Automata Theory - Lecture 4

Introduction to Logical Equivalence

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Lecture learning outcomes

At the end of the lecture you will be able to:

(i) Define logical equivalence.

(ii) Describe the various connectives used in logically equivalent statements.

(iii) Differentiate conditional from bi-conditional statement

(iv) Solve problems involving equivalent and conditional statements, using truth tables

4.1 Logical Equivalence

When two propositions P (p, q …) and Q (p, q …) are said to be ***logically equivalent*** they are also said to be ***simply equivalent*** or ***equal***.

This equivalence of statements is denoted by the symbol ≡ written as: -
P (p, q …) ≡ Q (p, q …).

This means that the two propositions have identical truth/falsity values in the last column of their truth tables.

<u>Example One:</u>

Consider the statements:

(i) $\sim (P \Lambda Q)$ and

(ii) $\sim P \vee \sim Q$,

Consider the two statements P = "Roses are red" Q = "Violets are blue"

The statement (i) above can be read as: "It is false that roses are red ***and*** violets are blue" written in form of $\sim (P \Lambda Q)$, where P is "Roses are red" and Q is "Violets are blue".

The statements (ii) above represent the outcome of opening the brackets whereby negating "and" we get "or" as follows: Roses are not red or violets are not blue written as $(\sim P \vee \sim P)$.

However, the truth tables below show that $\sim (P \Lambda Q)$ is logically equivalent to $(\sim P \vee \sim P)$. Thus, the given statement has the same meaning as the statement:" Roses are not red" and "Violets are not blue".

The truth tables of the above statements are as shown below and the values in the last column of the truth tables are similar indicating that the two statements are logically equivalent.

(i) $\sim (P \land Q)$

P	Q	$P \land Q$	$\sim(P \land Q)$
True	True	True	False
True	False	False	True
False	True	False	True
False	False	False	True

Equal values

(ii) $(\sim P \lor \sim Q)$

P	Q	$\sim P$	$\sim Q$	$\sim P \lor \sim Q$
True	True	False	False	False
True	False	False	True	True
F	True	True	False	True
False	False	True	True	True

Since the truth tables are the same, i.e. both propositions are false in the first case and true in the other instances, the propositions $\sim (P \land Q)$ and $\sim P \lor \sim P$, are logically equivalent meaning $\sim (P \land Q) \equiv (\sim P \lor \sim P)$.

4.2 Conditional & Bi-Conditional Statements

Many statements, particularly in mathematics, are of the form; "**If P then Q**".

Such statements are called ***conditional statements*** and are denoted by ***P → Q***.

The condition P → Q is frequently read as "***P implies Q***" or "***P only if Q***".

Another common statement is of the form; "**P if and only if Q**". Such statements are denoted by ***P ↔Q*** and are called ***bi-conditional statements***.

Example: The truth values of P → Q and P ↔Q are as shown below: -

P → Q

P	Q	P → Q
True	True	True
True	False	False
False	True	True
False	False	True

$P \leftrightarrow Q$

P	Q	$P \leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

Review Questions

1. If p V ~p is a tautology, substituting q Λ r for p, we obtain proposition (q Λ r) V~ (q Λ r), which by the principle of substitution, should also be a tautology. Verify this with the use of a truth table.

2. Let p be "Roses are Red" and let q be "Violets are Blue". Write verbal sentences that describes each of the following statements: -

a) ~p

Roses are not red.

b) ~ (p Λ q)

Its false that Roses are red and violets are blue

Upon opening the brackets we get ~p v ~q

Roses are not red or Violets are not blue

c) p v q

Roses are Red or Violets are Blue

d) q v ~p

Violets are Blue or Roses are not Red

e) ~p Λ ~q

Roses are not red and violets are not Blue

f) (vi.) ~~q

V*iolets are blue*

3. Construct a truth table for each of the following statements:

(i) $q \wedge (p \to q) \to p$

P	q	$p \to q$	$(p \to q) \to p$	$q \wedge (p \to q) \to p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

(ii) $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

4. Conditional statements are those of the form “If p then q” They are the fundamental principles of logical reasoning. Given the conditional statement: - “If p implies q and q implies r then p implies r” verify this fact by showing that the following proposition is a tautology $[(p\rightarrow q) \Lambda (q\rightarrow r)] \rightarrow (p\rightarrow r)$.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$[(p\rightarrow q)\Lambda(q\rightarrow r)]$	$[(p\rightarrow q)\Lambda(q\rightarrow r)]\rightarrow(p\rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

References

1. Rowan G. & John T., (2009), *Discrete Mathematics: Proofs, Structures and Applications*, CRC Press, ISBN: 9781439812808.

2. W. D. Wallis (2003), *A Beginners Guide to Discrete Mathematics*, Springer Science & Business Media, ISBN: 978-0817642693.

3. Introduction to the theory of computation (3rd ed.), Michael, S. Boston, Cengage Learning. ISBN-13: 978-1133187790, (2012).

4. Introduction to languages and the theory of computation (3rd ed.), Martin, J., New York: McGraw-Hill. ISBN-13: 978-0072322002, (2002)