Course: Automatic Control System Technology

Lecture 3: Determine Inverse Laplace transform

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Session objectives

By the end of this session, students will be able to :

- Define inverse Laplace transform
- Determine inverse Laplace transform by using Laplace transform table
- Determine Inverse Laplace Transform by using partial fraction expansion

Define inverse Laplace Transform

- ❖ The operation of obtaining $\boldsymbol{f(t)}$ from the Laplace transform **F(s)** is called **inverse Laplace transform**
- ❖ It is denoted by:

$$\boldsymbol{f(t) = \text{Inverse Laplace transform of } F(s) = \mathcal{L}^{-1}[F(s)]}$$

- ❖ And defined as: $\boldsymbol{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds}$,

  Where c is a real constant that is greater than real parts of the poles of F(s). Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 54.

Determine inverse Laplace transform by using Laplace transform table

- Note that you do not need to find the inverse Laplace transform by using directly its ***line integral definition***.
- For simple functions, the inverse Laplace transform operation can be carried out simply by referring to the Laplace transform table or pairs of common functions

Laplace Transform Table or pairs

Table 1: Laplace transform table or pairs of common functions

$f(t);\ t \geq 0$	$F(s)$	$f(t);\ t \geq 0$	$F(s)$
$\delta(t)$: unit impulse	1	$1-e^{-at}$	$\frac{1}{s(s+a)}$
$\mu(t)$: unit step	$\frac{1}{s}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$		

Gopal M (2008),Control Systems: Principles and Design, 3rd Edition, Tata McGraw-Hill, page 43.

Determine Inverse Laplace Transform by using partial fraction expansion

- For difficult functions, the inverse Laplace transform can be carrier out by first performing a partial-fraction expansion and then using the Laplace transform table.
- In the next slides, we will first perform the partial fraction expansion of a Laplace transformed function G(s) and then determine its inverse Laplace transform g(t)

Determine Inverse Laplace Transform by using partial fraction expansion

First case: G(s) has simple poles

If all the poles of G(s) are simple and real, then G(s) can be

written: $\boldsymbol{G(s)} = \frac{\boldsymbol{Q(s)}}{\boldsymbol{P(s)}} = \frac{\boldsymbol{Q(s)}}{(\boldsymbol{s+s_1})(\boldsymbol{s+s_2})\ldots(\boldsymbol{s+s_n})}$,

where $s_1 \neq s_2 \neq s_3 \ldots . \neq s_n$ $(all\ \ s_i\ different)$

Applying the partial fraction expansion,

G(s) is written: $\boldsymbol{G(s)} = \frac{\boldsymbol{K_{-s_1}}}{\boldsymbol{s+s_1}} + \frac{\boldsymbol{K_{-s_2}}}{\boldsymbol{s+s_2}} \ldots . + \frac{\boldsymbol{K_{-s_n}}}{\boldsymbol{s+s_n}}$

Determine Inverse Laplace Transform by using partial fraction expansion

First case: G(s) has simple poles(cont.)

The coefficients K_{-s_i} are determined by multiplying both sides of the equation by the factor $(s + s_i)$ and then by setting s equal to $-s_i$.

$$\boldsymbol{K_{-s_i} = \left[(s + s_i) \frac{Q(s)}{P(s)} \right] \Bigg|_{s=-s_i}}$$

Determine Inverse Laplace Transform by using partial fraction expansion

First case: G(s) has simple poles(cont.)

Example: $G(s) = \frac{s+3}{(s+1)(s+2)}$

In the partial expanded form, G(s) is written:

$$G(s) = \frac{s+3}{(s+1)(s+2)} = \frac{K_{-1}}{s+1} + \frac{K_{-2}}{s+2}$$

Determine Inverse Laplace Transform by using partial fraction expansion

First case: G(s) has simple poles(cont.)

$$K_{-1} = \left[\cancel{(s+1)} \frac{s+3}{\cancel{(s+1)}(s+2)} \right] \Bigg|_{s=-1} = \left[\frac{s+3}{(s+2)} \right] \Bigg|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$K_{-2} = \left[\cancel{(s+2)} \frac{s+3}{(s+1)\cancel{(s+2)}} \right] \Bigg|_{s=-2} = \left[\frac{s+3}{(s+1)} \right] \Bigg|_{s=-2} = \frac{-2+3}{-2+1} = -1$$

Therefore: $G(s) = \frac{2}{s+1} + \frac{-1}{s+2}$

Determine Inverse Laplace Transform by using partial fraction expansion

First case: G(s) has simple poles(cont.)

The inverse Laplace transform of the given Laplace transform can be obtained by using some equations in Table.

$$g(t) = \mathcal{L}^{-1}\left(\frac{2}{s+1} + \frac{-1}{s+2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= 2e^{-1t}u(t) - 1e^{-2t}u(t) = (2e^{-t} - e^{-2t})u(t)$$

$$= 2e^{-t} - e^{-2t}, t \geq 0$$

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles

If $\boldsymbol{r}$ poles of the $\boldsymbol{n}$ poles of G(s) are identical, the concerned pole $\boldsymbol{s = -s_i}$ is of multiplicity $\boldsymbol{r}$.

G(s) can be written as:

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\ldots(s+s_{n-r})(s+s_i)^r}$$

$$(i \neq 1,2,..,n-r)$$

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

Then G(s) can be expended as:

$$G(s) = \frac{K_{-s_1}}{s+s_1} + \frac{K_{-s_2}}{s+s_2} + \cdots + \frac{K_{-s_{n-r}}}{s+s_{n-r}} + \frac{A_1}{s+s_i} + \frac{A_2}{(s+s_i)^2} + \cdots + \frac{A_r}{(s+s_i)^r}$$

The (n-r) coefficients $K_{-s_1}, K_{-s_2}, \ldots, K_{-s_{n-r}}$ which correspond to simple poles may be evaluated as described in the first case.

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

The determination of the coefficients A_1, A_2, … A_r that correspond to the multiple order poles may be done as follows:

$$A_r = [(s+s_i)^r G(s)]\Big|_{s=-s_i}$$

$$A_{r-1} = \frac{1}{1!}\frac{d}{ds}[(s+s_i)^r G(s)]\Bigg|_{s=-s_i}$$

$$A_{r-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+s_i)^r G(s)]\Bigg|_{s=-s_i}$$

$$\vdots$$

$$A_1 = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}[(s+s_i)^r G(s)]\Bigg|_{s=-s_i}$$

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

Example: $\boldsymbol{G(s) = \frac{1}{s^2(s+1)}}$

In the partial expanded form, G(s) is written:

$$G(s) = \frac{1}{s^2(s+1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{K_{-1}}{s+1}$$

$$\boldsymbol{K_{-1}} = \left[\cancel{(s+1)}\frac{1}{s^2\cancel{(s+1)}}\right]\Bigg|_{\boldsymbol{s=-1}} = \left[\frac{1}{s^2}\right]\Bigg|_{\boldsymbol{s=-1}} = \frac{\mathbf{1}}{(-\mathbf{1})^{\mathbf{2}}} = \mathbf{1}$$

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

$$A_2 = s^2 * \frac{1}{s^2(s+1)}\bigg|_{s=0} = \frac{1}{(s+1)}\bigg|_{s=0} = 1$$

$$A_1 = \frac{1}{1!} * \frac{d}{ds} s^2 G(s) = \frac{1}{1!}\frac{d}{ds}\left[s^2 * \frac{1}{s^2(s+1)}\right]\bigg|_{s=0}$$

$$= \frac{1}{1}\frac{d}{ds}\left[\frac{1}{(s+1)}\right]\bigg|_{s=0} = 1 * \left[\frac{-1}{(s+1)^2}\right]\bigg|_{s=0} = -1$$

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

- **Therefore:**

$$G(s) = \frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

- The inverse Laplace transform of the given Laplace transform can be obtained by using equations in Laplace transform Table:

Determine Inverse Laplace Transform by using partial fraction expansion

Second case: G(s) has multiple poles(cont.)

$$g(t) = \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)$$

$$= -\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= -u(t) + tu(t) + e^{-t}u(t) = (-1 + t + e^{-t})u(t)$$

$$= -1 + t + e^{-t},\ t \geq 0$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles

G(s) can be written: $\boldsymbol{G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+p_1)(s^2+as+b)\cdots}}$**,** where

the order of Q(s) is less than the order of P(s), p_1 is real

and s^2+as+b has complex or pure imaginary roots.

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles (cont.)

G(s) can be expanded as:

$$G(s) = \frac{Q(s)}{(s+p_1)(s^2+as+b)\cdots} = \frac{K_1}{s+p_1} + \frac{K_2 s + K_3}{s^2+as+b} + \cdots$$

The complex or pure imaginary roots are expanded with K_2s+K_3 terms in the numerator rather than simply K_i as in the case of real roots.

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

Example: $\mathrm{G}(s) = \frac{3}{s(s^2+2s+5)}$; poles of G(s): s=0, and

$s^2 + 2s + 5 = 0$ because $\Delta = \mathbf{-16 = \boldsymbol{j}^2 16 \leq 0}$ **,** this shows that

G(s) contains complex poles.

Thus, it can be expand as : $\mathrm{G(s)} = \frac{3}{s(s^2+2s+5)} = \frac{K_0}{s} + \frac{K_1 s + K_2}{s^2+2s+5}$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

$$\Rightarrow \frac{3}{s(s^2+2s+5)} = \frac{K_0(s^2+2s+5)+s(K_1 s+K_2)}{s(s^2+2s+5)}$$

$$= \frac{(K_0+K_1)s^2+(2K_0+K_2)s+5K_0}{s(s^2+2s+5)}$$

$$\Rightarrow \begin{cases} K_0+K_1=0 \Longrightarrow K_1=-K_0=-\dfrac{3}{5} \\ 2K_0+K_2=0 \Longrightarrow K_2=-2K_0=-\dfrac{6}{5} \\ 5K_0=3 \Longrightarrow K_0=\dfrac{3}{5} \end{cases}$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

Thus:

$$\mathrm{G(s)} = \frac{3}{s(s^2+2s+5)} = \frac{3}{5} * \frac{1}{s} + \frac{-\frac{3}{5}s - \frac{6}{5}}{s^2+2s+5}$$

$$= \frac{3}{5} * \frac{1}{s} - \frac{3}{5}\left[\frac{s+2}{\boldsymbol{s^2+2s+1}+4}\right] = \frac{3}{5} * \frac{1}{s} - \frac{3}{5}\left[\frac{s+2}{(\boldsymbol{s+1})^{\boldsymbol{2}}+2^2}\right]$$

$$= \frac{3}{5} * \frac{1}{s} + \frac{3}{5}\left[\frac{s+1}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{3}{5} * \frac{1}{s} - \frac{3}{5}\left[\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right]$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

Thus:

$$G(s) = \frac{3}{5} * \frac{1}{s} - \frac{3}{5}\left[\frac{s+1}{(s+1)^2 + 2^2} + \frac{1}{2}\frac{2}{(s+1)^2 + 2^2}\right]$$

$$\boldsymbol{g(t) = \frac{3}{5}u(t) - \frac{3}{5}\left[e^{-t}\cos 2t\, u(t) + \frac{1}{2}e^{-t}\sin 2t\ u(t)\right]}$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

- ❖ The case of pure imaginary roots arises when a=0 and the calculation is the same.

- ❖ **Example:** $G(s) = \frac{3}{s(s^2+5)}$, poles are given by s=0, $s^2 + 5 = 0$

  Therefore poles are: s=0 which is a real pole, and s=j$\sqrt{5}$ and s=-j$\sqrt{5}$ which are pure imaginary conjugate pair of poles.

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

- Hence, it can be expand as :

$$\mathrm{G(s)} = \frac{3}{s(s^2+5)} = \frac{K_0}{s} + \frac{K_1 s + K_2}{s^2+5}$$

- Putting it on common denominator, we obtain:

$$\mathrm{G(s)} = \frac{3}{s(s^2+5)} = \frac{K_0(s^2+5) + s(K_1 s + K_2)}{s(s^2+5)}$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

- By equating both sides of expanded and non – expanded expression of G(s) , we obtain:

$$K_0(s^2+5)+s(K_1s+K_2)=3$$

$$K_0s^2+5K_0+K_1s^2+K_2s=3$$

$$(K_0+K_1)s^2+K_2s+5K_0=3$$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

- ❖ By identification we get:

$$\Rightarrow \begin{cases} K_0 + K_1 = 0 \\ K_2 = 0 \\ 5K_0 = 3 \implies K_0 = \dfrac{3}{5} \end{cases}$$

- ❖ From the above $K_1 = -K_0 = -\frac{3}{5}$

Determine Inverse Laplace Transform by using partial fraction expansion

Third case: G(s) has complex or pure imaginary poles(cont.)

- And finally :$G(s) = \frac{K_0}{s} + \frac{K_1 s + K_2}{s^2+5} = \frac{3}{5}\left(\frac{1}{s}\right) - \frac{3}{5}\left(\frac{s}{s^2+5}\right)$
- Utilizing Laplace transform table usual or common function

  to find $g(t)$ from $G(s)$, we get:

$$\boldsymbol{g(t) = \frac{3}{5}u(t) - \frac{3}{5}\cos\left(\sqrt{5}\,t\right)u(t)}$$

- That is : $\boldsymbol{g(t) = \frac{3}{5} - \frac{3}{5}\cos\left(\sqrt{5}\,t\right), t \geq 0}$

References

1. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education.
2. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons.
3. Gopal M (2008),Control Systems: Principles and Design, 3rd Edition, Tata McGraw-Hill.

THANK YOU