

# **Course: Automatic Control System Technology**

**Lecture 7:** Determine the transfer function model of a LTI control system

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# **Determine the transfer function model of a LTI control system**

## **Session objectives:**

**By the end of this session, students will be able to :**

- ❖ Define a transfer function of a LTI system
- ❖ Outline the transfer function properties
- ❖ Outline procedures to derive a transfer function of LTI systems
- ❖ Determine transfer functions of LTI systems-Examples

# Define a transfer function of a LTI control system

- ❖ The transfer function of a LTI system is defined as the Laplace transform of its impulse response, with all initial conditions set to zero.
- ❖ Let  $\mathbf{G}(s)$  denote the transfer function of a system with input  $\mathbf{u}(t)$ , output  $\mathbf{y}(t)$ , and impulse response  $\mathbf{g}(t)$ . The transfer function  $\mathbf{G}(s)$  is defined as :

$$\mathbf{G}(s) = \mathcal{L}[\mathbf{g}(t)]$$

$$\mathbf{G}(s) = \text{Laplace transform of } \mathbf{g}(t)$$

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems, 9<sup>th</sup> Edition, John Wiley & Sons, page 70.

# Define a transfer function of a LTI control system

- ❖ Additionally, the Lecture 2 defines the output  $\mathbf{y}(\mathbf{t})$  of a LTI system as the convolution of its impulse response  $\mathbf{g}(\mathbf{t})$  and the driving input  $\mathbf{u}(\mathbf{t})$ .
- ❖ The convolution of impulse response  $\mathbf{g}(\mathbf{t})$  and the driving input  $\mathbf{u}(\mathbf{t})$  is given by:  $\mathbf{y}(\mathbf{t}) = \mathbf{g}(\mathbf{t}) * \mathbf{u}(\mathbf{t}) = \int_0^t \mathbf{g}(\boldsymbol{\tau})\mathbf{u}(\mathbf{t} - \boldsymbol{\tau})d\boldsymbol{\tau}$ .

# Define a transfer function of a LTI control system

- ❖ And according to Laplace transform properties, the Laplace transform of a real convolution equals to multiplication in s-domain.

$$\mathcal{L}[y(t)] = \mathcal{L}[g(t) * u(t)] = \mathcal{L}\left[\int_0^t g(\tau)u(t - \tau)d\tau\right] = G(s)U(s)$$

- ❖ Which gives:  $Y(s) = G(s)U(s)$

Gopal M (2008), Control Systems: Principles and Design, 3<sup>rd</sup>

Edition, Tata McGraw-Hill, page 46.

# Define a transfer function of a LTI control system

❖ Therefore, the transfer function  $G(s)$  is related the Laplace transform of the input and the output through the following relation:

$$G(s) = \frac{Y(s)}{U(s)}$$

with all initial conditions set to zero, and  $Y(s)$  and  $U(s)$  respectively Laplace transforms of  $y(t)$  and  $u(t)$ .

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems, 9<sup>th</sup> Edition, John Wiley & Sons, page 70.

# Define a transfer function of a LTI control system

- ❖ Although, the Laplace transform of a LTI system is defined in terms of impulse response, in practice, the input-output relation of a linear time-invariant system with continuous time input is often described by a differential equation,
- ❖ So it is more convenient to derive the transfer function directly from the differential equation.

# Define a transfer function of a LTI control system

- ❖ Let us consider that the input-output relation of a linear time-invariant system is described by the below  $n^{th}$ -order differential equation with constant real coefficients.

$$\begin{aligned} & \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ &= b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

*The coefficients  $a_0, a_1, \dots, a_{n-1}$  and  $b_0, b_1, \dots, b_m$  are real constants.*

# Define a transfer function of a LTI control system

- ❖ To obtain the transfer function of the linear time invariant system represented by  $n^{\text{th}}$ -order differential equation in the previous slide,
- ❖ We take the Laplace transform of both sides of the equation assuming zero initial conditions.

$$\begin{aligned} & (s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0)Y(s) \\ &= (b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0)U(s) \end{aligned}$$

# Define a transfer function of a LTI control system

- ❖ The transfer function  $G(s)$  of the system whose input and output are  $u(t)$  and  $y(t)$  respectively, is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

With  $n > m$

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems, 9<sup>th</sup> Edition, John Wiley & Sons, page 70.

# Define a transfer function of a LTI control system

- ❖ Thus, the transfer function of a linear time-invariant system is the ratio of the Laplace transforms of its output and input variables, assuming zero initial conditions.

$$G(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} \Big|_{\text{initial conditions}=0} = \frac{Y(s)}{U(s)} \Big|_{\text{initial conditions}=0}$$

Gopal M (2008), Control Systems: Principles and Design, 3<sup>rd</sup> Edition, Tata McGraw-Hill, page 47.

# Define a transfer function of a LTI control system

- ❖ The impulse response  $y(t)$  is the output of the system when the driving input  $u(t)$  is a unit impulse signal.
- ❖ As per previous definition, the transfer function  $G(s)$  of a LTI system is  $\mathbf{G(s)} = \frac{\mathbf{Y(s)}}{\mathbf{U(s)}}$ .
- ❖ Referring to the Laplace transform table, the Laplace transform of a unit-impulse signal is 1, ie: ( $\mathbf{U(s)} = 1$ ), hence  $\mathbf{G(s)} = \mathbf{Y(s)}$ .
- ❖ Therefore, the transfer function  $\mathbf{G(s)}$  of a LTI system is the Laplace transform of its impulse response,  $\mathbf{Y(s)}$ .

# Outline the transfer function properties

- ❖ The below outlined are properties of a transfer function:
  - The transfer function  $G(s)$  is the Laplace transform of the unit impulse response  $g(t)$  of the system.
  - A transfer function of a system is its mathematical model. In that, it is an operational method of expressing the differential equation that relates the output variable to the input variable.

# Outline the transfer function properties

- ❖ The below outlined are properties of a transfer function (*cont.*):
  - All initial conditions of the system are set to zero while finding the Laplace Transform of a system.
  - The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. Thus, transfer functions of many physically different systems can be identical.

# Outline the transfer function properties

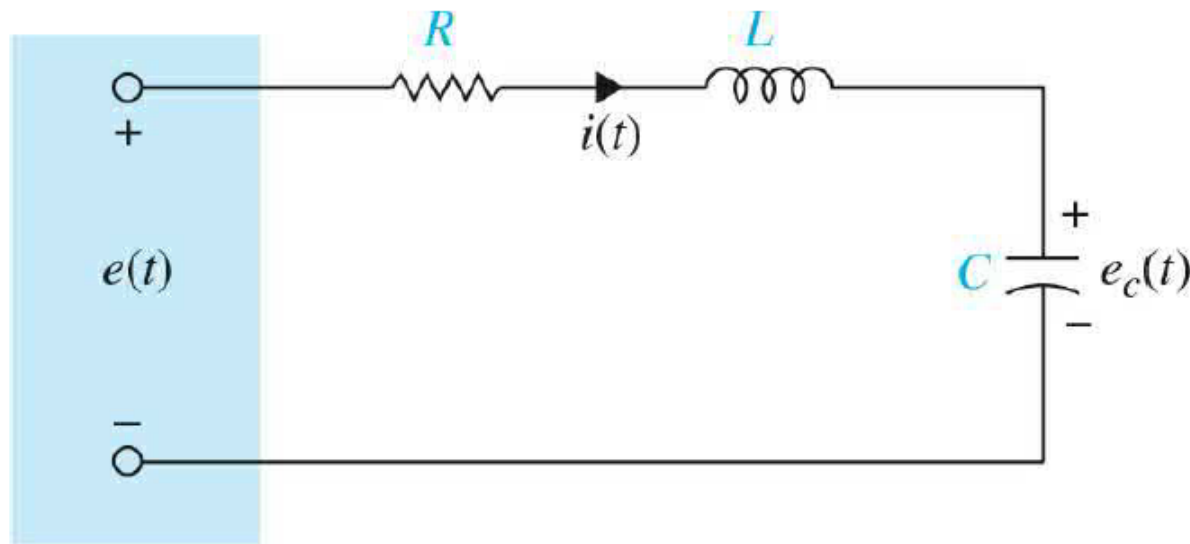
- ❖ The below outlined are properties of a transfer function (*cont.*):
  - If the transfer function of a system is known, the output or response can be studied for various forms of inputs to understand the nature of the system.
  - If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system.

# Outline procedures to derive the transfer function of LTI systems

- ❖ To derive the transfer function of a system, we use the following procedures:
  1. Develop the differential equation for the system by using the physical laws, e.g. Newton's laws, Kirchhoff's laws, etc.
  2. Take the Laplace transform of the differential equation under the zero initial conditions.
  3. Take the ratio of the output  $Y(s)$  to the input  $U(s)$ . This ratio is the transfer function.

# Determine transfer functions of LTI systems-Examples

**Example 1:** Consider the RLC circuit shown in the figure 1, where  $e(t)$  and  $e_c(t)$  are respectively input and output voltages of the circuit. Determine the transfer function of the circuit.



**Figure 1. RLC circuit**

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10<sup>th</sup> Edition, McGraw-Hill Education, page 104.

# Determine transfer functions of a LTI systems-Examples

- ❖ **Step 1:** Develop the differential equation of the RLC circuit:
- ✓ By applying Kirchhoff's voltage law to the RLC circuit shown in figure 1, we get:

$$e(t) = Ri(t) + L \frac{di(t)}{dt} + e_C(t) \text{ --- eq1}$$

$$i(t) = C \frac{de_C(t)}{dt} \text{ ---eq2}$$

# Determine transfer functions of LTI systems-Examples

- ✓ By substituting the current  $i(t)$  into equation eq1 and arranging the resultant equation from the highest derivative of the output, we get the differential equation of the RLC circuit:

$$LC \frac{d^2 e_c(t)}{dt^2} + RC \frac{de_c(t)}{dt} + e_c(t) = e(t) \text{ ---eq3}$$

- ❖ **Step 2:** Take the Laplace transform of both sides of this differential equation under the zero initial conditions.

# Determine transfer functions of LTI systems-Examples

✓ We get:

$$LCs^2 E_c(s) + RCs E_c(s) + E_c(s) = E(s)$$

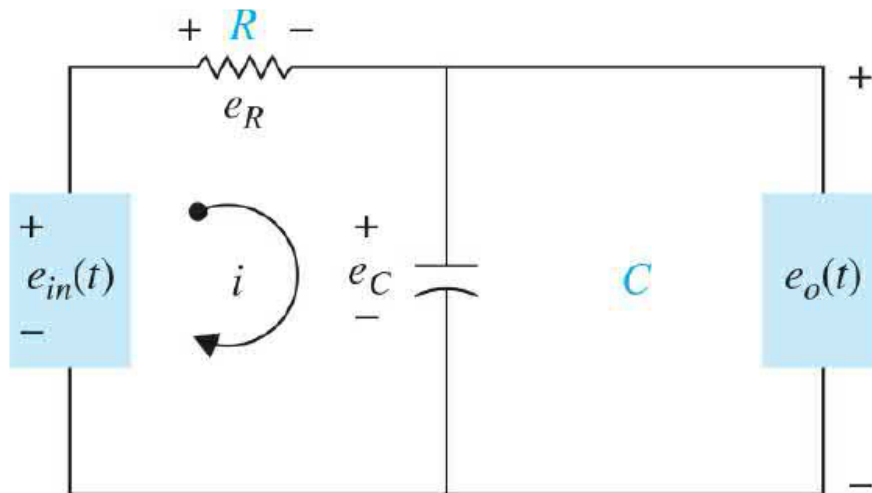
$$(LCs^2 + RCs + 1)E_c(s) = E(s)$$

❖ **Step 3:** Take the ratio of the output to the input to find the transfer function of the circuit:

$$G(s) = \frac{E_c(s)}{E(s)} = \frac{1}{LCs^2 + RCs + 1}$$

# Determine transfer functions of LTI systems-Examples

**Example 2:** Consider the RC circuit shown in figure 2, where  $e_{in}(t)$  and  $e_o(t)$  are respectively input and output voltages of the circuit. Determine the transfer function of the circuit.



**Figure 2. RC circuit**

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10<sup>th</sup> Edition, McGraw-Hill Education, page 106.

# Determine transfer functions of LTI systems-Examples

- ❖ **Step 1:** Develop the differential equation of the RC circuit:
- ✓ By applying Kirchhoff's voltage law to the RC circuit shown in figure 2, we get:

$$e_{in}(t) = Ri(t) + e_o(t) \text{ --- eq4}$$

$$i(t) = C \frac{de_o(t)}{dt} \text{ ---eq5}$$

# Determine transfer functions of LTI systems-Examples

- ✓ By substituting the current  $i(t)$  into equation eq4 and arranging the resultant equation from the highest derivative of the output, we get the following differential equation of the RC circuit:

$$RC \frac{de_o(t)}{dt} + e_o(t) = e_{in}(t) \text{--- eq5}$$

- ❖ **Step 2:** Take the Laplace transform of both sides of this differential equation under the zero initial conditions.

# Determine transfer functions of LTI systems-Examples

✓ We get:

$$RCsE_o(s) + E_o(s) = E_{in}(s)$$

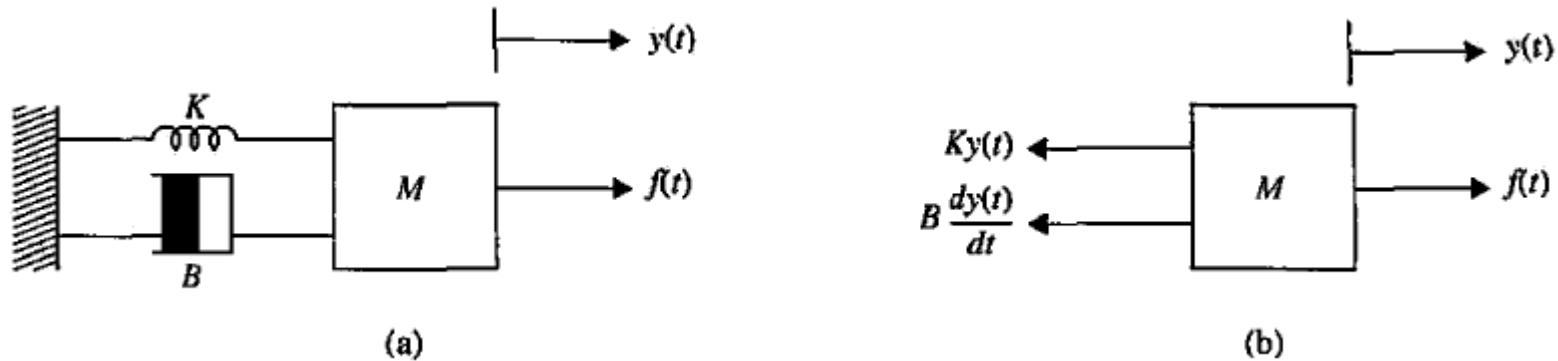
$$(RCs + 1)E_o(s) = E_{in}(s)$$

❖ **Step 3:** Take the ratio of the output to the input to find the transfer function of the circuit:

$$G(s) = \frac{E_o(s)}{E_{in}(s)} = \frac{1}{RCs + 1}$$

# Determine transfer functions of LTI systems-Examples

**Example 3:** Consider the mass spring friction system shown in figure 3 (a) and the corresponding free body diagram shown in figure 3 (b). Determine the transfer function of the system. Note that  $f(t)$  is input and  $y(t)$  output of the system.



**Figure 3. (a) Mass spring friction system, (a) Free body diagram**

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic

Control Systems, 9<sup>th</sup> Edition, John Wiley & Sons, page 151.

# Determine transfer functions of LTI systems-Examples

- ❖ **Step 1:** Develop the differential equation of the system
- ✓ By applying the Newton's second law of motion on mass  $M$ , we get:

$$f(t) - B \frac{dy(t)}{dt} - Ky(t) = M \frac{d^2y(t)}{dt^2}$$

- ✓ This is the required differential equation and it can be rearranged as follows:

$$M \frac{d^2y(t)}{dt^2} + B \frac{dy(t)}{dt} + Ky(t) = f(t)$$

# Determine transfer functions of LTI systems-Examples

- ❖ **Step 2:** Take the Laplace transform of both sides of this differential equation under the zero initial conditions, we get:

$$Ms^2Y(s) + BsY(s) + KY(s) = F(s)$$

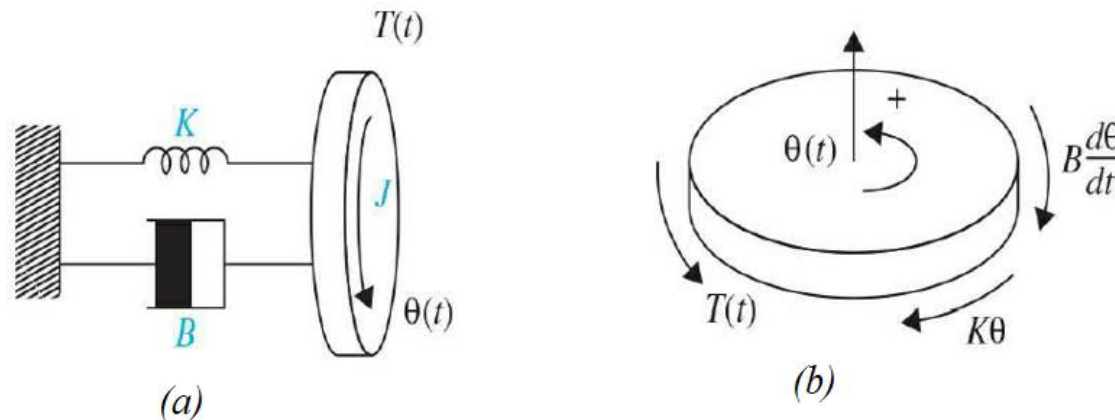
$$(Ms^2 + Bs + K)Y(s) = F(s)$$

- ❖ **Step 3:** Take the ratio of the output to the input to find the transfer function of the system:

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

# Determine transfer functions of LTI systems-Examples

**Example 4 :** Consider the torque torsional spring system (a) and its Free body diagram (b) shown in figure 4, where input is the applied torque  $T(t)$  and the output is the angular displacement  $\theta(t)$ , and determine the transfer function model of the system.



**Figure 4.(a) Torque torsional spring system, (b) Free body diagram**

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10<sup>th</sup> Edition, McGraw-Hill Education, page 87.

# Determine transfer functions of LTI systems-Examples

- ❖ **Step 1:** Develop the differential equation of the system
- ✓ By applying the Newton's second law of motion, we get:

$$T(t) - B \frac{d\theta(t)}{dt} - K\theta(t) = J \frac{d^2\theta(t)}{dt^2}$$

- ✓ Which is the required differential equation and it can be rearranged as follows:

$$J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} + K\theta(t) = T(t)$$

# Determine transfer functions of LTI systems-Examples

- ❖ **Step 2:** Take the Laplace transform of both sides of this differential equation under the zero initial conditions, we get:

$$Js^2\theta(s) + Bs\theta(s) + K\theta(s) = T(s)$$

$$(Js^2 + Bs + K)\theta(s) = T(s)$$

- ❖ **Step 3:** Take the ratio of the output to the input to find the transfer function of the system:

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

# References

1. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10<sup>th</sup> Edition, McGraw-Hill Education.
2. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems, 9<sup>th</sup> Edition, John Wiley & Sons.
3. Gopal M (2008), Control Systems: Principles and Design, 3<sup>rd</sup> Edition, Tata McGraw-Hill.

**THANK YOU**