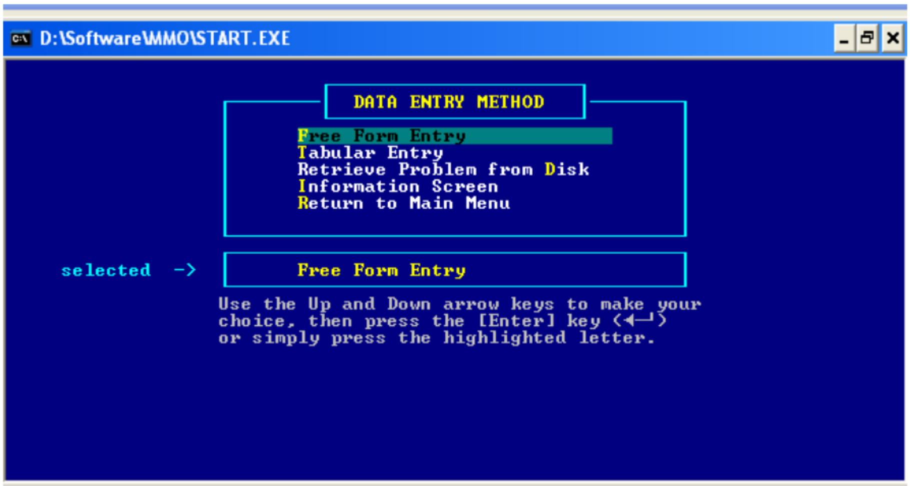


Fig. 4 Screen for "Data Entry Method"

Data entry may be done by either of two different ways.

1. Free Form Entry: You have to write the equation when prompted for input.
2. Tabular Entry: Data can be input in spreadsheet style. Only the coefficients are to be entered, not the variables.

Note that all variables must appear in the objective function (even those with a 0 coefficient); if a variable name is repeated in the objective function, an error message will indicate that it is a duplicate and allow you to change the entry. Constraints can be entered in any order; variables with 0 coefficients do not have to be entered; if a constraint contains a variable not found in the objective function, an error message indicates this and allows you to make the correction; constraints may not have negative right-hand-sides (multiply by -1 to convert them before entering); when entering inequalities using < or >, it is not necessary to add the equal sign (=); non-negativity constraints are assumed and do not have to be entered. However, this information can be made available by selecting "Information Screen".

Let us take following problem

$$
\begin{aligned}
&\textit{Maximize} \quad && Z = 2x_1 + 3x_2 \\
&\textit{Subject to} \quad && x_1 \le 5, \\
& && x_1 - 2x_2 \ge -5, \\
& && x_1 + x_2 \le 6 \\
& && x_1, x_2 \ge 0
\end{aligned}
$$

Thus, the second constraint is to be multiplied by -1 while entering, i.e., $-x_1 + 2x_2 \le 5$. After entering the problem the screen will appear as Fig. 5.

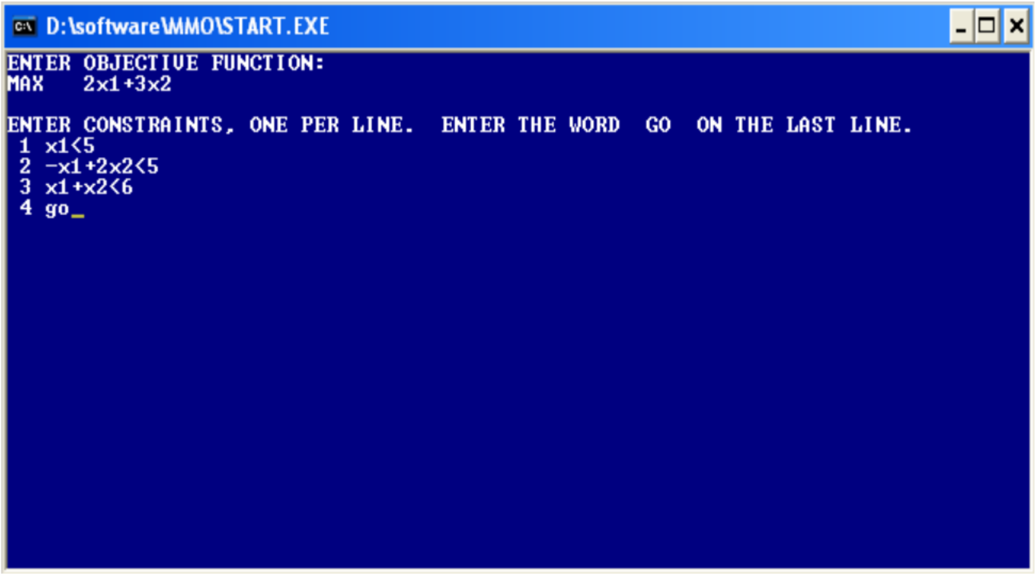


Fig. 5 Screen after Entering the Problem.

Once you run the problem, it will show the list of slack, surplus and artificial variables as shown in Fig. 6. Note that there are three additional slack variables in the above problem. Press any key to continue.

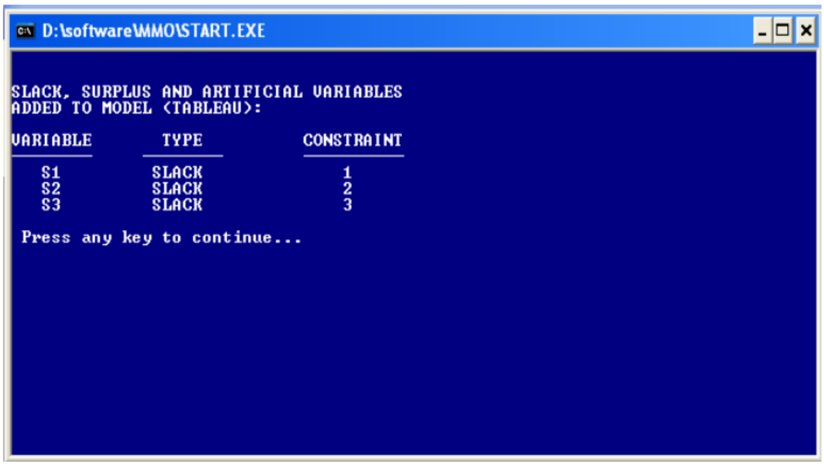


Fig. 6 List of slack, surplus and artificial variables

It will show three different options (Fig. 7):

1. No Tableau: Shows direct solutions
2. All Tableau: Shows all simplex tableau one by one
3. Final Tableau: Shows only the final simplex tableau directly

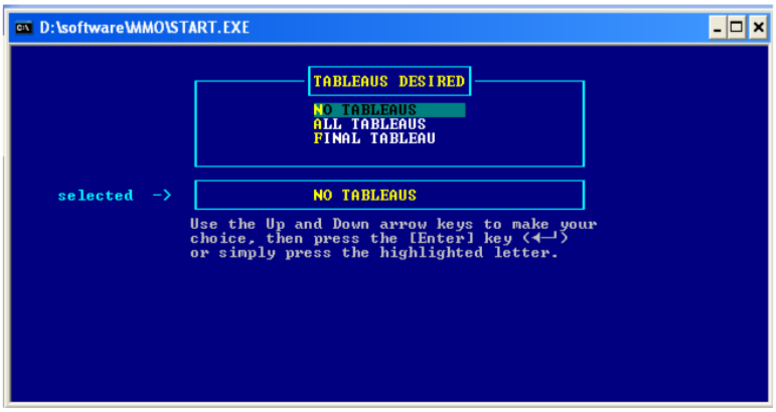


Fig. 7 Different Options for Simplex Solution

Final simplex tableau for the present problem is shown in Fig. 8 and the final solution is obtained as: Optimal $Z = 15.667$ with $x_1 = 2.333$ and $x_2 = 3.667$.

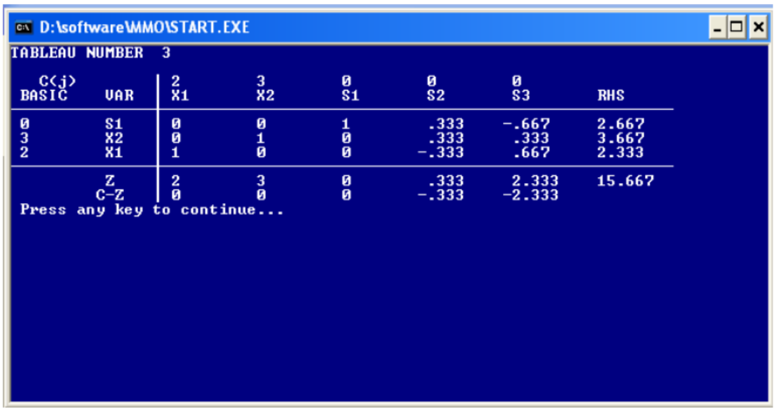


Fig. 8 Final Simplex Tableau

It may be noted that although MMO is good for classroom problems, it has limitation for solving large-scale LP problems. For such problems, commercially available software such as MATLAB, LINGO may be more useful.

<u>MATLAB Toolbox for Linear Programming</u>

Optimization toolbox of MATLAB (2001) is very popular and efficient. It includes different types of optimization techniques. In this lecture notes, we will briefly introduce the use of MATLAB toolbox for Simplex Algorithm. However, it is assumed that the users are aware of basics of MATLAB.

To use the simplex method, you have to set the option as 'LargeScale' to 'off' and 'Simplex' to 'on' in the following way.

options = optimset('LargeScale', 'off', 'Simplex', 'on')

Then a function called 'linprog' is to be used. A brief MATLAB documentation is shown in Fig. 9 for linear programming (linprog).

Optimization Toolbox

linprog

Solve a linear programming problem

$$\min_{x} f^T x \quad \text{such that} \quad \begin{aligned} A \cdot x &\le b \\ Aeq \cdot x &= beq \\ lb \le x &\le ub \end{aligned}$$

where *f*, *x*, *b*, *beq*, *lb*, and *ub* are vectors and *A* and *Aeq* are matrices.

Syntax

```
x = linprog(f,A,b,Aeq,beq)
x = linprog(f,A,b,Aeq,beq,lb,ub)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0,options)
[x,fval] = linprog(...)
[x,fval,exitflag] = linprog(...)
[x,fval,exitflag,output] = linprog(...)
[x,fval,exitflag,output,lambda] = linprog(...)
```

Description

`linprog` solves linear programming problems.

`x = linprog(f,A,b)` solves min `f'*x` such that `A*x <= b`.

`x = linprog(f,A,b,Aeq,beq)` solves the problem above while additionally satisfying the equality constraints `Aeq*x = beq`. Set `A=[]` and `b=[]` if no inequalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub)` defines a set of lower and upper bounds on the design variables, `x`, so that the solution is always in the range `lb <= x <= ub`. Set `Aeq=[]` and `beq=[]` if no equalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub,x0)` sets the starting point to `x0`. This option is only available with the medium-scale algorithm (the `LargeScale` option is set to `'off'` using `optimset`). The default large-scale algorithm and the simplex algorithm ignore any starting point.

`x = linprog(f,A,b,Aeq,beq,lb,ub,x0,options)` minimizes with the optimization options specified in the structure `options`. Use `optimset` to set these options.

`[x,fval] = linprog(...)` returns the value of the objective function`fun` at the solution `x`: `fval = f'*x`.

`[x,lambda,exitflag] = linprog(...)` returns a value `exitflag` that describes the exit condition.

`[x,lambda,exitflag,output] = linprog(...)` returns a structure `output` that contains information about the optimization.

Fig. 9 MATLAB Documentation for Linear Programming

Further details may be referred from the toolbox. However, with this basic knowledge, simple LP problems can be solved. Let us consider the same problem as considered earlier.

$$\begin{aligned} &\textit{Maximize} && Z = 2x_1 + 3x_2 \\ &\textit{Subject to} && x_1 \le 5, \\ &&& x_1 - 2x_2 \ge -5, \\ &&& x_1 + x_2 \le 6 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

Following MATLAB code will give the solution using simplex algorithm.

```
clear all
f=[-2 -3]; %Converted to minimization problem
A=[1 0;-1 2;1 1];
b=[5 5 6];
lb=[0 0];
options = optimset('LargeScale', 'off', 'Simplex', 'on');
[x,fval]=linprog(f,A,b,[],[],lb);
z=-fval   %Multiplied by -1
x
```

Note that objective function should be converted to a minimization problem before entering as done in line 2 of the code. Finally, solution should be multiplied by -1 to the optimized (maximum) solution as done in last but one line. Solution will be obtained as $Z = 15.667$ with $x_1 = 2.333$ and $x_2 = 3.667$ as in the earlier case.

LINGO

LINGO is tool to solve linear, nonlinear, quadratic, stochastic and integer optimization models**.** LINGO can be downloaded from http://www.lindo.com. The key benefits of LINGO are: easy model expression, convenient data options, powerful solvers, extensive documentation and help.

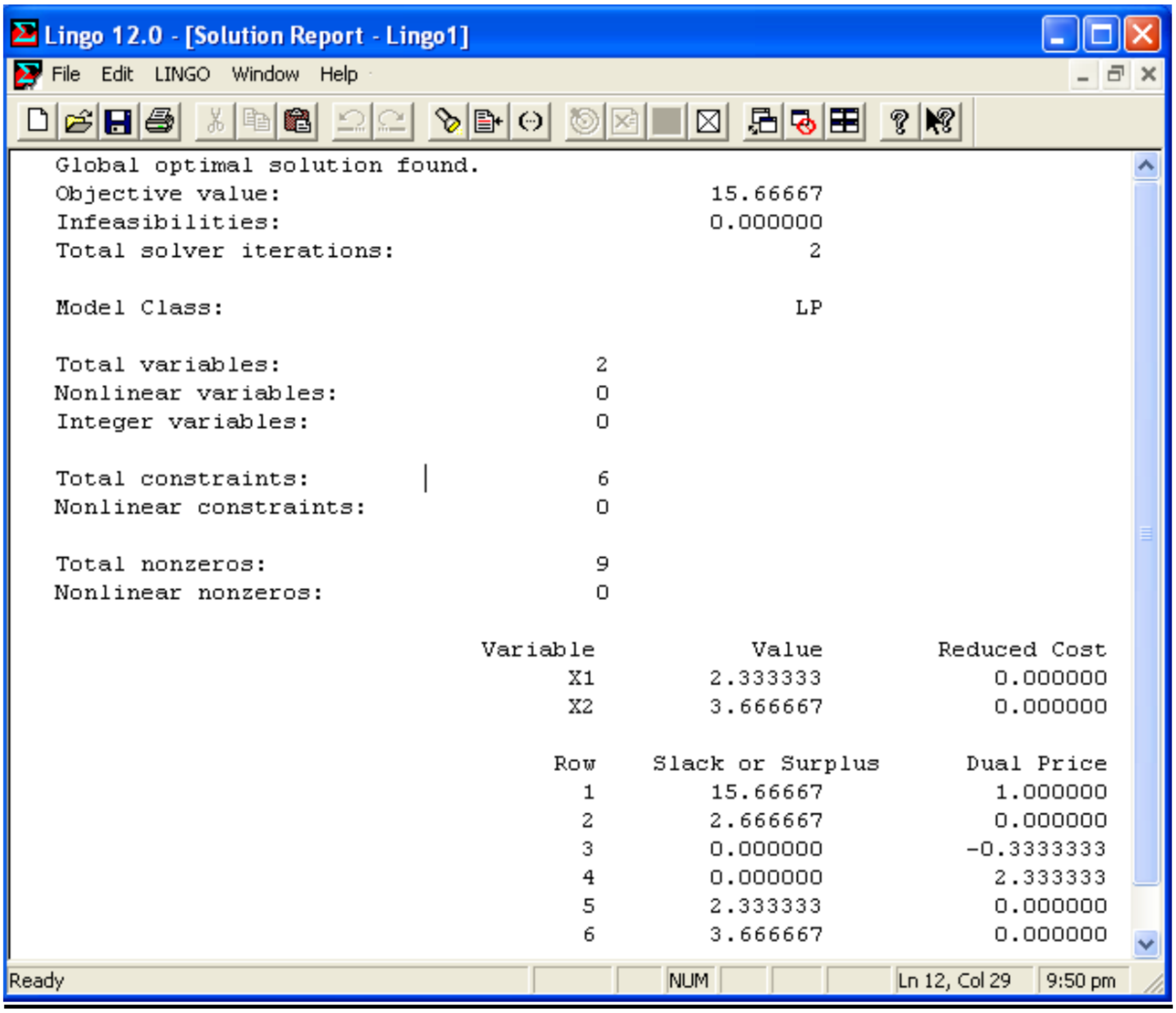
Let us consider the same problem

$$\begin{aligned} &\textit{Maximize} && Z = 2x_1 + 3x_2 \\ &\textit{Subject to} && x_1 \le 5, \\ &&& x_1 - 2x_2 \ge -5, \\ &&& x_1 + x_2 \le 6 \\ &&& x_1, x_2 \ge 0 \end{aligned}$$

The LINGO formulation is:

```
Max = 2*x+ 3*y;
x<=5;
x-2*y>=-5;
x+y<=6;
x>=0;
y>=0;
```

The solution report from LINGO is shown in figure 10.




```
Global optimal solution found.
Objective value:                              15.66667
Infeasibilities:                              0.000000
Total solver iterations:                             2

Model Class:                                        LP

Total variables:                      2
Nonlinear variables:                  0
Integer variables:                    0

Total constraints:                    6
Nonlinear constraints:                0

Total nonzeros:                       9
Nonlinear nonzeros:                   0

                        Variable           Value        Reduced Cost
                              X1        2.333333            0.000000
                              X2        3.666667            0.000000

                             Row    Slack or Surplus      Dual Price
                               1        15.66667            1.000000
                               2        2.666667            0.000000
                               3        0.000000          -0.3333333
                               4        0.000000            2.333333
                               5        2.333333            0.000000
                               6        3.666667            0.000000
```

Fig. 10 Solution report from LINGO

Solution obtained is $Z = 15.667$ with $x_1 = 2.333$ and $x_2 = 3.667$ as in the earlier case.

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