ENG 302 - SOLID MECHANICS

ST. PAUL'S UNIVERSITY
DEPARTMENT OF ENGINEERING
SECOND SEMESTER 2016-2017 EXAM SERIES

ATTEMPT SIX QUESTIONS ONLY

QUESTION ONE - 25 Marks

In 1875, representatives of 17 nations (including the UK) met in Paris to sign the Treaty of the Metre. The treaty established the metric system. The convention for the metre was that the circumference of the Earth should be forty million metres and a prototype bar of one metre was created (unfortunately, the measurement of the Earth was not accurate enough at the time and the circumference going through the poles is 40,007,863m). Eventually, in 1889 a convention was established for the metre as the length of one prototype bar (No 6) made of 90% platinum and 10% irridium measured at the melting point of ice. This bar remained the official definition of the metre until 1960 (when it was replaced by a multiple of a wavelength of Krypton-86 emission, then by a fraction of the distance travelled by light in vacuum in one second). As an exercise, assume that the bar is, in the absence of external loads, a cuboid of platinum of length 1m (obviously) and of section 10cm by 10cm. To obtain an estimate of its deformation due to its own weight, compute the shortening of the bar when held vertically by replacing its self-weight (which would vary along the length) by a single load on the top face of the same weight and assuming an homogeneous deformation. Now, compute the lengthening of the bar when held horizontally (again by replacing its own self-weight by a weight acting on top of it). *The actual bar is not a cuboid but has a X-shape section (See Fig. 1). *Why? *Why was it made of the combination platinum/irridium? and *why should it be measured at the melting point of ice? **How much longer would it be at ambient temperature (in Paris, say 300K)?

SOLUTION

For the vertical bar, assume the force magnitude F is applied at the cross-section of area $A = a \times a$ of a bar of length L. The bar has density ρ and the gravitational constant is g. The Cauchy stress is due to the weight of the bar is $\sigma = F/A = \rho L g$ where σ is the only nonzero component of the Cauchy stress tensor (the zz or 33 component). The Hookean constitutive law is $\sigma = E\left(\lambda_V - 1\right)$ where E is Young's Modulus and λ_V is the dimensionless stretch in vertical direction. We compute the latter as

$$\lambda_V = \frac{\rho L g}{E} + 1 \tag{1}$$

In the horizontal case, the force F is no longer acting on a cross section of area $a \times a$ but instead on $a \times L$. Then $\sigma = F/A = \rho a g$ and the horizontal stretch is

$$\lambda_H = \frac{a\rho g}{E} + 1 \tag{2}$$

We choose the values

$$\begin{aligned} \rho &= 21.43\text{g cm}^{-3} = 21.43 \cdot 10^3 \text{kg m}^{-3} && (3)\\ E &= 168G\text{Pa} = 168 \cdot 10^9 \text{N m}^{-2} && (4)\\ a &= 0.1\text{m} && (5)\\ L &= 1\text{m} && (6) \end{aligned}$$

and the results are (note that stretches λ are dimensionless!)

$$\lambda_V = 1 + 1.25 \cdot 10^{-6} \qquad \lambda_H = 1 + 1.25 \cdot 10^{-7} \tag{7}$$

For the case of the vertical bar, we introduce the strain $\left[z\left(L\right) - L\right]/L$ where $z\left(L\right)$ is the top of the bar in the current configuration. Since the vertical stretch is simply $\lambda_V = \partial z/\partial Z =$ const., we have $z = \lambda_V L$. The strain is

$$\frac{\left|z\left(L\right) - L\right|}{L} = \frac{\rho L g}{E} \tag{8}$$

QUESTION TWO

If you model the metre bar as a one-dimensional elastic medium, you can use the theory developed in the Lecture Notes (Chapter 1) to obtain a better estimate of the shortening. Compute the deformation of the bar under its own-weight. Is the Hookean model sufficient?

SOLUTION

Let us once again assume the bar is aligned with the Z direction and gravity acts in $-Z$ direction. The bar starts at $Z = 0$ and reaches up until $Z = L$ in reference configuration. First, we need to find how the force n acting on a cross-section of area A is distributed as a function of Z, i.e. $n(Z)$. To do this, we write the force balance

$$\frac{\mathrm{d}n}{\mathrm{d}Z} + f = 0 \tag{9}$$

where (in this case) f is a force per length due to gravity. we have $f = -\rho A g$ where ρ is the mass density in reference configuration and g is the gravitational constant. We demand that there is no force at $Z = L$ (at the top of the bar), i.e. $n(L) = 0$. Solving this ODE, we find

$$n(Z) = \rho A g (Z - L) \tag{10}$$

We ass ume that the material is Hookean,

$$n(Z) = EA(\lambda - 1) \tag{11}$$

where E is Young's modulus and $\lambda = \partial z / \partial Z$ is the elastic stretch in Z direction. As an initial condition, we choose $z(0) = 0$ as the bottom point of the bar is not moving during deformation. Combining (10), (11) and $z(0) = 0$ we obtain

$$z(Z) = Z + \frac{\rho g}{E}\left(\frac{Z^2}{2} - LZ\right) \tag{12}$$

The strain of the bar is

$$\frac{|z(L) - L|}{L} = -\frac{\rho g L}{2E} \tag{13}$$

This is half the shortening which the bar undergoes in problem 2.1, see eq. (8).

If we want to compare this result with a neo-Hookean model, we must substitute (11) with $n(Z) = EA\left(\lambda^2 - \lambda^{-1}\right)$ and compute $(z(L) - L)/L$. For values of ρ, E and g as in problem 2.1, we should find that the Hookean and the neo-Hookean model are in good agreement.

QUESTION THREE: 25 Marks

Consider the motion given in component form by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ where

$$x_1 = X_1 e^{-t} \qquad x_2 = X_2 e^{t} \qquad x_3 = X_3 + X_2\left(e^{-t} - 1\right) \tag{14}$$

(a) Determine the velocity in material form: $\mathbf{V} = \mathbf{V}(\mathbf{X}, t)$.

(b) Invert (14) to express $\mathbf{X}$ in terms of $\mathbf{x}$ and to find the velocity in spatial form $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.

(c) Check that $\operatorname{div}\mathbf{v} = 0$ and interpret this equality.

(d) Check that the acceleration $\mathbf{a}$ can be computed in the two following ways,

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{v}}{dt} = \mathbf{v}\cdot\operatorname{grad}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}. \tag{15}$$

SOLUTION

(a) The deformation map and the velocity are

$$\mathbf{x} = \boldsymbol{\chi}\left(\mathbf{X}, t\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 e^{-t} \\ X_2 e^{t} \\ X_3 + X_2\left(e^{-t} - 1\right) \end{pmatrix} \qquad \mathbf{V}\left(\mathbf{X}, t\right) = \frac{\partial \boldsymbol{\chi}\left(\mathbf{X}, t\right)}{\partial t} = \begin{pmatrix} -X_1 e^{-t} \\ X_2 e^{t} \\ -X_2 e^{-t} \end{pmatrix} \tag{16}$$

(b) We need $\mathbf{v} = \mathbf{v}\left(\mathbf{x}, t\right) = \left[\mathbf{V}\left(\mathbf{X}, t\right)\right]_{\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t)}$. First invert $\boldsymbol{\chi}$, then compute $\mathbf{v}$:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} x_1 e^{t} \\ x_2 e^{-t} \\ x_3 - x_2 e^{-t}\left(e^{-t} - 1\right) \end{pmatrix} \qquad \mathbf{v}\left(\mathbf{x}, t\right) = \begin{pmatrix} -x_1 \\ x_2 \\ -x_2 e^{-2t} \end{pmatrix} \tag{17}$$

(c) The motion is isochoric (locall volume preserving) since $\operatorname{div}\mathbf{v} = \partial v_i / \partial x_i = -1 + 1 + 0 = 0$.

(d) The left hand side of the expression given in the problem is

$$\mathbf{a} = \frac{\partial \mathbf{V}\left(\mathbf{X}, t\right)}{\partial t} = \begin{pmatrix} X_1 e^{-t} \\ X_2 e^{t} \\ X_2 e^{-t} \end{pmatrix} \tag{18}$$

For the right hand side we can verify in a cartesian basis $\mathbf{v}\cdot\operatorname{grad}\mathbf{v} = \left(\operatorname{grad}\mathbf{v}\right)\mathbf{v}$ and compute

$$\mathbf{a} = \left(\operatorname{grad}\mathbf{v}\right)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{-2t} & 0 \end{pmatrix}}_{\operatorname{grad}\mathbf{v}} \underbrace{\begin{pmatrix} -x_1 \\ x_2 \\ -x_2 e^{-2t} \end{pmatrix}}_{\mathbf{v}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2x_2 e^{-2t} \end{pmatrix}}_{\partial \mathbf{v}/\partial t} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 e^{-2t} \end{pmatrix} \tag{19}$$

Considering $X_1 = x_1 e^{t}$ and $X_2 = x_2 e^{-t}$ in (17), we see that the last expression for $\mathbf{a}$ matches (18).

QUESTION FOUR: 25 Marks

Consider the scalar field $\phi(\mathbf{x}) = (x_1)^2 x_3 + x_2(x_3)^2$ and the vector field $\mathbf{v}(\mathbf{x}) = x_3\mathbf{e}_1 + x_2\sin(x_1)\mathbf{e}_3$. Find the components of $\operatorname{grad}\phi$ and $\operatorname{grad}\mathbf{v}$.

SOLUTION

We have $\phi(\mathbf{x}) = x_1^2 x_3 + x_2 x_3^2$. Then

$$\operatorname{grad}\phi(\mathbf{x}) = \frac{\partial\phi(\mathbf{x})}{\partial x_i}\mathbf{e}_i = 2x_1x_3\mathbf{e}_1 + x_3^2\mathbf{e}_2 + \left(x_1^2 + 2x_2x_3\right)\mathbf{e}_3 \tag{20}$$

Also, we have $\mathbf{v}(\mathbf{x}) = x_3\mathbf{e}_1 + x_2\sin x_1\mathbf{e}_3$. Then

$$\operatorname{grad}\mathbf{v}(\mathbf{x}) = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j = \mathbf{e}_1\otimes\mathbf{e}_3 + (x_2\cos x_1)\,\mathbf{e}_3\otimes\mathbf{e}_1 + (\sin x_1)\,\mathbf{e}_3\otimes\mathbf{e}_2\ . \tag{21}$$

QUESTION FIVE: 25 Marks

Show that $\mathbf{u}\cdot\mathbf{M}\mathbf{v}=\mathbf{v}\cdot\mathbf{M}^T\mathbf{u}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors and $\mathbf{M}$ is a second-order tensor. Use this relation to prove that the following motion is a *rigid* motion,

$$\mathbf{x}(t)=\mathbf{c}(t)+\mathbf{Q}(t)\mathbf{X}, \tag{22}$$

i.e. the distance between any two points remains unchanged during the motion. Here $\mathbf{x}$ is the current position of a point which was initially at $\mathbf{X}$, $\mathbf{c}$ is a vector and $\mathbf{Q}$ is a proper orthogonal second-order tensor.

SOLUTION

In a cartesian basis, $\mathbf{u}=u_i\mathbf{e}_i$, $\mathbf{v}=v_i\mathbf{e}_i$ and $\mathbf{M}=M_{ij}=\mathbf{e}_i\otimes\mathbf{e}_j$. In components,

$$\mathbf{u}\cdot\mathbf{M}\mathbf{v}=u_iM_{ij}v_j=v_jM^T_{ji}u_i=\mathbf{v}\cdot\mathbf{M}^T\mathbf{u} \tag{23}$$

We want so show that $|\mathbf{y}-\mathbf{x}|=|\mathbf{Y}-\mathbf{X}|$ where $\mathbf{y}\left(t\right)=\mathbf{c}\left(t\right)+\mathbf{Q}\left(t\right)\mathbf{Y}$ and $\mathbf{x}\left(t\right)=\mathbf{c}\left(t\right)+\mathbf{Q}\left(t\right)\mathbf{X}$ for proper orthogonal $\mathbf{Q}$, i.e. $\mathbf{Q}^T\mathbf{Q}=\mathbf{1}$.

$$\begin{aligned}
|\mathbf{y}-\mathbf{x}|^2 &= (\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}-\mathbf{x}) && (24)\\
&= \underbrace{\mathbf{Q}\left(\mathbf{X}-\mathbf{Y}\right)}_{\mathbf{u}}\cdot\underbrace{\mathbf{Q}}_{\mathbf{M}}\underbrace{\left(\mathbf{Y}-\mathbf{X}\right)}_{\mathbf{v}} && (25)\\
&= \left(\mathbf{X}-\mathbf{Y}\right)\cdot\underbrace{\mathbf{Q}^T\mathbf{Q}}_{\mathbf{1}}\left(\mathbf{X}-\mathbf{Y}\right) && (26)\\
&= \left(\mathbf{X}-\mathbf{Y}\right)\cdot\left(\mathbf{X}-\mathbf{Y}\right) && (27)\\
&= |\mathbf{X}-\mathbf{Y}|^2 && (28)
\end{aligned}$$

QUESTION SIX: 25 Marks

The motion of a body is given for $t \geq 0$ by

$$\mathbf{x}(\mathbf{X},t) = (X_1 + ktX_3, X_2 + ktX_3, X_3 - kt(X_1 + X_2)), \tag{29}$$

where $k > 0$ is a constant. Show that the path of an arbitrary material point with reference position $\mathbf{X} \neq 0$ is a straight line orthogonal to $\mathbf{X}$.

Show that a material plane initially at $X_1 = h$ is mapped to another plane and compute its normal unit vector. Conclude that asymptotically as $t \to \infty$, all planes $X_1 = h$ become parallel.

SOLUTION

We can write the deformation map as $\mathbf{x}(\mathbf{X},t) = \mathbf{X} + t\mathbf{V}(\mathbf{X})$ where $\mathbf{V}(\mathbf{X}) = \partial\mathbf{x}(\mathbf{X},t)/\partial t = (kX_3, kX_3, -k(X_1, X_2))$ is the material velocity. This parametrises a straight line. Computing $\mathbf{X} \cdot \mathbf{V}(\mathbf{X}) = 0$ shows that $\mathbf{x}(\mathbf{X})$ is a straight line orthogonal to $\mathbf{X}$ for all t.

In the material configuration, consider the plane $\mathbf{P}$ parameterised by scalars R, U: $\mathbf{P}(R,U) = h\mathbf{E}_1 + R\mathbf{E}_2 + U\mathbf{E}_3$ where $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are cartesian basis. We now map $\mathbf{P}$ to

$$\mathbf{p}(R,U,t) = \mathbf{x}(\mathbf{P}(R,U),t) = \begin{pmatrix} h \\ 0 \\ -kt \end{pmatrix} + R\begin{pmatrix} 0 \\ 1 \\ -kth \end{pmatrix} + U\begin{pmatrix} kt \\ kt \\ 1 \end{pmatrix} \tag{30}$$

which is the parameterisation of a plane. Its normal vector is

$$\mathbf{n} = \frac{1}{\sqrt{1 + 2(kt)^4 + 3(kt)^2}} \begin{pmatrix} 1 + \frac{1}{(kt)^2} \\ -1 \\ -\frac{1}{kt} \end{pmatrix} \tag{31}$$

You can compute the limit

$$\lim_{t\to\infty} \mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{32}$$

showing that all planes thave the same normal vector for $t \to \infty$, meaning they are parallel.

You can practice your *Mathematica* skills by experimenting with these commands:

```
n = {1 + (k t)^2, -(k t)^2, -k t}
temp = Limit[n/Norm[n], t -> \[Infinity]]
FullSimplify[temp, Assumptions -> {k \[Element] Reals}]
```

QUESTION SEVEN: 25 Marks

Consider a cylindrical tube and invert it by turning it inside out (so that the inner surface is now the outer surface - think of it as a sock). Assuming that a radial fibre does not deform and that the everted shape is a cylinder, write the deformation mapping. Show that if you do it twice, you will recover the initial shape.

SOLUTION:

The deformation map for a cylinder is

$$\mathbf{X} = R\mathbf{E}_R + \Theta\mathbf{E}_\Theta \qquad \boldsymbol{\chi}\left(\mathbf{X}\right) = f\left(R\right)\mathbf{e}_r + \theta\mathbf{e}_\theta \tag{33}$$

where $\mathbf{E}_R = \mathbf{e}_r$, $\mathbf{E}_\Theta = \mathbf{e}_\theta$ and $\Theta = \theta$. We need to find $f\left(R\right)$. To do this, let us consider a cylindrical shell with $A \leq R \leq B$ in the initial reference configuration. Based on the sketch in figure (1), we define the quantity

$$\delta := R - \frac{A+B}{2} \tag{34}$$

As should be evident from the figure, we demand that the eversion map fulfills $f\left(R\right) = R - 2\delta$. Using the definition (34), we find

$$f\left(R\right) = A + B - R \tag{35}$$

We can easily check that $f\left(f\left(R\right)\right) = f\left(A+B-R\right) = A + B - \left(A+B-R\right) = R$ as should be.

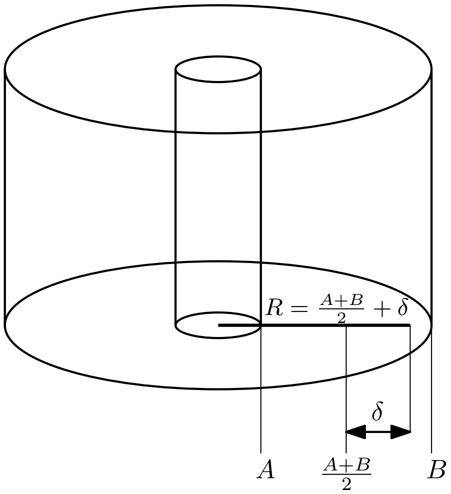

Figure 1: Sketch of setup for eversion of a cylinder.

QUESTION EIGHT: 25 Marks

Let $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ be three mutually orthogonal unit vectors. Consider the second-order tensor $\mathbf{A}$ with components

$$A_{ij} = r_i r_j + s_i s_j + t_i t_j. \tag{36}$$

Now, any vector $\mathbf{u}$ can be written as $\mathbf{u} = \alpha\mathbf{r} + \beta\mathbf{s} + \gamma\mathbf{t}$ for some scalars α, β and γ. Show that $\mathbf{Au} = \mathbf{u}$ and hence, that $\mathbf{A}$ is the identity.

We assume a cartesian basis throughout. We want to show that $u_i = A_{ij}u_j$. Since $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are orthogonal, $\mathbf{r}\cdot\mathbf{s} = 0$, $\mathbf{r}\cdot\mathbf{t} = 0$ and $\mathbf{s}\cdot\mathbf{t} = 0$ which in components reads

$$r_i s_i = 0 \qquad r_i t_i = 0 \qquad s_i t_i = 0 \tag{37}$$

Similarly, since $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are normalised, we have

$$r_i r_i = 1 \qquad s_i s_i = 1 \qquad t_i t_i = 1 \tag{38}$$

We now proof that A_{ij} are the components of the identity.

$$\begin{aligned} A_{ij}u_j &= (r_i r_j + s_i s_j + t_i t_j)(\alpha r_j + \beta s_j + \gamma t_j) && (39)\\ &= \alpha r_i r_j r_j + \beta s_i s_j s_j + \gamma t_i t_j t_j && (40)\\ &= u_i && (41) \end{aligned}$$

From the first to the second line, we made use of the orthogonality (37), and from the second to the third line, we used the normalisation (38). Q.E.D.

So $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is the identity. This is particularly simple if $\mathbf{r} = \mathbf{e}_1$, $\mathbf{s} = \mathbf{e}_2$, $\mathbf{t} = \mathbf{e}_3$ in which case $A_{ij} = \delta_{ij}$ is the Kronecker delta.