

PROBABILIY AND STATISTICS I

LECTURE NINE

Introduction to probability

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INTRODUCTION

This lecture will focus on definition of terms, assigning probabilities and laws of probability.

Intended learning outcomes

At the end of this lecture, you will be able to define probability, describe the different approaches to probability determination and perform elementary probability operations.

References

These lecture notes should be supplemented with relevant topics from the book listed in the Bibliography at the end of the lecture.

Probability

Probability is a very popular concept in everyday life. This is because it covers the risks which may be involved in certain situations. It is a fact that when an event is being organized, the outcome is definitely uncertain. The concept of probability is then used to describe the degree of uncertainty of a particular outcome.

Probability therefore can be defined as the chances of occurrence of a given event. This simply means that probability is a number that describes the chance that something will happen. Probability values lie between zero and one inclusive. Probability of one means that the event is sure to happen, zero means that the event cannot happen. The values of probability can be expressed as fractions, percentages or decimals.

Application of probability:

1. Games of chance e.g. betting, lotteries e.t.c
2. Insurance firms use the concept of probability especially when a new client or property is being insured. The insurer needs to ascertain the chances of the insured risks.
3. Business decision making regarding viability of projects.
4. Opinion polls.
5. Clinical trials.
6. Politics.
7. Weather forecasting.

Definition of terms

Experiment - A process that leads to the occurrence of one and only one of several possible observations

Outcome - A particular result of an experiment

Event - A collection of one or more outcomes of an experiment

Sample space - The set of all possible outcomes of an experiment

Example

A probability experiment consists of rolling a single fair die.

- i. Identify the outcomes of the probability experiment.
- ii. Determine the sample space.
- iii. Define the event "roll an even number"

The outcomes are the possible results of the experiment. The sample space is a list of all possible outcomes.

Solution

- i. The outcomes from rolling a single fair die are rolling a one $E_1 = \{1\}$, a two $E_2 = \{2\}$, a three $E_3 = \{3\}$, a four $E_4 = \{4\}$, a five $E_5 = \{5\}$, and a six $E_6 = \{6\}$
- ii. The set of all possible outcomes forms the sample space, $S = \{1, 2, 3, 4, 5, 6\}$.
There are 6 outcomes in the sample space.
- iii. The event $E = \text{"roll an even number"} = \{2, 4, 6\}$.

Approaches to probability

Empirical Approach

Empirical probability is based on the number of times an event occurs as a proportion of a known number of trials.

Probabilities computed in this manner rely on empirical evidence, that is, evidence based on the outcomes of an experiment. An experiment has to be conducted to get this probability

The probability of an event A is approximately the number of times event A is observed divided by the number of repetitions of the experiment.

$$P(A) = \frac{\text{Frequency of } A}{\text{Number of trials of the experiment}}$$

Example

James wanted to approximate the probability of rolling a seven using a pair of dice that have been in use for a while. To do this he rolls the dice 100 times and records 15 sevens. The probability of rolling a seven is approximately $\frac{15}{100} = 0.15$

Classical approach

Classical probability assumes that all the outcomes are equally likely.

An experiment is said to have equally likely outcomes when each outcome has the same probability of occurring. For example, in throwing a fair die once, each of the six outcomes in the sample space, has an equal chance of occurring.

The classical method of computing probabilities does not require that a probability experiment actually be performed. Rather, we use counting techniques to determine the probability of an event.

If an experiment has n equally likely outcomes and if the number of ways that an event A can occur is m , then the probability of A , $P(A)$, is

$$P(A) = \frac{\text{Number of ways that } A \text{ occurs}}{\text{Number of possible outcomes}} = \frac{N(A)}{N(S)}$$

Example

A pair of fair dice is rolled.

- i. Compute the probability of rolling a sum of seven.
- ii. Compute the probability of rolling “snake eyes”; that is, compute the probability of rolling a sum of two.
- iii. Comment on the likelihood of rolling a sum of seven versus rolling a sum of two.

Solution

To compute probabilities using the classical method, we count the number of outcomes in the sample space and count the number of ways the event can occur.

- i. In rolling a pair of fair dice there are 36 equally likely outcomes in the sample space

$$\text{So } N(S) = 36$$

The event $A =$ roll a sum of seven $= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$ has six outcomes

$$\text{So } N(A) = 6$$

Using Formula, the probability of rolling a sum of seven is

$$P(A) = \frac{N(A)}{N(S)}$$

- ii. The event $A = \text{roll a sum of two} = \{(1, 1)\}$ has one outcome

$$P(A) = \frac{N(A)}{N(S)} = \frac{1}{36}$$

- iii. From the computed probabilities, rolling a sum of seven is six times as likely as rolling a sum of two since out of the 36 rolls of the dice, we expect to observe about 6 sums of seven and only 1 sum of two.

Subjective approach

A subjective probability of an outcome is a probability obtained on the basis of personal judgment.

Example

A financial reporter may ask an economist about the likelihood the economy will fall into recession next year. The economist must use his or her knowledge of the current conditions of the economy and make an educated guess as to the likelihood of recession.

Definitions

Mutually exclusive event (disjoint event) - Two events are said to be disjoint or mutually exclusive if they have no outcomes in common. This means that the two events cannot occur at the same time in a single trial.

For example when tossing a coin, the events are a head or a tail. These two events are mutually exclusive since the occurrence of the heads automatically deters the occurrence of the tail.

Independent events - Two events are said to be independent if the occurrence one event in a probability experiment does not affect the probability of the other event. Two

events are dependent if the occurrence of event in a probability experiment affects the probability of the other event.

For instance the outcome of tossing a coin is independent of the preceding or succeeding toss.

Laws of probability

Addition Rule for Disjoint Events

This is the rule used to calculate the probability of two or more mutually exclusive events. The probability of the separate events are added.

If A and B are disjoint (or mutually exclusive) events, then $P(A \text{ or } B) = P(A) + P(B)$

Example

What is the probability of throwing a 1 or a 5 with a throw of a die?

Solution

$$P(\text{throwing a 1 or a 5}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Exercise

Suppose that a single card is selected from a standard 52-card deck,

- Compute the probability of the event $A = \text{"drawing a king"}$
- Compute the probability of the event $A = \text{"drawing a king"}$ or $B = \text{"drawing a queen"}$
- Compute the probability of the $A = \text{"drawing a king"}$ or $B = \text{"drawing a queen"}$ or $C = \text{"drawing a jack"}$

Solution

$$N(S) = 52$$

a) $P(A) = \frac{N(A)}{N(S)} = \frac{4}{52}$

$$b) P(A \text{ or } B) = P(A) + P(B) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$$

$$c) P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52}$$

The general Addition Rule

For any two events A and B $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Multiplication Rule for Independent Events

This is used when there is a string of independent events for which individual probability is known and it is required to know the overall probability.

If A and B are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example

What is the probability of throwing a 1 and a 5 with two throws of a die?

Solution

$$P(1) \text{ and } P(5) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Note

- In probability the word and is replaced by multiplication.
- $P(A) \text{ and } P(B) \neq P(A \text{ and } B)$ the first part implies A happening and B happening.

If the order which the events happen is unimportant, then $P(A \text{ and } B)$ is used. In the above example:

$$\begin{aligned} P(1 \text{ and } 5) &= P(1 \text{ followed by } 5) \text{ or } P(5 \text{ followed by } 1) \\ &= [P(1)P(5)] \text{ or } [P(5)P(1)] = \frac{1}{36} + \frac{1}{36} = \frac{1}{18} \end{aligned}$$

Conditional probability Rule

This is the probability associated with combinations of events but given that some prior event has already occurred with known results. It is denoted as $P(A/B)$ and read as “the

probability of event A given event B ." It is therefore the probability of event A occurring given that event B has already occurred

If A and B are any two events, then

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{N(A \text{ and } B)}{N(A)}$$

General Multiplication Rule

The probability that two events A and B both occur is

$$P(A \text{ and } B) = P(B) \cdot P(A/B)$$

Two events A and B are independent if $P(A/B) = P(B)$

Example

In a competitive examination. 30 candidates are to be selected. In all 600 candidates appear in a written test, and 100 will be called for the interview.

- i. What is the probability that a person will be called for the interview?

Let event A be that the person is called for the interview and event B that the person is selected.

$$P(A) = \frac{100}{600} = \frac{1}{6}$$

- ii. Determine the probability of a person getting selected if he has been called for the interview

$$P(B/A) = \frac{30}{100} = \frac{3}{10}$$

- iii. What is the probability that a person is called for the interview and is selected?

$$P(AB) = P(A) \times P(B/A) = \frac{1}{6} \times \frac{3}{10} = \frac{1}{20}$$

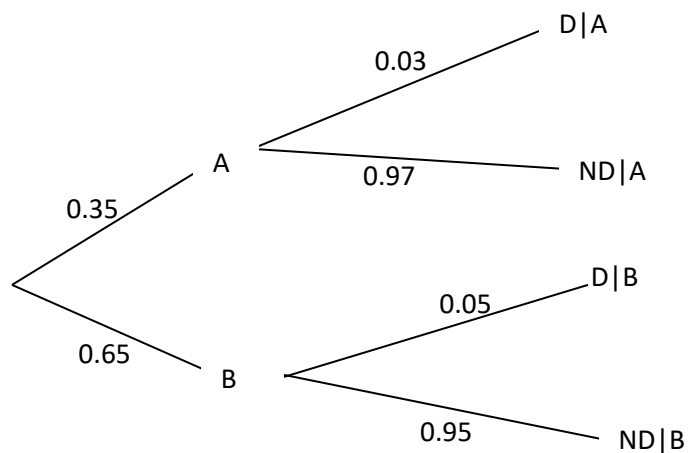
Bayes theory

Example

A company has two production plants A and B . Both plants produce a total of 1000 items per week. A produces 350 items and B produces 650 items. Plant A produces 3% defective items and plant B produces 5% defective items.

The information above can be represented on a tree diagram as follows:

A represents plant A , B plant B , D represents defective items and ND non-defective items.



The probabilities are written along the branches. If an item is picked at random the probability that it was produced at plant A is $0.35 = \left(\frac{350}{1000}\right)$. Probability it was produced at plant B is $0.65 = \left(\frac{650}{1000}\right)$.

Probability that an item is defective given it is from A is given as $P(D|A) = 0.03$

Probability that an item is defective given it is from B is given as $P(D|B) = 0.05$

Probability that an item came from A and is defective is obtained by multiplying probabilities along the branches, thus

$$P(A \text{ and } D) = P(A) \times P(D|A) = 0.35 \times 0.03 = 0.0105$$

Similarly, the probability that an item came from B and is defective is given as

$$P(B \text{ and } D) = P(B) \times P(D/B) = 0.65 \times 0.05 = 0.0325$$

Let us then answer the question:

Given an item was defective, what is the probability it was produced at A ? Denoted by $P(A/D)$

The probability an item is defective means that it either came from A and is defective or it came from B and is defective, that is, the probability $P(A \text{ and } D)$ or $P(B \text{ and } D)$. This is then given as:

$$P(D) = P(A) \times P(D/A) + P(B) \times P(D/B)$$

Now, given that an item is defective, the probability that it is from A that is, $P(A/D)$ is given as:

$$P(A/D) = \frac{P(A) \times P(D/A)}{P(D)} = \frac{P(A) \times P(D/A)}{P(A) \times P(D/A) + P(B) \times P(D/B)}$$

The formula above is called **Bayes formula** developed by English Presbyterian minister Rev Thomas Bayes (1702 – 1761). This formula is frequently used in decision making where information is given is in the form of conditional probabilities and the reverse of these probabilities must be found.

$$P(A/D) = \frac{0.0105}{0.0105 + 0.0325} = 0.244$$

Therefore given an item is defective there is a 24.4% chance it came from A .

Example

Analysis of questionnaires completed by holiday makers showed that 0.75 classified their holiday as good at Malindi. The probability of hot weather in the resort is 0.6. If

the probability of regarding the holiday as good given hot weather is 0.9, what is the probability that there was hot weather if a holiday maker considers the holiday good?

Solution

Let A represent hot weather and B good

$$P(B) = 0.75, \quad P(A) = 0.6, \quad \text{and}$$

$$P(B/A) = 0.9 - \text{probability of regarding the holiday as good given hot weather.}$$

The probability that there was hot weather given that the holiday has been rated as good is given by:

$$P(A/B) = \frac{P(A)P(B/A)}{P(B)} = \frac{(0.6)(0.9)}{0.75} = 0.72$$

Bibliography

Gupta, SP (Dr.), (2014). *Statistical methods* (43rd Ed.). Sultan Chand & Sons.

S. C. Gupta and V. K. Kapoor, (2020). *Fundamentals of mathematical Statistics* (12th Ed).
Sultan Chand & Sons.