LECTURE 2: REYNOLDS' TRANSPORT THEOREM

Consider a time-dependent volume $V(t)$ that is convected by the fluid, so that it always consists of the same fluid particles. Then, for any function $f(\boldsymbol{x},t)$ that is continuously differentiable with respect to all of its arguments,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V(t)} \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\boldsymbol{u})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \qquad (1.22)$$

To prove this important result, we transform the integral on the left-hand side into Lagrangian variables to obtain

$$I(t) := \iiint_{V(t)} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V(0)} fJ\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z, \qquad (1.23)$$

where J again denotes the Jacobian (1.2). In (1.23), the Lagrangian integral is over the fixed initial domain $V(0)$ corresponding to the moving volume $V(t)$. We can therefore differentiate through the integral to obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \iiint_{V(0)} \frac{\mathrm{D}}{\mathrm{D}t}(fJ)\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z, \qquad (1.24)$$

where the time derivative is taken with the integration variables (X,Y,Z) held fixed.

Now we expand out the derivative in (1.24) and use Euler's identity (1.20) to obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) J\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z = \iiint_{V(t)} \frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \qquad (1.25)$$

The definition (1.9) of the convective derivative then leads to Reynolds' Transport Theorem (1.22).

1.2.8 Conservation of mass

It is instructive to consider mass conservation from both Eulerian and Lagrangian viewpoints. To begin with, consider a volume D which is fixed in space, so that fluid flows in and out through its boundary ∂D. The net mass of fluid inside D is given by

$$\iiint_D \rho(\boldsymbol{x},t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

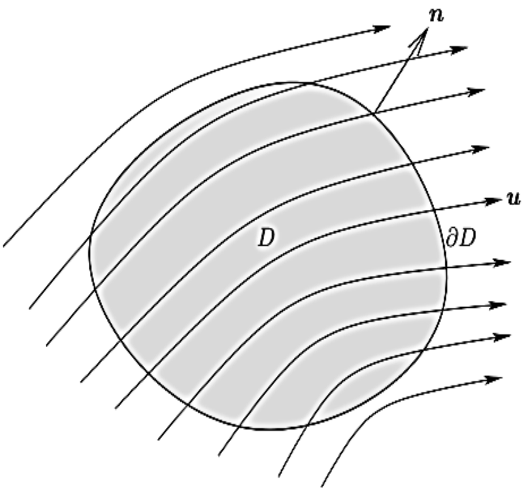


Figure 1.6: Schematic of a region D, fixed in space, with fluid flowing in and out through its boundary ∂D.

where ρ is the *density*, which may in general vary with both position and time. The net rate at which mass flows *out* of D is given by

$$\iint_{\partial D} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S,$$

where $\boldsymbol{n}$ is the unit outward-pointing normal to ∂D. Since mass can neither be created nor destroyed inside D, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_D \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = -\iint_{\partial D} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{1.26}$$

We can commute the differentiation and integration on the left-hand side to write

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_D \rho(\boldsymbol{x}, t) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_D \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.27}$$

Note that, when we differentiate through the integral, the time derivative $\partial/\partial t$ is performed while holding the integration variables (x, y, z) constant. Applying the Divergence Theorem to the right-hand side of (1.26), we therefore obtain

$$\iiint_D \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0. \tag{1.28}$$

This result must hold for *any* fixed volume D, and it follows that (assuming it is continuous) the integrand must be zero. We therefore deduce the equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{1.29}$$

relating the density and velocity in any continuous medium. We can expand out the divergence here to write (1.29) in the equivalent form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{1.30}$$

which demonstrates how the rate of change of the density and the divergence of the velocity field are intimately related.

Using a Lagrangian approach, we would instead consider the mass of a material volume $V(t)$ that is convected by the flow, so that it always consists of the same fluid elements. As above, we can write the net mass inside V in the form

$$\iiint_{V(t)} \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Now the integration region V varies with t, so we cannot directly differentiate through the integral. Instead, we can use the Transport Theorem (1.22) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_{V(t)} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u})\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.31}$$

Since the volume $V(t)$ is defined to consist always of the same fluid elements, its mass cannot change with time. This must be true for all material volumes, and, as above, we deduce that the integrand on the right-hand side of (1.31) must be zero (assuming it is continuous). Hence we reproduce the mass conservation equation (1.29).

We can use (1.29) to deduce the following useful corollary of the transport theorem. If $f = \rho h$ in (1.22), where h is any continuously differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho h \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_{V(t)} \rho \frac{\mathrm{D}h}{\mathrm{D}t} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.32}$$

1.3 The Euler equations

1.3.1 Conservation of momentum

Thus far we have been concerned just with describing the motion of a continuous medium, without considering what sort of medium it is (*e.g.* solid, liquid, gas, *etc.*) or what is causing it to move. Next we derive an equation linking the velocity of the fluid to the applied forces by applying Newton's second law, namely "force equals rate of change of momentum" to a material volume $V(t)$. The net momentum of such a volume is

$$\iiint_{V} \rho \boldsymbol{u} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

while the applied force has two ingredients. First there is the *external* body force $\boldsymbol{g}$ per unit mass, which contributes a net force

$$\iiint_{V} \rho \boldsymbol{g} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

In this course, we will ususally think of $\boldsymbol{g}$ as being the acceleration due to gravity, although it might also incorporate other effects such as electromagnetic forces on a liquid metal.

Second there is the *internal* force exerted on each volume V by the surrounding fluid. We suppose that this may be accounted for by a *pressure*, p, which acts in the inward normal direction at each point, so the net internal force on V is

$$\iint_{\partial V} -p\boldsymbol{n}\,\mathrm{d}S = \iiint_V -\boldsymbol{\nabla} p\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

using a well-known corollary of the divergence theorem. It is at this stage that we have restricted ourselves to considering *inviscid fluids.* Other continuous media, such as viscous fluids[3] or elastic solids,[4] can transmit tangential as well as normal internal forces.

Now we can formulate Newton's second law in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_V -\boldsymbol{\nabla} p\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_V \rho\boldsymbol{g}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.33}$$

To calculate the left-hand side, we apply the transport theorem corollary (1.32) and hence obtain

$$\iiint_V \left(\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p - \rho\boldsymbol{g}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \boldsymbol{0}, \tag{1.34}$$

which must hold for all material volumes V. It follows that (assuming it is continuous) the integrand must be zero, and we therefore obtain the *momentum equation*

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho\boldsymbol{g}. \tag{1.35}$$

1.3.2 Incompressible flow

We have shown that conservation of mass and momentum for an inviscid fluid leads to the scalar equation (1.29) and the vector equation (1.35). In total, we therefore have four scalar equations for five unknowns: ρ, p and the three components of $\boldsymbol{u}$. We therefore need more information to close the system. One possibility is to try and obtain a relation between the pressure and the density. This is the focus of *compressible* fluid dynamics, which describes such phenomena as sound waves and shock waves in gases.[5] However, it is an empirical observation that, in most liquids, the density varies by only a few per cent under typical variations in temperature and pressure. It is therefore common to assume that liquids have *constant density*, and we will see this approximation allows many familiar and important flows to be described.

With $\rho = \text{constant}$, we deduce from (1.30) that

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{1.36}$$

and we recall from (1.21) that the flow is therefore incompressible. The implication does not quite go the other way. If the flow is incompressible, then (1.30) reduces to

$\mathrm{D}\rho/\mathrm{D}t = 0$, so that ρ is preserved following the flow, but need not be constant (this can occur for example in *stratified* fluids). However, the term *incompressible* is often slightly abused to refer to constant-density fluids, and we will assume that ρ is constant throughout the remainder of this course.

By expanding out the convective derivative, we can write the momentum equation (1.35) in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g}. \tag{1.37}$$

Now (1.36) and (1.37) amount to a closed system of four scalar equations for p and the three components of $\boldsymbol{u}$, known as the *Euler equations.*

If we assume that the body force is *conservative*, then it may be written as $\boldsymbol{g} = -\boldsymbol{\nabla}\chi$ in terms of a potential χ. For example, a constant gravitational acceleration in the $-z$-direction corresponds to $\boldsymbol{g} = -g\boldsymbol{e}_z$ and hence $\chi = gz$. We also note the vector identity

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \equiv \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u}, \tag{1.38}$$

whose proof is a straightforward exercise.

We can therefore rearrange (1.37) to the alternative form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right). \tag{1.39}$$

For steady flow, the first term on the left-hand side is zero. If we dot the whole equation with $\boldsymbol{u}$, then the second term also disappears, since we end up with a triple scalar product $[\boldsymbol{u}, \boldsymbol{\nabla} \times \boldsymbol{u}, \boldsymbol{u}]$ with two repeated entries. It follows that

$$\boldsymbol{u} \cdot \boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right) = 0 \tag{1.40}$$

when $\partial\boldsymbol{u}/\partial t = \boldsymbol{0}$, and from this we deduce that

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi \text{ is constant along streamlines in steady flow.} \tag{1.41}$$

This is known as **Bernoulli's Theorem for steady flow**. We will see shortly that various different versions of Bernoulli's Theorem may apply when the flow is not steady.

1.3.3 Boundary conditions

If the fluid is in contact with a fixed rigid boundary B, then the normal velocity of the fluid there must be zero, that is

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } B, \tag{1.42}$$

where $\boldsymbol{n}$ denotes the unit normal to B. This condition states that the fluid can neither flow through B nor separate from B, leaving behind a vacuum. However, it says nothing about the *tangential* velocity.[6]

Free boundaries will be introduced later in the course when we study water waves.

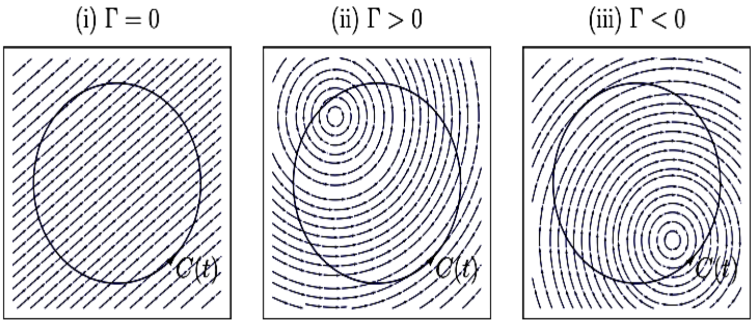


Figure 1.7: A closed curve $C(t)$ in flows with (i) zero circulation, (ii) positive circulation, (iii) negative circulation.

1.4 Vorticity and circulation

1.4.1 The vorticity equation

The *vorticity* $\boldsymbol{\omega}$ is defined to be the curl of the velocity field:

$$\boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u}. \tag{1.43}$$

The vorticity is a measure of the local rotation of the flow. We can obtain an equation for $\boldsymbol{\omega}$ by taking the curl of the momentum equation in the form (1.39), recalling that curl grad $\equiv \mathbf{0}$, so that

$$\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \mathbf{0}. \tag{1.44}$$

The partial derivative $\partial/\partial t$ commutes with the curl operator, since it is taken with the Eulerian coordinates (x, y, z) held constant, and we can expand out the second term in (1.44) by using the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) \equiv (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{1.45}$$

We recall that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ for incompressible flow, and $\boldsymbol{\omega}$ is likewise divergence-free, since div curl $\equiv 0$. Hence (1.44) may be rearranged to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}, \tag{1.46}$$

which is known as the *vorticity equation.*

We see that $\boldsymbol{\omega}$ is not in general preserved following the flow. However, equation (1.46) suggests that, if $\boldsymbol{\omega}$ is initially zero, then it will remain zero for all time. To establish this fact, it is helpful first to introduce the concept of *circulation.*

1.4.2 Kelvin's Circulation Theorem

Consider a closed curve $C(t)$ that is convected by the flow, for example a smoke ring. We define the *circulation* around such a curve by

$$\Gamma(t) = \oint_{C(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}. \tag{1.47}$$

The circulation is thus the net flow along the closed curve $C(t)$. Figure 1.7 shows schematically how circulation is related to rotation in the flow. In diagram (i) there is no rotation. The net clockwise and anticlockwise flows around C will cancel, resulting in a net circulation of zero. In diagram (ii), there is an anti clockwise rotation in the flow, resulting in a positive circulation about C. Finally, in diagram (iii) we see that a clockwise-rotating flow leads to a negative circulation about C.

We can also relate circulation to vorticity, since Stokes' Theorem implies that

$$\Gamma(t) = \iint_{S(t)} (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S = \iint_{S(t)} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{1.48}$$

where S is any surface spanning C. This reinforces the connection between vorticity and rotation in the flow alluded to in §1.4.1.

Kelvin's Circulation Theorem states that Γ is independent of t, and we will prove it by showing that $\mathrm{d}\Gamma/\mathrm{d}t$ is zero. To differentiate Γ, it is helpful to transform the integral to Lagrangian variables, using the chain rule:

$$\Gamma = \oint_{C(t)} \sum_i u_i \, \mathrm{d}x_i = \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \, \mathrm{d}X_j. \tag{1.49}$$

With respect to Lagrangian variables, the integral is taken around the fixed initial curve $C(0)$. We can therefore now differentiate through the integral to obtain

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \, \mathrm{d}X_j = \oint_{C(0)} \frac{\mathrm{D}}{\mathrm{D}t} \left(\sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \right) \mathrm{d}X_j, \tag{1.50}$$

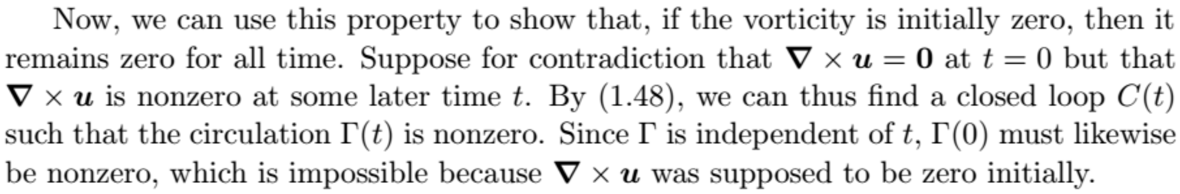
holding the integration variables $\boldsymbol{X}$ constant when performing the time derivative $\mathrm{D}/\mathrm{D}t$. We expand out the derivative in the integrand, using the fact that $\mathrm{D}/\mathrm{D}t$ commutes with $\partial/\partial X_j$, to obtain

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= \oint_{C(0)} \sum_{i,j} \left(\frac{\mathrm{D}u_i}{\mathrm{D}t} \frac{\partial x_i}{\partial X_j} + u_i \frac{\partial u_i}{\partial X_j} \right) \mathrm{d}X_j \\ &= \oint_{C(t)} \sum_i \frac{\mathrm{D}u_i}{\mathrm{D}t} \, \mathrm{d}x_i + \oint_{C(t)} \sum_{i,j} u_i \frac{\partial u_i}{\partial x_j} \, \mathrm{d}x_j. \end{aligned} \tag{1.51}$$

The second integrand may be rewritten as $\partial_j(\frac{1}{2}|\boldsymbol{u}|^2)$, and we use (1.37) to substitute for the acceleration in the first integral. Swapping i and j in the second integral and combining it with the first integral gives

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_{C(t)} \sum_i \frac{\partial}{\partial x_i} \left(-\frac{p}{\rho} - \chi + \frac{1}{2}|\boldsymbol{u}|^2 \right) \mathrm{d}x_i = \left[-\frac{p}{\rho} - \chi + \frac{1}{2}|\boldsymbol{u}|^2 \right]_{C(t)}, \tag{1.52}$$

where $[\cdot]_{C(t)}$ denotes the change in $\cdot$ as the closed loop C is traversed. Since p, χ and $\boldsymbol{u}$ are all single-valued functions of position, we deduce that the right-hand side is zero and, hence, that Γ is constant.



Now, we can use this property to show that, if the vorticity is initially zero, then it remains zero for all time. Suppose for contradiction that $\boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0}$ at $t = 0$ but that $\boldsymbol{\nabla} \times \boldsymbol{u}$ is nonzero at some later time t. By (1.48), we can thus find a closed loop $C(t)$ such that the circulation $\Gamma(t)$ is nonzero. Since Γ is independent of t, $\Gamma(0)$ must likewise be nonzero, which is impossible because $\boldsymbol{\nabla} \times \boldsymbol{u}$ was supposed to be zero initially.