Classical Mechanics

LECTURE 10

5.3 Poisson brackets

Suppose we have a function $f(\mathbf{q}, \mathbf{p}, t)$ on $\mathscr{P} \times \mathbb{R}$ (with $\mathbb{R}$ the time direction). Then its evolution in time, when evaluated on a solution $(\mathbf{q}(t), \mathbf{p}(t))$ to the Hamilton equations of motion, is computed as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f &= \sum_{a=1}^{n} \left(\frac{\partial f}{\partial q_a} \dot{q}_a + \frac{\partial f}{\partial p_a} \dot{p}_a \right) + \frac{\partial f}{\partial t} \\ &= \sum_{a=1}^{n} \left(\frac{\partial f}{\partial q_a} \frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial H}{\partial q_a} \right) + \frac{\partial f}{\partial t} \ . \end{aligned} \tag{5.26}$$

This motivates defining the *Poisson bracket* of two functions f, g on phase space as[12]

$$\{f, g\} \equiv \sum_{a=1}^{n} \frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q_a} \ . \tag{5.27}$$

Here f and g may also depend on time t, *i.e.* $f = f(\mathbf{q}, \mathbf{p}, t)$, $g = g(\mathbf{q}, \mathbf{p}, t)$ are more precisely functions on $\mathscr{P} \times \mathbb{R}$. The Poisson bracket has the following properties, for any functions f, g, h on $\mathscr{P} \times \mathbb{R}$:

1. *anti-symmetry*: $\{f, g\} = -\{g, f\}$,
2. *linearity*: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$, for all constant $\alpha, \beta \in \mathbb{R}$,
3. *Leibniz rule*: $\{fg, h\} = f\{g, h\} + \{f, h\}g$,
4. *Jacobi identity*: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

The anti-symmetry and linearity follow almost immediately from the definition, while the Leibniz rule follows straightforwardly from the usual Leibniz rule of calculus. The Jacobi identity is a little more tedious to verify – you simply substitute from the definition and check that all the terms do indeed cancel (see Problem Sheet 4). Notice also that $\{f, c\} = 0$ for any constant c.

With this definition we may immediately derive some very nice formulae. First, the dynamical evolution of the funtion f in (5.26) is

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{f, H\} + \frac{\partial f}{\partial t} \ . \tag{5.28}$$

In particular when $\partial f / \partial t = 0$ we have simply $\dot{f} = \{f, H\}$, and we say that the Hamiltonian H *generates the time-evolution* of the function (via the Poisson bracket). Two special cases are Hamilton's equations themselves. Namely putting $f = p_a$ or $f = q_a$ we have

$$\begin{aligned} \dot{p}_a &= \{p_a, H\} = -\frac{\partial H}{\partial q_a} \ , \\ \dot{q}_a &= \{q_a, H\} = \frac{\partial H}{\partial p_a} \ . \end{aligned} \tag{5.29}$$

The Poisson bracket is clearly quite fundamental! Moreover, it is straightforward to compute the Poisson brackets between the canonical coordinates themselves:

$$\{q_a, q_b\} = 0 = \{p_a, p_b\} \ , \qquad \{q_a, p_b\} = \delta_{ab} \ . \tag{5.30}$$

[12] Some authors define this with the opposite overall sign.

These are often called the *fundamental Poisson brackets*, or *canonical Poisson brackets*. Those who have taken any quantum mechanics courses may recognize that these Poisson bracket formulae look *very* similar to commutation relations. This is no accident, but we'll postpone any further discussion of this to the starred subsection at the end of this section.

Conserved quantities

Let's begin by looking at the Hamiltonian itself. Using (5.28) we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \dot{H} = \{H,H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} , \tag{5.31}$$

where $\{f,f\} = 0$ trivially for any function f due to the anti-symmetry of the Poisson bracket. Thus if H doesn't depend explicitly on time it is conserved. Of course this is simply the Hamiltonian version of the conservation of energy law we found in Lagrangian mechanics when $\partial L/\partial t = -\partial H/\partial t = 0$. In terms of the phase space picture we described in the last subsection, conservation of H means that H is constant along the flow trajectories $\gamma(t)$.

More generally a *constant of the motion*, or *conserved quantity*, is a function $f = f(\mathbf{q},\mathbf{p},t)$ that when evaluated on a solution to the Hamilton equations of motion is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \dot{f} = \{f,H\} + \frac{\partial f}{\partial t} = 0 . \tag{5.32}$$

In particular if $\partial f/\partial t = 0$, so that $f = f(\mathbf{q},\mathbf{p})$, then f will be conserved if

$$\dot{f} = \{f,H\} = 0 . \tag{5.33}$$

We then say that f *Poisson commutes* with the Hamiltonian H. An example would be an ignorable coordinate q_a (with the index a fixed). Being ignorable means that $\partial H/\partial q_a = 0$, and from (5.29) we see that the conjugate momentum p_a is conserved, $\dot{p}_a = 0$.

One consequence of the Jacobi identity is that if we have two functions f, g that Poisson commute with H, then also their commutator $\{f,g\}$ Poisson commutes with H. Specifically, putting $h = H$ in the Jaboci identity above we have

$$\{\{f,g\},H\} = \{f,\{g,H\}\} + \{g,\{H,f\}\} = 0 . \tag{5.34}$$

We may use this to prove

Poisson's theorem: If f, g are two constants of the motion, then $\{f,g\}$ is also conserved.

Proof: When $f = f(\mathbf{q},\mathbf{p})$, $g = g(\mathbf{q},\mathbf{p})$ don't explicitly depend on time t, this follows immediately from (5.33) and (5.34). More generally we can use (5.34) to rewrite (5.28) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} &= \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\} \\ &= \left\{f, \{g,H\} + \frac{\partial g}{\partial t}\right\} + \left\{\{f,H\} + \frac{\partial f}{\partial t}, g\right\} \\ &= \left\{f, \frac{\mathrm{d}g}{\mathrm{d}t}\right\} + \left\{\frac{\mathrm{d}f}{\mathrm{d}t}, g\right\} ,\end{aligned} \tag{5.35}$$

from which the result follows. Notice that we have used anti-symmetry of the Poisson bracket on the second term, and $\partial_t\{f,g\} = \{\partial_t f, g\} + \{f, \partial_t g\}$ where $\partial_t = \partial/\partial t$.

Thus from two conserved quantities we can generate a third conserved quantity! Although notice that $\{f,g\}$ might simply turn out to be a constant linear combination of f and g (including perhaps 0), and thus give nothing new. Since conserved quantities are rather special, this is what tends to happen. If $\{f,g\} = 0$ then the functions f and g are said to be *in involution*.

Example (angular momentum): Consider a particle moving in $\mathbb{R}^3$ with angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$. The Hamiltonian will thus be a function of $\mathbf{q} = \mathbf{r} = (x_1, x_2, x_3)$, and $\mathbf{p}$. The components of $\mathbf{L}$ are

$$L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} x_j p_k \, . \tag{5.36}$$

Then one can compute the following Poisson bracket relations:

$$\{L_i, x_j\} = \sum_{k=1}^{3} \epsilon_{ijk} x_k \, , \qquad \{L_i, p_j\} = \sum_{k=1}^{3} \epsilon_{ijk} p_k \, , \qquad \{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k \, . \tag{5.37}$$

Since the proofs of these are all quite similar, let's just look at the first identity:

$$\begin{aligned} \{L_i, x_j\} &= \sum_{k,l=1}^{3} \epsilon_{ikl} \{x_k p_l, x_j\} = \sum_{k,l=1}^{3} \epsilon_{ikl} \left(x_k \{p_l, x_j\} + \{x_k, x_j\} p_l \right) \\ &= \sum_{k,l=1}^{3} \epsilon_{ikl} \left(-x_k \delta_{lj} + 0 \right) = \sum_{k=1}^{3} \epsilon_{ijk} x_k \, . \end{aligned} \tag{5.38}$$

In the first equality we have substituted (5.36) and used linearity of the Poisson bracket to bring the alternating symbol outside of the bracket. In the second equality we have used the Leibniz rule. The third equality uses the canonical brackets (5.30) and anti-symmetry, while for the final equality $\epsilon_{ijk} = -\epsilon_{ikj}$ holds for all i, j, k.

We already know from section 2.4 that angular momentum is conserved for a particle moving under the influence of a central potential $V = V(|\mathbf{r}|)$. One can also derive this using the Poisson bracket formalism above (see Problem Sheet 4). Notice that the bracket $\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$ implies that if we know that L_1 and L_2 are conserved, then it follows from Poisson's theorem that $L_3 = \{L_1, L_2\}$ is conserved.

Example (Laplace-Runge-Lenz vector): Another interesting example is the Laplace-Runge-Lenz vector, considered at the end of section 2.4. This was defined in (2.57) to be

$$\mathbf{A} \equiv \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} \, , \tag{5.39}$$

where κ is a constant. Using the brackets (5.37), similar computations to (5.38) lead to the following Poisson brackets:

$$\{L_i, A_j\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} A_k \;, \qquad \{A_i, A_j\} \;=\; -2m\left(\frac{|\mathbf{p}|^2}{2m} - \frac{\kappa}{|\mathbf{r}|}\right)\sum_{k=1}^{3}\epsilon_{ijk} L_k \;. \tag{5.40}$$

In particular for the central potential $V = -\kappa/|\mathbf{r}|$ the Hamiltonian is $H = |\mathbf{p}|^2/2m - \kappa/|\mathbf{r}|$, and the second equation above reads $\{A_i, A_j\} = -2mH\sum_{k=1}^{3}\epsilon_{ijk}L_k$. One can also then verify that

$$\{A_i, H\} \;=\; 0 \;, \qquad i \;=\; 1,2,3 \;, \tag{5.41}$$

from which we deduce that $\mathbf{A}$ is conserved (which you checked rather more directly on Problem Sheet 1). We then see that the constants of motion $\mathbf{L}$, $\mathbf{A}$ and H form a closed algebra under the Poisson bracket. In particular if we look at solutions with negative energy $E = H < 0$, corresponding to the closed elliptic orbits in the Kepler problem, then we may define the new quantities

$$\mathbf{L}^{\pm} \;\equiv\; \frac{1}{2}\left(\mathbf{L} \pm \frac{1}{\sqrt{-2mE}}\mathbf{A}\right) \;. \tag{5.42}$$

From the above expressions one checks these diagonalize the Poisson brackets as

$$\{L_i^{\pm}, L_j^{\pm}\} \;=\; \sum_{k=1}^{3}\epsilon_{ijk}L_k^{\pm} \;, \qquad \{L_i^{+}, L_j^{-}\} \;=\; 0 \;. \tag{5.43}$$

The elliptical orbits of the planets thus have two sets of Poisson commuting conserved angular momenta $\mathbf{L}^{\pm}$! Just as the usual angular momentum $\mathbf{L}$ is a conserved quantity arising from rotational symmetry under the group $SO(3)$, the full set of conserved vectors for the elliptic orbits in the Kepler problem is associated to a "hidden" action of the the rotation group $SO(4)$ (this actually follows immediately from (5.43), but to understand that we need to go into more group theory than we have time for). One can analyse the parabolic and hyperbolic orbits similarly (in particular, the hidden symmetry of the hyperbolic orbits is coincidentally the *Lorentz group* $SO(3,1)$!).

5.4 Canonical transformations

Recall that we are free to make any choice for the generalized coordinates $\mathbf{q}$ (as long as they define uniquely the configuration of the system at a fixed time). Moreover, Lagrange's equations take the same form under the coordinate transformation[13]

$$\mathbf{q} \;\to\; \mathbf{Q}(\mathbf{q}, t) \;. \tag{5.44}$$

The change of variables (5.44) is sometimes called a *point transformation*. Since Lagrange's equations are invariant, so too are Hamilton's equations (5.11). However, the form of Hamilton's

[13] We will use the notation $\mathbf{Q}(\mathbf{q}, t)$ rather than $\tilde{\mathbf{q}}(\mathbf{q}, t)$ in order to avoid a proliferation of tildes in what follows.

equations is invariant under a much larger class of transformations, called *canonical transformations*. This fact can be traced to the more symmetric way in which the coordinates and momenta appear in the Hamiltonian formalism, and is one of the advantages of it. In particular, judicious choices of transformation may sometimes be used to simplify the form of the Hamiltonian function, and hence the equations of motion.

There are a number of ways to both describe and construct canonical transformations. We shall begin with a formulation that is closest to the modern (geometric) viewpoint on the subject. We first introduce coordinates $\mathbf{y} = (y_1, \ldots, y_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on the whole phase space $\mathscr{P}$, and label the indices of y_α by $\alpha = 1, \ldots, 2n$. We then define the $2n \times 2n$ matrix[14]

$$\Omega \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \tag{5.45}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix, and 0 is the $n \times n$ matrix with all entries 0. Notice that $\Omega^2 = -\mathbb{1}_{2n\times 2n}$ and $\Omega^T = -\Omega$. In this notation the Poisson bracket between two functions $f = f(\mathbf{y}, t), g = g(\mathbf{y}, t)$ on $\mathscr{P} \times \mathbb{R}$ becomes

$$\{f, g\} = \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial y_\alpha} \Omega_{\alpha\beta} \frac{\partial g}{\partial y_\beta} . \tag{5.46}$$

The transformation

$$\mathbf{q} \rightarrow \mathbf{Q}(\mathbf{q}, \mathbf{p}, t) , \qquad \mathbf{p} \rightarrow \mathbf{P}(\mathbf{q}, \mathbf{p}, t) , \tag{5.47}$$

is called a *canonical transformation* if it leaves the Poisson bracket invariant. More precisely, in terms of the $\mathbf{y}$ coordinates the transformation (5.47) may be written

$$\mathbf{y} \rightarrow \mathbf{Y}(\mathbf{y}, t) , \tag{5.48}$$

and by invariance of the Poisson bracket we mean that

$$\{f, g\}_y \equiv \sum_{\gamma,\delta=1}^{2n} \frac{\partial f}{\partial y_\gamma} \Omega_{\gamma\delta} \frac{\partial g}{\partial y_\delta} = \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial Y_\alpha} \Omega_{\alpha\beta} \frac{\partial g}{\partial Y_\beta} \equiv \{f, g\}_Y , \tag{5.49}$$

should hold for all functions f, g. Since by the chain rule

$$\frac{\partial f}{\partial y_\gamma} = \sum_{\alpha=1}^{2n} \frac{\partial f}{\partial Y_\alpha} \frac{\partial Y_\alpha}{\partial y_\gamma} , \tag{5.50}$$

this is equivalent to

$$\Omega_{\alpha\beta} = \sum_{\gamma,\delta=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma} \Omega_{\gamma\delta} \frac{\partial Y_\beta}{\partial y_\delta} . \tag{5.51}$$

[14] This is not to be confused with our use of the symbol Ω in (4.5). The latter use won't appear in the rest of these lectures.

To see this one first uses (5.50) and (5.51) to compute

$$\begin{aligned}
\{f,g\}_y &\equiv \sum_{\gamma,\delta=1}^{2n} \frac{\partial f}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial g}{\partial y_\delta} \\
&= \sum_{\alpha,\beta,\gamma,\delta=1}^{2n} \frac{\partial f}{\partial Y_\alpha}\frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial g}{\partial Y_\beta}\frac{\partial Y_\beta}{\partial y_\delta} \\
&= \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial Y_\alpha}\Omega_{\alpha\beta}\frac{\partial g}{\partial Y_\beta} \;\equiv\; \{f,g\}_Y \;, \qquad (5.52)
\end{aligned}$$

so that (5.49) holds for all f and g. Conversely, if (5.49) holds for all f and g then simply choose $f = Y_\alpha$, $g = Y_\beta$: then (5.49) reads

$$\{Y_\alpha, Y_\beta\}_y \;=\; \sum_{\gamma,\delta=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial Y_\beta}{\partial y_\delta} \;=\; \{Y_\alpha, Y_\beta\}_Y \;=\; \Omega_{\alpha\beta} \;, \qquad (5.53)$$

which is (5.51). In matrix notation (5.51) reads

$$\Omega \;=\; \mathcal{J}\,\Omega\,\mathcal{J}^T \;, \qquad \text{where} \qquad \mathcal{J}_{\alpha\beta} \;\equiv\; \frac{\partial Y_\alpha}{\partial y_\beta} \;. \qquad (5.54)$$

Here $\mathcal{J}$ is the Jacobian matrix for the change of coordinates (5.48). $2n \times 2n$ matrices $\mathcal{J}$ satisfying $\Omega = \mathcal{J}\,\Omega\,\mathcal{J}^T$ form a subgroup of $GL(2n, \mathbb{R})$ called the *symplectic group* $Sp(2n, \mathbb{R})$. Thus canonical transformations are symplectic, in that the Jacobian matrix $\mathcal{J} \in Sp(2n, \mathbb{R})$ holds at each point in $\mathscr{P} \times \mathbb{R}$.

Notice that equation (5.53) says that a transformation is canonical if and only if it preserves the *canonical* Poisson brackets. Indeed, the canonical brackets (5.30) read

$$\{y_\alpha, y_\beta\} \;=\; \Omega_{\alpha\beta} \;, \qquad (5.55)$$

and we may further rewrite (5.53) as

$$\{Y_\alpha, Y_\beta\}_y \;=\; \{y_\alpha, y_\beta\}_y \;. \qquad (5.56)$$

Notice that in the original coordinates on phase space the Jacobian is

$$\mathcal{J} \;=\; \begin{pmatrix} \partial Q_a/\partial q_b & \partial Q_a/\partial p_b \\ \partial P_a/\partial q_b & \partial P_a/\partial p_b \end{pmatrix} \;, \qquad (5.57)$$

where each entry is an $n \times n$ matrix ($a, b = 1, \ldots, n$), and

$$\mathcal{J}\,\Omega\,\mathcal{J}^T \;=\; \begin{pmatrix} \{Q_a, Q_b\} & \{Q_a, P_b\} \\ \{P_a, Q_b\} & \{P_a, P_b\} \end{pmatrix} \;, \qquad (5.58)$$

where the Poisson brackets are evaluated in the original $\mathbf{q}, \mathbf{p}$ coordinates. We thus again see that if $\{Q_a, Q_b\} = 0 = \{P_a, P_b\}$ and $\{Q_a, P_b\} = \delta_{ab}$, then $\mathcal{J}$ is symplectic, and hence the transformation is canonical.

Example (point transformation): Of course we expect the point transformation (5.44) to be canonical. It is interesting to see in detail how this works. Denote the transformed Lagrangian by $\tilde{L}(\mathbf{Q}, \dot{\mathbf{Q}}, t) = L(\mathbf{q}(\mathbf{Q}, t), \dot{\mathbf{q}}(\mathbf{Q}, \dot{\mathbf{Q}}, t), t)$. Then the transformed momentum is by definition

$$\begin{aligned} P_a &= \frac{\partial \tilde{L}}{\partial \dot{Q}_a} = \sum_{b=1}^{n} \frac{\partial L}{\partial \dot{q}_b} \frac{\partial \dot{q}_b}{\partial \dot{Q}_a} = \sum_{b=1}^{n} \frac{\partial L}{\partial \dot{q}_b} \frac{\partial q_b}{\partial Q_a} \\ &= \sum_{b=1}^{n} p_b \frac{\partial q_b}{\partial Q_a} . \end{aligned} \tag{5.59}$$

Here notice that $\mathbf{q} = \mathbf{q}(\mathbf{Q}, t)$ means

$$\dot{q}_b = \sum_{a=1}^{n} \frac{\partial q_b}{\partial Q_a} \dot{Q}_a + \frac{\partial q_b}{\partial t} \qquad \Rightarrow \qquad \frac{\partial \dot{q}_b}{\partial \dot{Q}_a} = \frac{\partial q_b}{\partial Q_a} . \tag{5.60}$$

In particular (5.59) gives

$$\frac{\partial P_a}{\partial p_b} = \frac{\partial q_b}{\partial Q_a} . \tag{5.61}$$

If we now use the formula (5.57) for the Jacobian, and noting that $\partial Q_a / \partial p_b = 0$, we can directly compute

$$\mathcal{J} \, \Omega \, \mathcal{J}^T = \begin{pmatrix} 0 & \sum_{c=1}^{n} \frac{\partial Q_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} \\ -\sum_{c=1}^{n} \frac{\partial P_a}{\partial p_c} \frac{\partial Q_b}{\partial q_c} & \{P_a, P_b\} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \Omega , \tag{5.62}$$

where we use (5.61) in the second equality in (5.62). The remaining fiddly part in proving the second equality is to show that $\{P_a, P_b\} = 0$. To see this we again use (5.61) to compute

$$\sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} = \sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial q_c}{\partial Q_b} . \tag{5.63}$$

Here $\mathbf{q} = \mathbf{q}(\mathbf{Q}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t)$. Using (5.59) and the chain rule we may write

$$\frac{\partial}{\partial q_c} P_a = \sum_{d,e=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_e} \frac{\partial Q_e}{\partial q_c} . \tag{5.64}$$

Substiting this into the right hand side of (5.63) we thus have

$$\begin{aligned} \sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} &= \sum_{c,d,e=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_e} \frac{\partial Q_e}{\partial q_c} \frac{\partial q_c}{\partial Q_b} \\ &= \sum_{d=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_b} . \end{aligned} \tag{5.65}$$

The right hand side of (5.65) is symmetric in a and b, which thus shows that $\{P_a, P_b\} = 0$. This concludes the proof of (5.62), and hence that point transformations are canonical.

Example (swapping position and momentum): A more interesting canonical transformation is

$$\mathbf{Q} \;=\; \mathbf{p}\,, \qquad \mathbf{P} \;=\; -\mathbf{q}\,. \tag{5.66}$$

We thus swap the coordinates and momenta, with an appropriate sign. The Jacobian (5.57) for this transformation is simply Ω itself, and Ω is symplectic since $\Omega\,\Omega\,\Omega^T = -\Omega^T = \Omega$. Of course one can also see the invariance of the Poisson bracket (5.27) and Hamilton's equations (5.11) very simply. What we call "position" and "momentum" is therefore somewhat arbitrary. It is thus better simply to talk of $\mathbf{p}$ and $\mathbf{q}$ as being *canonically conjugate* variables, meaning they satisfy the canonical Poisson bracket relations (5.30).

Next we turn to Hamilton's equations (5.11). In the new notation these read

$$\dot{y}_\alpha \;=\; \sum_{\beta=1}^{2n} \Omega_{\alpha\beta}\frac{\partial H}{\partial y_\beta}\,. \tag{5.67}$$

From this equation and the transformation (5.48) we compute

$$\begin{aligned}\dot{Y}_\alpha \;&=\; \sum_{\gamma=1}^{2n}\frac{\partial Y_\alpha}{\partial y_\gamma}\dot{y}_\gamma + \frac{\partial Y_\alpha}{\partial t} \;=\; \sum_{\gamma,\delta=1}^{2n}\frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial H}{\partial y_\delta} + \frac{\partial Y_\alpha}{\partial t}\\ &=\; \sum_{\beta=1}^{2n}\Omega_{\alpha\beta}\frac{\partial \tilde{H}}{\partial Y_\beta} + \frac{\partial Y_\alpha}{\partial t}\,,\end{aligned} \tag{5.68}$$

where in the last line we used the symplectic condition (5.51). We have also defined the Hamiltonian in the new coordinates as

$$\tilde{H}(\mathbf{Y},t) \;\equiv\; H(\mathbf{y}(\mathbf{Y},t),t)\,, \tag{5.69}$$

by inverting (5.48). Similarly, $\partial Y_\alpha/\partial t$ means more precisely $\partial_t Y_\alpha(\mathbf{y},t)\mid_{\mathbf{y}=\mathbf{y}(\mathbf{Y},t)}$, so that it is a function of $(\mathbf{Y},t)$.

We thus see from (5.68) that for *time-independent* canonical transformations (so that $\partial\mathbf{Y}/\partial t = \mathbf{0}$), Hamilton's equations take the same form in the transformed coordinates, with the Hamiltonian transforming simply as a scalar (5.69) – compare to the corresponding discussion of the Lagrangian around equation (2.15). In the more general time-dependent case we can still write the transformed equations in the form (5.67) by introducing a new Hamiltonian $K = K(\mathbf{Y},t)$. Specifically, one can show that there exists a function $\Lambda = \Lambda(\mathbf{Y},t)$ such that

$$\frac{\partial\Lambda}{\partial Y_\alpha} \;=\; -\sum_{\beta=1}^{2n}\Omega_{\alpha\beta}\frac{\partial Y_\beta}{\partial t}\,. \tag{5.70}$$

Given this, we may define the transformed Hamiltonian as

$$K \;\equiv\; \tilde{H} + \Lambda\,, \tag{5.71}$$

so that (5.68) takes the form of Hamilton's equations

$$\dot{Y}_\alpha \;=\; \sum_{\beta=1}^{2n} \Omega_{\alpha\beta} \frac{\partial K}{\partial Y_\beta} \;. \tag{5.72}$$

Since the existence of Λ satisfying (5.70) is a little technical, we have relegated it to the starred paragraph after the next example.

Example (Galilean boost): A simple example in dimension $n = 1$ that demonstrates the need for the additional Λ term in (5.71) is a Galilean boost $Q(q,t) = q + tv$, where v is constant. It follows that $\dot{Q} = \dot{q} + v$. Taking the Lagrangian for a free particle $L = \frac{1}{2}m\dot{q}^2$, the conjugate momentum is $p = m\dot{q}$ with Hamiltonian $H = p^2/2m$. The transformed Lagrangian is

$$\tilde{L} \;=\; \frac{1}{2}m(\dot{Q} - v)^2 \;=\; \frac{1}{2}m\dot{Q}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(mvQ - \frac{1}{2}mv^2 t\right) \;. \tag{5.73}$$

The total derivative term doesn't contribute to the Lagrange equations of motion – see the comment after (2.17). So the free particle maps to a free particle, as it should do. We then compute the conjugate momentum $P = \partial\tilde{L}/\partial\dot{Q} = m(\dot{Q} - v) = p$. Of course this is in accord with our general formula (5.61) for point transformations. On the other hand the transformed Hamiltonian is

$$\begin{aligned} K \;&=\; P\dot{Q} - \tilde{L} \;=\; m(\dot{Q} - v)\dot{Q} - \frac{1}{2}m\dot{Q}^2 + mv\dot{Q} - \frac{1}{2}mv^2 \;, \\ &=\; \frac{1}{2}m\dot{Q}^2 - \frac{1}{2}mv^2 \;=\; \frac{P^2}{2m} + Pv \;=\; \tilde{H} + Pv \;. \end{aligned} \tag{5.74}$$

Here $\tilde{H}(P) = H(p)\mid_{p=P} = P^2/2m$. Thus we explicitly find $\Lambda = \Lambda(Q,P,t) = Pv$. In this notation (5.70) reads $\partial\Lambda/\partial Q = -\partial P/\partial t$ and $\partial\Lambda/\partial P = \partial Q/\partial t$, which are both seen to hold.

* The existence of the function $\Lambda(\mathbf{Y}, t)$ satisfying (5.70) is analogous to the more familiar fact that if $\nabla \wedge \mathbf{F} = \mathbf{0}$ then there exists (locally) a function $V = V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$. That is, a vector field with zero curl is (locally) the divergence of a function. In the current setting our space is $2n$-dimensional, and the "zero curl" condition means

$$\frac{\partial}{\partial Y_\gamma}\left(\sum_{\beta=1}^{2n} \Omega_{\alpha\beta}\frac{\partial Y_\beta}{\partial t}\right) - \frac{\partial}{\partial Y_\alpha}\left(\sum_{\beta=1}^{2n} \Omega_{\gamma\beta}\frac{\partial Y_\beta}{\partial t}\right) \;=\; 0 \;, \tag{5.75}$$

holds for all $\alpha, \gamma = 1, \ldots, 2n$. Notice that (5.75) is certainly a *necessary* condition for (5.70) to hold, simply using symmetry of the mixed partial derivatives with respect to Y_α. We shall assume it is also sufficient (this result goes under the general name of the *Poincaré Lemma*). We then check explicitly that (5.75) does indeed hold. Using the chain rule one sees that the left hand side is the anti-symmetric part of the matrix $\Omega\,\partial_t\mathcal{J}\,\mathcal{J}^{-1} = -\Omega\,\mathcal{J}\,\partial_t(\mathcal{J}^{-1})$, where $\mathcal{J}$ is the Jacobian matrix in (5.54). But we can also show that the latter is symmetric:

$$\begin{aligned} -\left(\Omega\,\mathcal{J}\,\partial_t\left(\mathcal{J}^{-1}\right)\right)^T \;&=\; \left(\partial_t(\mathcal{J}^{-1})\right)^T\,\mathcal{J}^T\Omega \\ &=\; -\Omega\,\partial_t\mathcal{J}\,\Omega\,\mathcal{J}^T\,\Omega \;=\; \Omega\,\partial_t\mathcal{J}\,\mathcal{J}^{-1} \;. \end{aligned} \tag{5.76}$$

Here in the first equality of (5.76) we have used $\Omega^T = -\Omega$, the second equality arises by differentiating the symplectic relation $(\mathcal{J}^{-1})^T = -\Omega\,\mathcal{J}\,\Omega$ with respect to time t, and the final equality again uses the symplectic relation (5.54). This proves (5.75).

5.5 Generating functions

We have so far seen the definition of canonical transformations, and some of their properties, but how does one actually *construct* transformations (5.47) that are canonical? In this section we describe two different approaches that both involve *generating functions*. The first is somewhat more mathematical/geometrical, while the second is the more traditional physics approach.