Automata Theory - Lecture 2

Introduction to Sequences, Tuples, Functions and Relations

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Lecture learning outcomes

At the end of the lecture you will be able to:

(i) Define a sequence, tuple, function and relation

(ii) Explain the importance of these data structures to computing.

(iii) Differentiate functions and relations as they are used in automata theory

(iv) Solve problems involving sequences, functions and relations

2.1 Introduction to Sequences

A sequence is **an ordered list of objects.** A list arranged in some order either increasing order or decreasing order (ascending order or descending order).

Unlike in sets:

i. **repetition is allowed** in sequences. Example {1, 1, 2, 3, 3}

ii. o**rder is also important** in sequences. Example {1, 5, 7, 8}

2.2 Introduction to Tuples

A tuple is a sequence of elements. E.G A sequence of **k elements is a k-tuple. (Length property** has been added **(length = k)**).

A 2-tuple is a sequence of two elements (pair) example {1, 3}

A 3-tuple is a sequence of three elements (triple) example {1, 1, 2}

A 4-tuple is a sequence of four elements (quadruple) example {1, 2, 3, 3}

2.3 Introduction to Functions

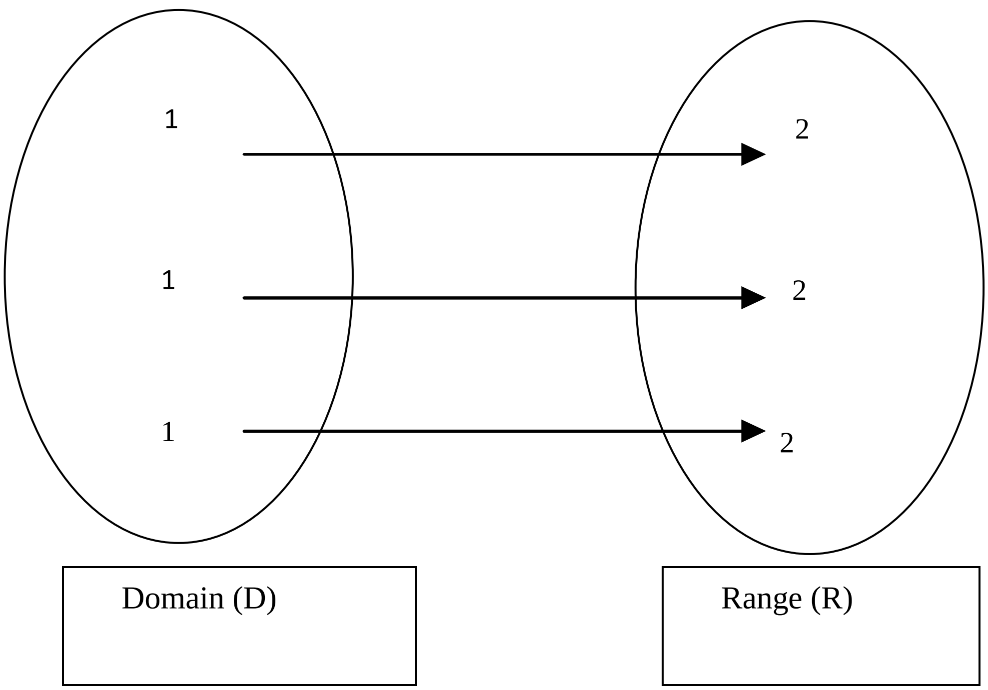
A function is a **special kind of relationship that maps one element from the input elements (domain) to only one other element of the output elements (range) (with none included).**

Domain – a collection of the input values

Range – a collection of the output values

Example One:

Suppose domain **D** is the set of possible **inputs of the integer one (1)** and range R is the set of possible **outputs of the integer two (2);**



A "***k-ary***" function is a function with **k arguments**.

A ***unary*** function is a function with one element i.e. k=1

A ***binary*** function is a function with two elements i.e. k=2

A function can be a partial function, total function or onto function.

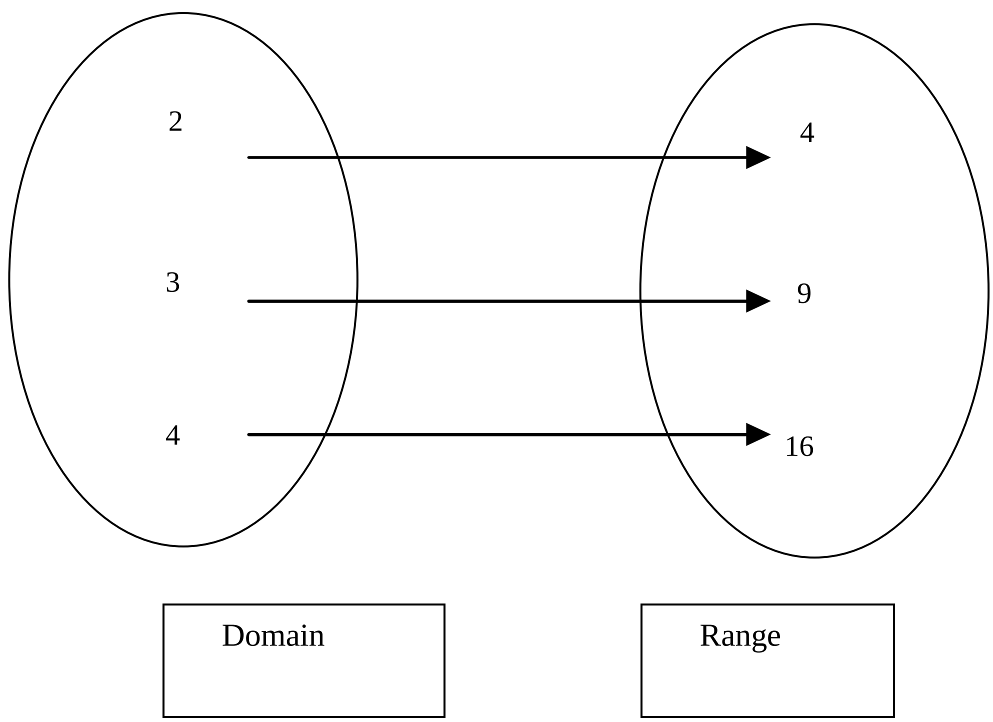
<u>Note:</u> One item in the domain cannot be mapped to more than one item in the range unless if it is a **relation**.

A function is a special kind of relation in which an element is mapped onto only one other element at most. (None included i.e. we cannot have an item in the range missing a mapping in the domain or an item in the domain missing a mapping from the range).

Example: if a function *f* on *a* is described as: - f $(a) = b \wedge c$

Or $f(a) = c \Rightarrow b = c$.

A function square *f(sqr)*, is a function that generates the square of a given number and can be demonstrated diagrammatically as follows: -

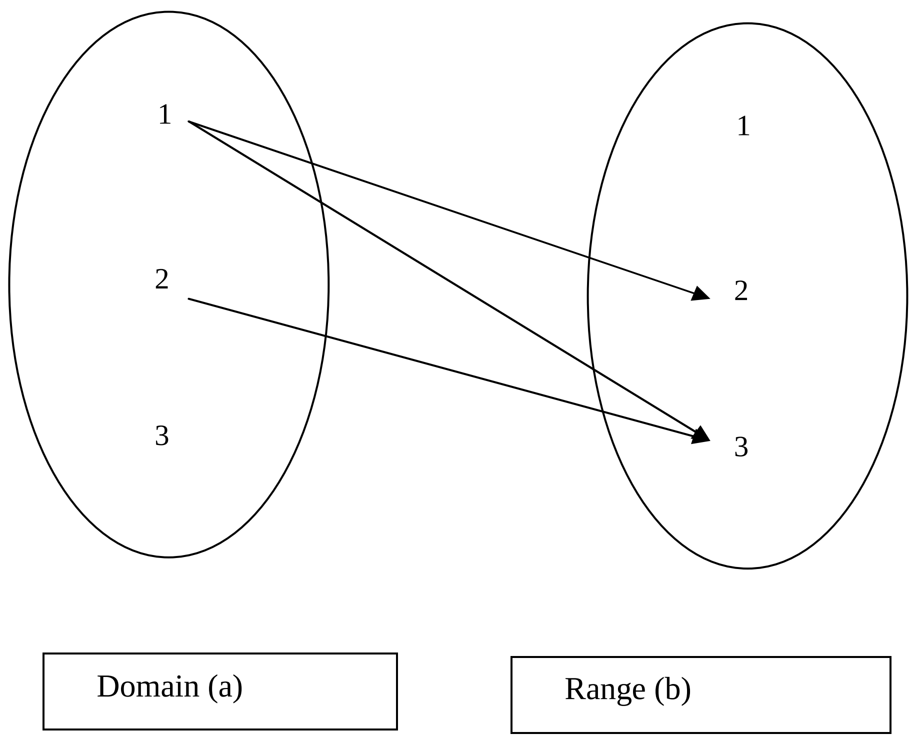


2.4 Introduction to Relations

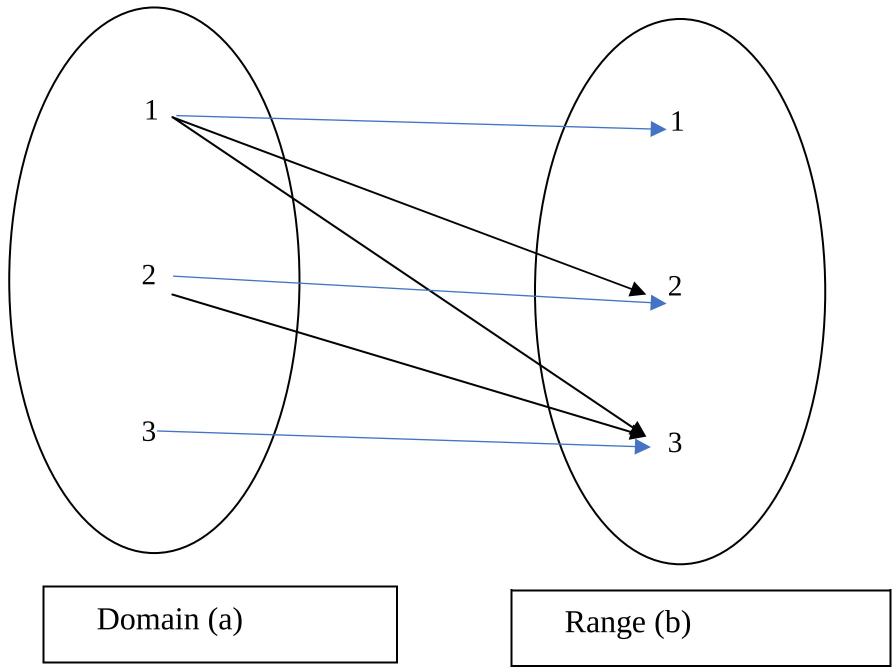
A relation describes the link between elements. Unlike a function, a relation can map one domain element to many other elements in the range.

A relation between two elements is said to be a ***binary relation*** and when a relation is binary, we use infix notation aRb meaning *a is relation to b*

Example One: The ***less than relation*** between element *a* and element *b* denoted as a<b means that a ***less than*** *b*. Possible domain and range values include {(1, 2), (1, 3), (2, 3)}



Example: The ***less than or equal to relation*** between elements *a* and *b* denoted as a<=b means *element a* is ***less than or equal to*** element *b*. Possible domain and range values include {(1, 2), (1,1) (1, 3), (2,2), (2, 3), (3,3)}



An ***equivalence relation*** describes two elements (objects) that are equal. There are four ways of looking at equivalence relations: -

a) ***Reflexive Relation*** – Each element is related to itself or maps onto itself. Example, 1+1, 1=1…the *less than relation is not reflexive.*

- In terms of set theory, the relation R on a set A is denoted as: for every $x \in$ A, (x, x) $\in$ R.

b) ***Symmetric Relation*** - ∀xy, xRy iff yRx - *For all x* and *y* elements, *x* is related to *y* ***if and only if*** *y* is related to *x.*

An example of a symmetric relation is the less than relation that can only mean that the not equal relation holds e.g. ∀ 2, 3, 2R3 ∧ 2R3; $2<3 \wedge 2\neq3$;

- From set theory we realize that this is a relation R on a set A if (x, y) ∈ R then (y, x) ∈ R, for all x & y ∈ A.

c) ***Transitive Relation*** - ∀xyz, xRy yRz ⇒ xRz - For all *x, y* and *z* elements, *x* is related to *y* ***and*** y is related to *z* meaning that *x* is related to *z*.

The ***less than*** relation is transitive;

∀1, 2, 3, 1R2 ∧ 2R3 ⇒ 1R3

If (x, y) ∈ R, (y, x) ∈ R, then (x, z) ∈ R, for all x, y, z ∈ A and this relation in set A is transitive.

d) ***Equivalence*** – When all the above three relations are brought together, an equivalence relation is formed i.e. a saturation of the above three relations.

- A relation that is reflexive, symmetric and transitive is called an equivalence relation.

Review Questions

1. Using examples, describe the following terms clearly explaining their relevance to the design and operations of hardware and software
    a. Set
    b. Function
    c. Alphabet
    d. Language
    e. Relation
2. If A = {0, 1, 2} and B = {a, b}, describe reflexive, transitive and symmetric relations over the sets A and B
3. Using examples, describe two differences between a set and an ordered pair

Solutions

Question One

Set: It is a well-defined collection of objects or elements e.g. set $A = \{a, e, i, o, u\}$ is the set of the English alphabets. A set is a non-repeating, unordered collection of objects (elements, members). E.g. the collection of four letters; a, b, c d is a set which is written as: L = {a, b, c, d}.

A **function** is a **special kind of relationship that maps one element from the input elements (domain) to only one other element of the output elements (range) (with none included).**

Alphabet – A finite set of symbols. It is frequently denoted by Σ , which is the set of letters in an alphabet.

Language - refers to a set of words formed by symbols in each alphabet.

A relation describes the link between elements. Unlike a function, a relation can map one domain element to many other elements in the range.

Question Three

A Set is not ordered but for the ordered pairs, order is important: A set is a non-repeating, unordered collection of elements.

References

1. Rowan G. & John T., (2009), *Discrete Mathematics: Proofs, Structures and Applications*, CRC Press, ISBN: 9781439812808.
2. W. D. Wallis (2003), *A Beginners Guide to Discrete Mathematics*, Springer Science & Business Media, ISBN: 978-0817642693.