

Course: Automata Theory

Lecture 3: Introduction to Logic and Truth Tables

Lecturer: Martha Gichuki

Course description

- The course begins with an introduction to logic and formal grammar where learners will do a recap on sets, logic and truth tables, sequences, relations and functions
- A coverage of finite state machines, Push Down automata and Turing Machines (The Church's thesis) will culminate the study of various models of computation.
- Formal language and grammar will then follow to enable learners differentiate regular and context free languages.
- An evaluation of the computability and complexity of practical computational problems which are the foundations of automata theory will then be done and the outcome will be problem description.

Learning outcomes:

Lecture 3: Introduction to Logic and Truth Tables

At the end of the lecture, you will be able to:

- i. Define a compound statement, a connective and a truth value of a statement.
- ii. Explain the various connectives used in statements.
- iii. Differentiate tautology propositions from contradictory propositions
- iv. Solve problems involving compound statements, using truth tables

3.0 Logic & Truth Tables

Introduction:

- A computer may be programmed to make decisions based on whether certain statements **are true or false** for example **“The number of students in this class exceeds 50”**.
- The truth or falsity of a statement is called its **truth value**.
- A statement is either true or false but not both.

Truth Tables

- A **truth table** lists whether something is true or false.
- The act of negation changes all the true values into false values and the false values to true values as shown in the truth table below.

P	$\neg P$
True	False
False	True

Constructing Truth Tables

To construct a truth table, the following guidelines apply:

- Create a column for every proposition in the expression
- Create a row for every possible true and false value
- This shows all the possible combinations that a given set of propositions can take.

Example: Two Propositions

A truth table for all the possible truth value combinations will have:

- Four rows for true and false possibilities
- Two columns for the propositions.
- In general, there will be 2^n rows for n different propositions.

A	B
True	True
True	False
False	True
False	False

Logical operators

- Logical operators are used to link together propositions and build more complex logical expressions
- They operate in the same way as $+$, $-$, \times , and \div operators used to link mathematical expressions.
- The basic logical operators, along with **negation**, are **conjunction**, **disjunction**, **conditional**, and **biconditional**.

Compound Statements

- Some statements are compound statements; meaning, they are composed of **sub-statements and various connectives**.
- For example, “He is intelligent or studies very hard” is an implicit statement.
- A compound statement and sub-statements would be; “He is intelligent” **and** “He studies very hard”
- Suppose $P =$ “He is intelligent” and $Q =$ “He studies very hard”
“ P or Q ” is an implicit statement.
“ P and Q ” is a compound statement with two sub-statements

Compound Statements

- The fundamental property of a compound statement is that its truth value is completely determined two factors:
 - i. The truth values of its sub-statements
 - ii. The way in which the sub-statements are **connected to form the compound statement.**

Example:

The compound statement “P or not P” ($P \vee \sim P$) has two sub-statements i.e. p and $\sim P$. The two are connected using the disjunction “or” and all these determine the value of the compound statement.

Connectives

- Several connectives are used to connect sub-statements and these include conjunction, disjunction and negation.

(i.) Conjunction ($P \wedge Q$)

- Any two statements can be combined by the word “*and*” to form a compound statement called the *conjunction of the original statement*.
- This is symbolically written as: $P \wedge Q$; which denotes the conjunction of the statements P and Q read as “*P and Q*”.

The truth values of the compound statement $P \wedge Q$ is given by the following truth table: -

P	Q	$P \wedge Q$
False	False	False
False	True	False
True	False	False
True	True	True

Here, the last line is a short way of saying **that if P is true and If Q is True, then $P \wedge Q$ is also true.** The other lines have analogous meaning and this table defines precisely the truth values of the compound statement $P \wedge Q$ as a function of the truth values of P and Q.

Consider the following statements which are represented in form of a truth table below: -

i). Paris is in France and $2 + 2 = 4$ (*True and True*)

ii). Paris is in France and $2 + 2 = 5$ (*True and False*)

iii). Paris is in England and $2 + 2 = 4$ (*False and True*)

iv). Paris is in England and $2 + 2 = 5$ (*False and False*)

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

Disjunction ($P \vee Q$)

- Any two statements can be combined by the word “*or*” to form a new statement which is called the *disjunction of the original two statements*, symbolically written as $P \vee Q$.
- This denotes the disjunction of the statements P and Q which is read as “*P or Q*”.
- The truth values of $P \vee Q$ are given by the following truth table: -

Disjunction ($P \vee Q$)

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

$P \vee Q$ is false only when both statements are false and is otherwise true.

(iii.) Negation ($\sim P$)

- Given any statement P, another statement known as *the negation of P* can be formed by writing “*It is false that...*” before P or if possible, by inserting in front of P the word “*not*”; symbolically written as “ $\sim P$ ”.
- This denotes the negation of P read as “*not P*” and the truth values of $\sim P$ are given by the following table: -

P	$\sim P$
True	False
False	True

If P is true, then $\sim P$ is false, thus the truth value of the negation of any statement is always the opposite of the truth value of the original statement.

Propositions & Truth Tables

- Through repetitive use of the logical connectives - \wedge , \vee , \sim and others, we are able to construct a compound statement(s) that are more involving.
- In the case where the sub-statements P, Q of compound statements P (P, Q ...) are variables; we call the compound statement a **proposition**.

Propositions & Truth Tables

- The truth value of a proposition depends exclusively upon the truth values of its variables
- i.e. the truth value of a proposition is known once the truth values of its variables are known.

A simple way to show its relationship is through a truth table and the truth table of the proposition $\sim(P \wedge \sim Q)$, for example, is constructed as follows:

P	Q	$\sim Q$	$P \wedge \sim Q$	$\sim(P \wedge \sim Q)$
True	True	False	False	True
True	False	True	True	False
False	True	False	False	True
False	False	True	False	True

Tautologies & Contradictions

- Some propositions contain *only true values in the last column of their truth tables* i.e. are true for any truth values of their variables.
- Such propositions are called **tautologies**.
- Similarly, a proposition $P(P, Q \dots)$ is called a **contradiction** if it contains *only false values in the final column of its truth table* i.e. the statement is false for any truth values of its variables.

Example, the proposition “P and not P” ($P \wedge \sim P$) is a contradiction as verified by constructing the truth tables below: -

P	$\sim P$	$P \wedge \sim P$
True	False	False
False	True	False

$P \wedge \sim P$ is a contradiction

Example, the proposition “P or not P” ($P \vee \sim P$) is a Tautology as verified by constructing the truth table below: -

P	$\sim P$	$P \vee \sim P$
True	False	True
False	True	True

$P \vee \sim P$ is a Tautology

The proposition $(Q \wedge R) \vee \sim (Q \wedge R)$, which by principle of substitution must also be a tautology is proven using the truth table below: -

Q	R	$Q \wedge R$	$\sim (Q \wedge R)$	$(Q \wedge R) \vee \sim (Q \wedge R)$
True	True	True	False	True
True	False	False	True	True
False	True	False	True	True
False	False	False	True	True

$(Q \wedge R) \vee \sim (Q \wedge R)$, by principle of substitution is a tautology

Review Questions

1) Construct truth tables for the following propositions: -

a) $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

b) $\sim(\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	$\sim(\sim p \vee \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

3) Let p be "Roses are Red" and let q be "Violets are Blue". Write verbal sentences that describes each of the following statements: -

a) $\sim p$

Roses are not red.

b) $\sim (p \wedge q)$

Its false that Roses are red and violets are blue

c) $p \vee q$

Roses are Red or Violets are Blue

d) $q \vee \sim p$

Violets are Blue or Roses are not Red

e) $\sim p \wedge \sim q$

Roses are not red and violets are not Blue

f) (vi.) $\sim \sim q$

Violets are blue

4.) Construct truth tables for the following propositions: -

a. $p \wedge \sim q$

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	F
F	T	F	F
F	F	T	F

b. $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

c. $\sim(\sim p \wedge q)$

p	q	$\sim p$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

5) Construct truth tables for the following statements

(i.) $\sim (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim (\sim p \vee \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(ii.) $\sim (\sim P \vee Q)$

p	q	$\sim p$	$(\sim p \vee q)$	$\sim (\sim p \vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

(iii.) $P \vee (\sim P \wedge Q)$

p	q	$\sim p$	$(\sim p \wedge q)$	$p \vee (\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

6.) Construct a truth table for each of the following statements:

(i.) $q \wedge (p \rightarrow q) \rightarrow p$

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow p$	$q \wedge (p \rightarrow q) \rightarrow p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

(ii.) $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

a) Conditional statements are those of the form “If p then q” They are the fundamental principles of logical reasoning. Given the conditional statement: - “If p implies q and q implies r then p implies r” verify this fact by showing that the following proposition is a tautology $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)]$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

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