Course: Automata Theory

Lecture 4: Introduction to Logical Equivalence

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Course description

- The course begins with an introduction to logic and formal grammar where learners will do a recap on sets, logic and truth tables, sequences, relations and functions
- A coverage of finite state machines, Push Down automata and Turing Machines (The Church's thesis) will culminate the study of various models of computation.
- Formal language and grammar will then follow to enable learners differentiate regular and context free languages.
- An evaluation of the computability and complexity of practical computational problems which are the foundations of automata theory will then be done and the outcome will be problem description.

Learning outcomes:

Lecture 4: Introduction to Logical Equivalence

At the end of the lecture you will be able to:

i. Define logical equivalence.
ii. Describe the various connectives used in logically equivalent statements.
iii. Differentiate conditional from bi-conditional statement
iv. Solve problems involving logically equivalent and conditional statements, using truth tables

4.1 Logical Equivalence

Introduction:

- Two or more propositions are said to be logically equivalent if they have identical truth/falsity values in the last column of their truth tables
- When two propositions P (P, Q …) and Q (P, Q …) are said to be ***logically equivalent*** they are also said to be ***simply equivalent*** or ***equal***.

Introduction…

- When two or more propositions have identical truth/falsity values in the last column of their truth tables they are said to be logically equivalent.
- This is true regardless of the number of columns involved while achieving the truth/false values of the last column of the truth table.

Denoting Logical Equivalence

- The equivalence of statements is denoted by the symbol ≡
- ***Logical equivalence is*** written as:- ***P (p, q …) ≡ Q (p, q …)***
- Where p and q are ***sub-statement***s of the ***compound statements*** P and Q without any limitation of having only two sub statements, thus the three dots therein…

Compound Statements

- The fundamental property of a compound statement is that its truth value is completely determined two factors:
  1. The truth values of its sub-statements
  2. The way in which the sub-statements are **connected to form the compound statement**.

Example:

- The compound statement “P or not P” (P V ~ P) has two sub-statements i.e. P and ~ P.
- The two sub-statements are connected using the **disjunction connective “or”**.
- The number of sub-statements and the connective used determine the value of the compound statement.

Propositions & Truth Tables

- Through repetitive use of the logical connectives - Ʌ, V, ~ and others, we are able to construct a compound statement(s) that are more involving.

- In the case where the sub-statements P, Q of compound statements P (P, Q …) are variables; we call the compound statement a **proposition**.

Recap: Constructing truth tables

To construct a truth table, the following guidelines apply:

- Create a column for every proposition in the expression
- Create a row for every possible true and false value
- This takes care of ***all the possible*** combinations that a given set of propositions can take.

Example: with two propositions

A truth table for all the possible truth value combinations will have:

- Four rows for true and false possibilities
- Two columns for the propositions.
- In general, there will be **2^n rows for *n* different propositions**.

P	Q
True	True
True	False
False	True
False	False

Example One:

- Consider the two statements:
  i. $\sim (P \land Q)$ and
  ii. $\sim P \lor \sim Q$,
- Suppose the variable P represents the statement "Roses are red" and the variable Q represents the statement "Violets are blue"
  - P = "Roses are red"
  - Q = "Violets are blue"

Example One continued…

- The statement ~ (P Λ Q) above can be read as: "It is false that roses are red ***and*** violets are blue" written in form of ~ (P Λ Q), where P is "Roses are red" and Q is "Violets are blue".
- The truth table of the statement ~ (P Λ Q) is shown below:-

P	Q	P Λ Q	~(P Λ Q)
True	True	True	False
True	False	False	True
False	True	False	True
False	False	False	True

Example One continued…

- The statements ~ P V ~ Q above represents the outcome of opening the brackets whereby ***negating "and" we get "or"*** as follows: *Roses are not red or violets are not blue written as (~P V ~ P).*
- The truth table of the statement ~ P V ~ Q is shown below:-

P	Q	~P	~Q	~P V ~Q
True	True	False	False	False
True	False	False	True	True
F	True	True	False	True
False	False	True	True	True

Comparing the Truth Tables

- The truth tables of the above statements are as shown below and the values in the last column of the truth tables are similar indicating that the two statements are logically equivalent.
- The truth table values show that ~ (P Λ Q) is logically equivalent to (~P V ~ Q).

i. (~P V ~ Q)

P	Q	~P	~Q	~P V ~Q
True	True	False	False	False
True	False	False	True	True
F	True	True	False	True
False	False	True	True	True

Equal values

ii. ~ (P Λ Q)

P	Q	P Λ Q	~(P Λ Q)
True	True	True	False
True	False	False	True
False	True	False	True
False	False	False	True

Comparing the statements

- The statement ***"It is false that roses are red and violets are blue"*** has the <u>same meaning</u> as the statement: ***" Roses are not red" or "Violets are not blue".***
- *This comes as a result of negating the statement* (P Λ Q) written as ~ (P Λ Q).
- When we negate the conjunction connective we get a disjunction connective indicating that the statement ~ (P Λ Q) is logically equivalent to (~P V ~ Q).

Conclusion

- Since the truth tables are the same, i.e. both propositions are false in the first case and true in the other instances.

- The propositions ~ (P Λ Q) and ~P V ~ Q, are **logically equivalent meaning:**

  ~ (P Λ Q) ≡ (~P V ~ Q).

4.2 Conditional & Bi-Conditional Statements

- Many statements, particularly in mathematics, are of the form; "**If P then Q**".

- Such statements are called ***conditional statements*** and are denoted by $P \rightarrow Q$.

- The condition P → Q is frequently read as "***P implies Q***" or "***P only if Q***".

Example:

- The truth table values of the conditional statement P → Q are as shown below where only a true value does not imply a false value.

P	Q	P → Q
True	True	True
True	False	False
False	True	True
False	False	True

P → Q

- Another common statement is of the form; "**P if and only if Q**".

- Such statements are denoted by $P \leftrightarrow Q$ and are called ***bi-conditional statements.***

- A true value is achieved if and only if the two values are true or false, else a false value is implied.

Example: The truth table values of P ↔ Q are as shown below where both true values must either be true or false for the condition to hold true.

P	Q	P ↔ Q
True	True	True
True	False	False
False	True	False
False	False	True

P ↔ Q

Review Questions

1. If p V ~p is a tautology, substituting q Λ r for p, we obtain proposition (q Λ r) V~ (q Λ r), which by the principle of substitution, should also be a tautology. Verify this with the use of a truth table.

q	r	(q Λ r)	~(q Λ r)	(q Λ r) V~ (q Λ r),
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

All True values (Tautology)

2) Let p be “Roses are Red” and let q be “Violets are Blue”. Write verbal sentences that describes each of the following statements: -

a) ~p

Roses are not red.

b) ~ (p Λ q)

Its false that Roses are red and violets are blue

Upon opening the brackets we get ~p v ~q

Roses are not red or Violets are not blue

c) p v q

Roses are Red or Violets are Blue

d) q v ~p

Violets are Blue or Roses are not Red

e) ~p Λ ~q

Roses are not red and violets are not Blue

f) (vi.) ~~q

V*iolets are blue*

Construct a truth table for each of the following statements:

$q \wedge (p \rightarrow q) \rightarrow p$

P	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow p$	$q \wedge (p \rightarrow q) \rightarrow p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

Construct a truth table for the following statement and show whether it is a tautology or a contradiction

(i) $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

p	q	~p	~q	(p Λ q)	~ (p Λ q)	~p V ~q	~ (p Λ q) ↔ (~p V ~
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

All True values (Tautology)

a) Conditional statements are those of the form "If p then q" They are the fundamental principles of logical reasoning. Given the conditional statement: - "If p implies q and q implies r then p implies r" verify this fact by showing that the following proposition is a tautology $[(p\to q) \land (q\to r)] \to (p\to r)$.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$[(p\to q)\land(q\to r)]$	$[(p\to q)\land(q\to r)]\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

All True values (Tautology)

Show that [(p→q) Λ (q→r)] →(p→r) is a tautology

p	q	r	p → q	q → r	p → r	[(p→q)Λ(q→r)]	[(p→q)Λ(q→r)] →(p→r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

All True values (Tautology)

References

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