Statistical Digital Signal Processing

Week 15 Revision of Major Topics

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Previous Topic (Week-14)

IIR Adaptive Filter

- IIR Adaptive Filtering: Introduction
- IIR Steepest Decent Adaptive Filter
- IIR LMS Adaptive Filter

Lecture Learning Outcomes

1. Review and explain fundamental concepts of signal modeling, including deterministic and stochastic signal modeling
2. Apply the Levinson–Durbin recursion algorithm and analyze lattice filter structures for efficient parameter estimation and signal prediction
3. Evaluate the principles of optimum filter design, including Wiener filtering and Kalman filtering concepts
4. Interpret and compare different spectrum estimation techniques, and determine their suitability for analyzing the frequency characteristics of signals
5. Integrate concepts of adaptive filtering with signal modeling, optimum filtering, and spectrum estimation to analyze and solve practical signal processing problems

Week 15: Revision of Major Topics

Outline

- Signal modeling
- Levinson-Durbin Recursion and Lattice Filters
- Optimum filters
- Spectrum estimation
- Adaptive filtering

Signal Modeling

- **Signal Modeling:** is an important problem encountered across various engineering and scientific applications
- It deals with creating concise and effective representations of signals

Broad classifications of Signal modeling

1. Non-Parametric Signal Modeling:

- ✓ a data-driven approach through learning of the signal characteristics directly from the available data.
- ✓ Practically, a sufficiently large number of recorded data is required to acceptably represent the system

Signal Modeling

2. Parametric Signal Modeling:

- ✓ a process of representing a signal with a mathematical model that is defined by a fixed number of parameters.
- ✓ parametric signal modeling is economical and powerful since it uses limited parameters to represent the signal
- ✓ In this course we focused on the parametric signal modeling due to the following motivations:

  - ❖ **Capability of Efficient Signal Transmission and Signal Storage**
  - ❖ **Signal Prediction and Estimation**

Parametric Signal Modeling Steps

Choosing the parametric form of the model

↓

Determining the model parameters

↓

Evaluating the model performance

Parametric Signal Modeling Using Filter

- Modeling the signal as an output of linear shift invariant filter that has a rational system function $(H(z))$:

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q(k) z^{-k}}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \qquad (1)$$

Estimation of the model for $x(n)$

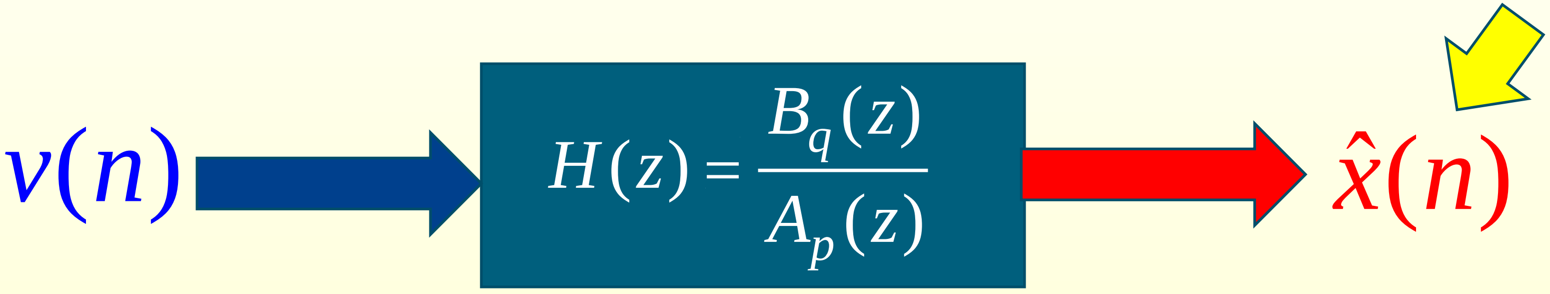

Figure 1: Modeling of a signal as an output of linear shift invariant filter

Deterministic Signal Modeling

- The signal to be modeled could be deterministic signal or random signal
- For the modeling of deterministic signal, the input of the filter is unit sample signal:

$$v(n) = \delta(n) \qquad (2)$$

Types of Deterministic Signal modeling

- Direct/Least Square Method
- Padé Approximation
- Prony Method
- Shank Approximation

- The objective the above deterministic signal modeling algorithm is to minimize the squared estimation error

Deterministic Signal Modeling: Direct Method

- Requires solving of p+q+1 non linear equations

$$\frac{\partial \varepsilon_{LS}}{\partial a_p^*(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[X(e^{j\omega}) - \frac{B_q(e^{j\omega})}{A_p(e^{j\omega})}\right]\frac{B_q^*(e^{j\omega})}{[A_p^*(e^{j\omega})]^2}e^{jk\omega}d\omega = 0 \qquad for\ k = 1,2,\ldots,p \tag{3}$$

$$\frac{\partial \varepsilon_{LS}}{\partial b_q^*(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[X(e^{j\omega}) - \frac{B_q(e^{j\omega})}{A_p(e^{j\omega})}\right]\frac{e^{jk\omega}}{A_p^*(e^{j\omega})}d\omega = 0 \qquad for\ k = 0,1,2,\ldots,q \tag{4}$$

Weakness of Direct Method

- High computational complexity
- Not applicable for real time and delay sensitive applications

Deterministic Signal Modeling: Padé Approximation

- Requires solving of p+q+1 linear equations

$$\begin{pmatrix} x(q) & x(q-1) & \cdots & x(q-p+1) \\ x(q+1) & x(q) & \cdots & x(q-p+2) \\ \vdots & \vdots & \cdots & \vdots \\ x(q+p-1) & x(q+p) & \cdots & x(q) \end{pmatrix} \begin{pmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{pmatrix} = - \begin{pmatrix} x(q+1) \\ x(q+2) \\ \vdots \\ x(q+p) \end{pmatrix} \qquad (5)$$

$$\begin{pmatrix} x(0) & 0 & & \cdots & 0 \\ x(1) & x(0) & 0 & \cdots & 0 \\ x(2) & x(1) & x(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x(q) & x(q-1) & x(q-2) & \cdots & x(q-p) \end{pmatrix} \begin{pmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{pmatrix} = \begin{pmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{pmatrix} \qquad (6)$$

Deterministic Signal Modeling: Padé Approximation

- Produce an exact fit for data in the **[0,q+p]** interval

Weakness of Padé Approximation

- Does not give guarantee of good modeling estimation for data values of $n>q+p$
- The model generated is not stable

Deterministic Signal Modeling: Prony's Method

- Requires solving of p+q+1 linear equations

$$\begin{bmatrix} r_x(1,1) & r_x(1,2) & r_x(1,3)\cdots & r_x(1,p) \\ r_x(2,1) & r_x(2,2) & r_x(2,3)\cdots & r_x(2,p) \\ r_x(3,1) & r_x(3,2) & r_x(3,3)\cdots & r_x(3,p) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p,1) & r_x(p,2) & r_x(p,3)\cdots & r_x(p,p) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ a_p(3) \\ \vdots \\ a_p(p) \end{bmatrix} = - \begin{bmatrix} r_x(1,0) \\ r_x(2,0) \\ r_x(3,0) \\ \vdots \\ r_x(p,0) \end{bmatrix} \quad (7)$$

Where: $r_x(k,l) = \sum_{n=q+1}^{\infty} x(n-l)x^*(n-k)$

$$\begin{pmatrix} x(0) & 0 & & \cdots & 0 \\ x(1) & x(0) & 0 & \cdots & 0 \\ x(2) & x(1) & x(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x(q) & x(q-1) & x(q-2) & \cdots & x(0) \end{pmatrix} \begin{pmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{pmatrix} = \begin{pmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{pmatrix} \quad (8)$$

Deterministic Signal Modeling: Prony's Method

Minimum Modeling Error

$$\varepsilon_{p,q} = r_x(0,0) + \sum_{k=1}^{p} a_p(k) r_x(0,k) \qquad (9)$$

- Prony method minimize the modeling error over the entire available data

Weakness of Prony's Method

- ❖ The Prony's minimum modeling error deviates from the true modeling error
- ❖ The true modeling error is grater than the Prony's minimum modeling error

Deterministic Signal Modeling: Shank's Method

- Enhanced version of Prony's method and use Prony's method to find the filter denominator coefficient

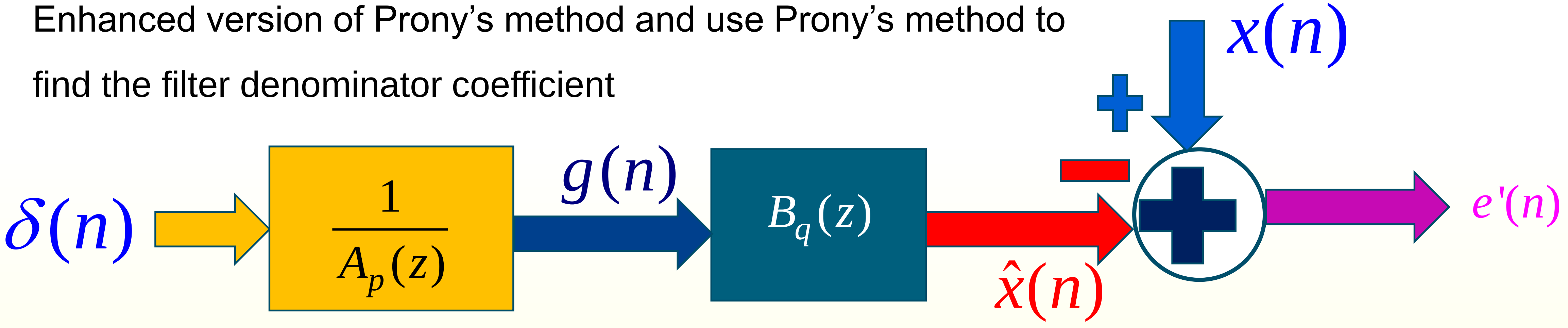

Figure 2: Shank method illustration

- It requires to the following equation to find filters nominator coefficient:

$$\begin{bmatrix} r_g(0,0) & r_g(0,1) & r_g(0,2)\cdots & r_g(0,q) \\ r_g(1,0) & r_g(1,1) & r_g(1,2)\cdots & r_g(1,q) \\ r_g(2,0) & r_g(2,1) & r_g(2,2)\cdots & r_g(2,q) \\ \vdots & \vdots & \vdots & \vdots \\ r_g(q,0) & r_g(q,1) & r_g(q,2)\cdots & r_g(q,q) \end{bmatrix} \begin{bmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{bmatrix} = \begin{bmatrix} r_{xg}(0) \\ r_{xg}(1) \\ r_{xg}(2) \\ \vdots \\ r_{xg}(q) \end{bmatrix} \quad (10)$$

Deterministic Signal Modeling: Shank's Method

Minimum Modeling Error

$$\{\varepsilon_S\}_{\min} = r_x(0) - \sum_{k=0}^{q} b_q(k) r_{xg}^*(k) \qquad (11)$$

- Shank is better than Prony method in minimizing the modeling error

Weakness of Shank's Method

- Shank's method introduce extra computation of sequence $g(n)$, autocorrection of $g(n)$, cross correlation of $x(n)$ and $g(n)$

Stochastic Signal Modeling

- Linear modeling of stationary random signal $(x(n))$ using rational filter $H(z)$ using autocorrelation matching method

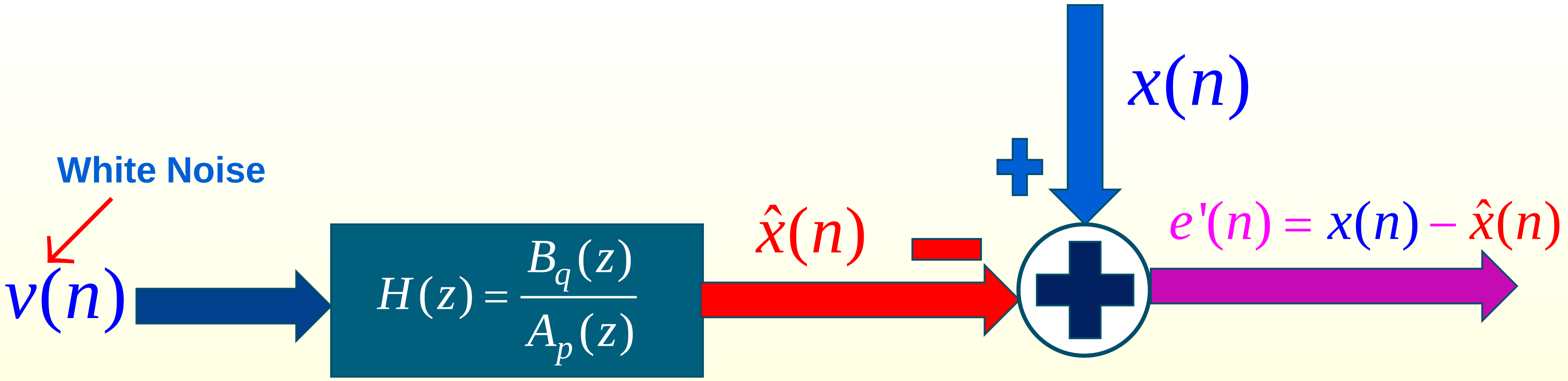

Figure 3: Illustration of Linear Modeling of Stationary Random Signal

Stochastic Signal Modeling: ARMA Model

- A wide sense stationary ARMA (p,q) process, $x(n)$, having p poles and q zeros can be modeled using linear shift invariant filter, $H(z)$, with a white noise input $v(n)$

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q(k) z^{-k}}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \qquad (12)$$

- Requires solving of the following **Yule-Walker equations:**

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = c_q(k) \qquad (13)$$

Where:

$$c_q(k) = \begin{cases} \sum_{l=0}^{q} b_q(l)\, h^*(l-k)\ , \text{ for } 0 \le k \le q \\ 0\ , \text{ for } k > q \end{cases} \qquad (14)$$

Stochastic Signal Modeling: AR Model

- An order p wide-sense stationary autoregressive process is a special case of ARMA(p,q) when q=0.

$$H(z) = \frac{B_0(z)}{A_p(z)} = \frac{b_0(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \qquad (15)$$

- Requires solving of the following equations to find the model parameters

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = |b_0(0)|^2\, \delta(k)\ ; \ \text{for } k \geq 0 \qquad (16)$$

- And

$$|b_0(0)|^2 = r_x(0) + \sum_{l=1}^{p} a_p(l)\, r_x(-l) = r_x(0) + \sum_{l=1}^{p} a_p(l)\, r_x^*(l) \qquad (17)$$

Levinson-Durbin Recursion and Lattice Filters

- **Levinson Algorithm:** A fast recursive method for solving the normal equations in signal modeling (e.g., Prony's all-pole method
- In 1961, Durbin improved the Levison recursion for special case in which the right hand side of Toeplitz equation is unit vector
- For example let's consider all-pole modeling using Prony's method, we have

$$\mathrm{R}_p \mathrm{a}_p = \varepsilon_p \mathrm{u}_1 \qquad (18)$$

Where:

$\mathrm{R}_p \rightarrow (p+1)\times(p+1)$ Hermitian Toeplitz matrix

$\mathrm{u}_1 \rightarrow$ Unit vector with 1 in the first position

- Levinson-Durbin Recursion algorithm is used to solve eq (18) through recursive method in the model order

Levinson-Durbin Recursion and Lattice Filters

- The steps of the Levinson-Durbin recursion Algorithm are as follows :

1. Initialize the recursion

(a). $a_0(0) = 1$

(b). $\varepsilon_0(0) = r_x(0)$

2. For $j = 0, 1, \ldots, p-1$

(a). $\gamma_j = r_x(j+1) + \sum_{i=1}^{j} a_j(i) r_x(j-i+1)$

(b). $\Gamma_{j+1} = -\gamma_j / \varepsilon_j^*$

(c). **For** $i = 1, 2, \ldots, j$

$$a_{j+1}(i) = a_j(i) + \Gamma_{j+1} a_j^*(j-i+1)$$

(d). $a_{j+1}(j+1) = \Gamma_{j+1}$

(e). $\varepsilon_{j+1} = \varepsilon_j \left[1 - |\Gamma_{j+1}|^2 \right]$

3. $b(0) = \sqrt{\varepsilon_{j+1}}$

Lattice Filters

- One of the important application of Levinson-Durbin recursion algorithm is the development of Lattice structure in digital filters
- Levinson order-update equation is used to drive the lattice filter structure for FIR digital filters
- Set of coupled difference equations which describe two port network

$$a_{j+1}(n) = a_j(n) + \Gamma_{j+1} a_j^R(n-1)$$
$$a_{j+1}^R(n) = a_j^R(n-1) + \Gamma_{j+1}^* a_j(n) \qquad (19)$$

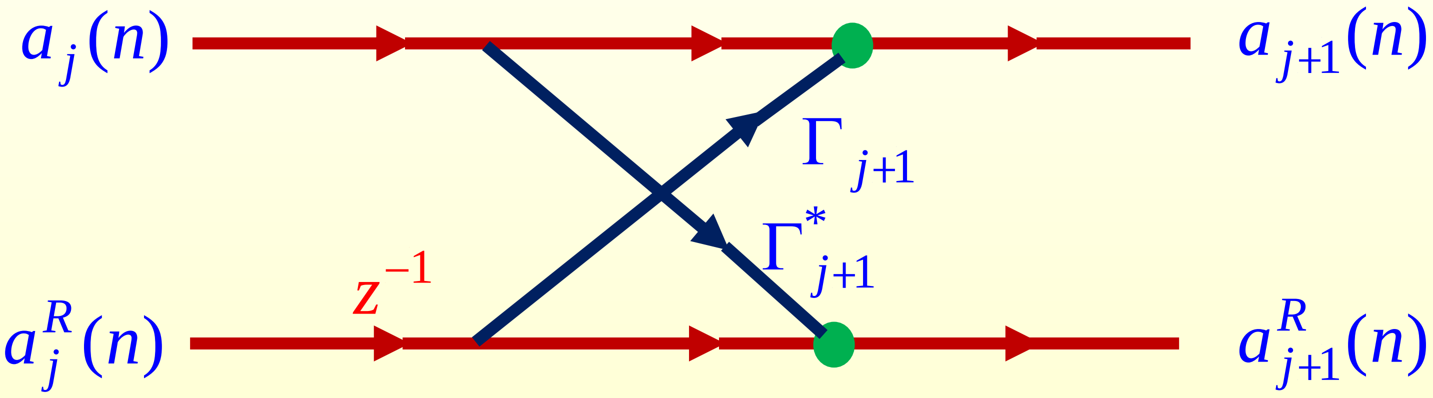

Figure 4: two port network

Optimum Filters

- In most practical applications such as radar communication, wireless communication etc., the desired signals are not observed directly
- In most practical cases classical filters are not optimum in producing the best estimate of the desired signal from its noisy observations
- To solve such problems, Digital Optimum Filters such as Wiener Filter and Discrete Kalman Filter become very important

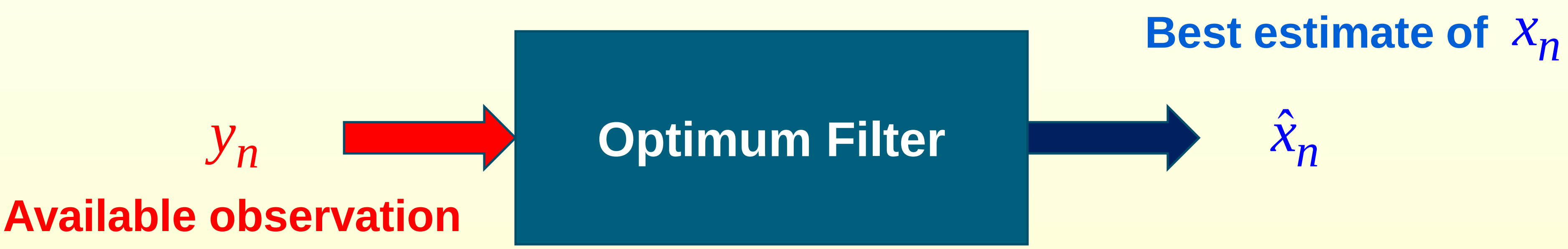

Figure 5: Linear Optimum Filtering

Optimum Filters: Wiener Filter

- The Wiener filtering problem, is to design a filter to recover a signal $d(n)$ from noisy measurement $x(n)$ given by:

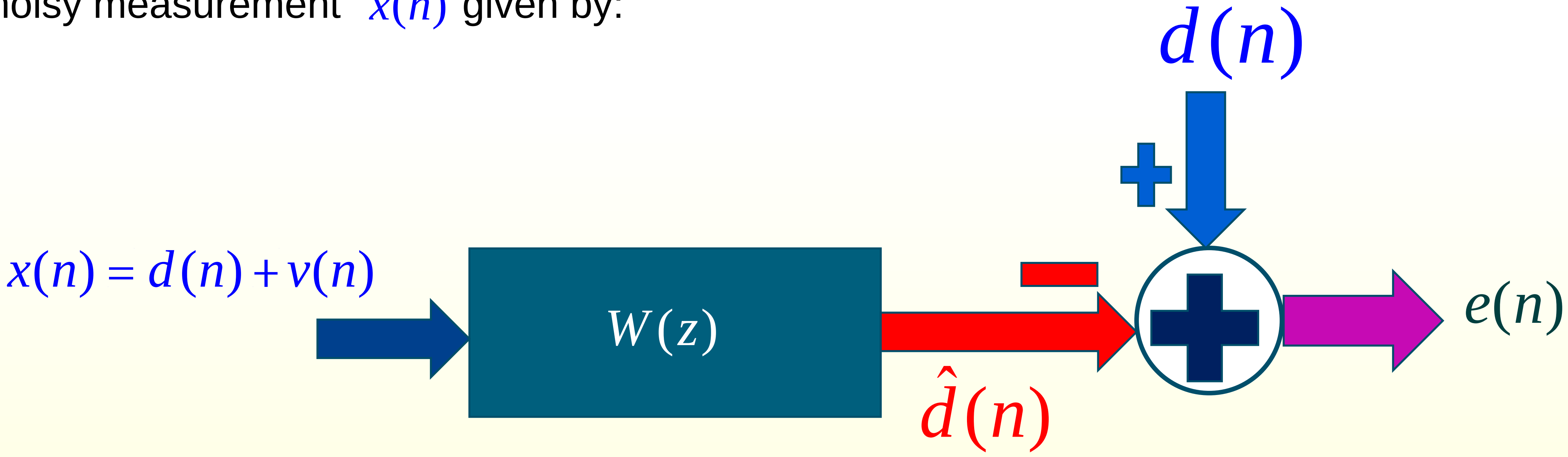

Figure 6: Illustration of Wiener Filtering

- Thus, the problem is to find the linear shift invariant (LSI) filter that minimizes the mean square error ξ

$$\xi = E\ |e(n)|^2 \qquad (20)$$

Optimum Filters: FIR Wiener Filter

- The FIR Wiener filtering problem, it is required to solve the following ***Wiener-Hopf Equations***

$$\sum_{l=0}^{p-1} w(l) r_x(k-l) = r_{dx}(k) \; ; \; k = 0,1,\ldots, p-1 \qquad (21)$$

- In matrix form:

$$\underbrace{\begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p-1) \\ r_x(1) & r_x(0) & \cdots & r_x^*(p-2) \\ r_x(2) & r_x(1) & \cdots & r_x^*(p-3) \\ \vdots & \vdots & & \vdots \\ r_x(p-1) & r_x(p-2) & \cdots & r_x(0) \end{bmatrix}}_{\mathbf{R}_x} \underbrace{\begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ \vdots \\ w(p-1) \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \\ \vdots \\ r_{dx}(p-1) \end{bmatrix}}_{\mathbf{r}_{dx}} \qquad (22)$$

Optimum Filters: FIR Wiener Filter

- In compact Form, we have

$$\mathbf{R}_x \mathbf{w} = \mathbf{r}_{dx} \qquad (23)$$

- The minimum mean square error $\xi_{\min}$ is given by:

$$\xi_{\min} = r_d(0) - \sum_{l=0}^{p-1} w(l) r_{dx}^{*}(l) \qquad (24)$$

- Different applications can be casted to the Wiener filtering problems:
  - Linear Prediction
    - Single Step Linear Prediction
    - Multi-Step Linear Prediction
  - Noise Cancelation

Optimum Filters: IIR Wiener Filter

- We have infinite filter coefficients to be determined in the IIR Wiener filter
- In many applications, an IIR (recursive) filter is preferred over an FIR filter for performing linear operations since the design of IIR filter has fewer parameters [1]
- Two classes of IIR Winer filter:
  - **Non-Causal IIR Wiener Filters**
  - **Causal IIR Wiener Filters**
- For **non-causal or unconstrained IIR filter**, the goal is to find the unit sample response, $h(n)$, of the IIR filter:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \qquad (25)$$

- The system function, $H(z)$, of the non-causal IIR wiener filter is:

$$H(z) = \frac{P_{dx}(z)}{P_x(z)} \qquad (26)$$

Optimum Filters: IIR Wiener Filter

- For non-causal or unconstrained IIR filter, the minimum mean square estimation error is given by:

$$\xi_{\min} = \frac{1}{2\pi j}\oint_C \left[P_d(z) - H(z)P_{dx}^*(1/z^*)\right]z^{-1}dz \qquad (27)$$

- In case of **causal IIR Wiener filter**, we have seen 3 cascaded filters

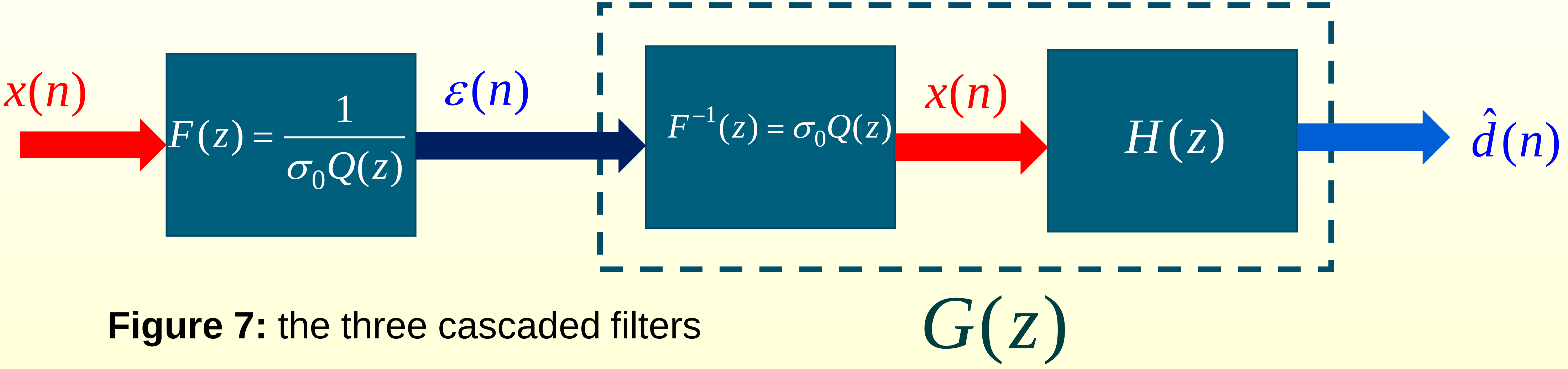

Figure 7: the three cascaded filters

Where: $P_x(z) = \sigma_0^2 Q(z) Q^*(1/z^*)$ (28)

Optimum Filters: IIR Wiener Filter

- The system function of causal IIR Wiener filter is given by:

$$H(z) = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{P_{dx}(z)}{Q^*(1/z^*)} \right]_+ \qquad (29)$$

- The minimum mean square estimation error:

$$\xi_{\min} = \frac{1}{2\pi j} \oint_C \left[P_d(z) - H(z) P_{dx}^*(1/z^*) \right] z^{-1} dz \qquad (30)$$

- **Casual Linear Prediction** is also another application of causal IIR Wiener filter and the causal linear predictor is given by:

$$H(z) = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{z P_x(z)}{Q^*(1/z^*)} \right]_+ = \frac{1}{Q(z)} \left[z Q(z) \right]_+ = z \left[1 - \frac{1}{Q(z)} \right] \qquad (31)$$

- The minimum mean square prediction error:

$$\xi_{\min} = \frac{1}{2\pi j} \oint_C P_x(z) \left[1 - H(z) z^{-1} \right] z^{-1} dz \qquad (32)$$

Optimum Filters: Discrete Kalman Filter

- **Discrete Kalman** Filter is the appropriate filter type for non-stationary signal estimation
- The derivation of discrete Kalman filter is summarized as follows:

$$\textbf{State Equation} \rightarrow \mathbf{x}(n) = \mathbf{A}(n-1)\mathbf{x}(n-1) + \mathbf{w}(n)$$

$$\textbf{Observation Equation} \rightarrow \mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}(n)$$

$$\textbf{Initialization} \rightarrow \hat{\mathbf{x}}(0/0) = E\ \ \mathbf{x}(0/0)$$

$$\mathbf{P}(0/0) = E\ \ \mathbf{x}(0)\mathbf{x}^{H}(0)$$

$$\textbf{Computation} \rightarrow \textit{for } n = 1, 2, \ldots \ \textit{compute}$$

$$\mathbf{x}(n/n-1) = \mathbf{A}(n-1)\hat{\mathbf{x}}(n-1/n-1)$$

$$\mathbf{P}(n/n-1) = \mathbf{A}(n-1)\mathbf{P}(n-1/n-1)\mathbf{A}^{H}(n-1) + \mathbf{Q}_{w}(n)$$

$$\mathbf{K}(n) = \mathbf{P}(n/n-1)\mathbf{C}^{H}(n)\left[\mathbf{C}(n)\mathbf{P}(n/n-1)\mathbf{C}^{H}(n) + \mathbf{Q}_{v}(n)\right]^{-1}$$

$$\hat{\mathbf{x}}(n/n) = \hat{\mathbf{x}}(n/n-1) + \mathbf{K}(n)\ \ \mathbf{y}(n) - \mathbf{C}(n)\hat{\mathbf{x}}(n/n-1)$$

$$\mathbf{P}(n/n) = \ \ \mathbf{I} - \mathbf{K}(n)\mathbf{C}(n)\ \ \mathbf{P}(n/n-1)$$

Spectrum Estimation

- In general, there are **two broad categories** of spectrum estimation techniques;

I. Classical or Nonparametric Methods

- Based on estimating the autocorrelation sequence from a given set of data and;
- Estimating the power spectrum by taking the Fourier transform of the estimated autocorrelation sequence

II. Nonclassical or Parametric Methods

- Based on using a model for the process to estimate the power spectrum
- For example, suppose $x(n)$ is pth order AR process, measured value of $x(n)$ can be used to estimate the model parameter $a_p(k)$
- Then the power spectrum can be estimated from the model

Non Parametric Spectrum Estimation: Periodogram Method

- The periodogram is given by:

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\left|X_N(e^{j\omega})\right|^2 \qquad (33)$$

Where:

$$X_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_N(n)e^{-jn\omega} = \sum_{n=0}^{N-1} x(n)e^{-jn\omega} \qquad (34)$$

$$x_N(n) = w_R(n)x(n) = \begin{cases} x(n) \;\; ; \;\; 0 \le n < N \\ 0 \;\; ; \;\; Otherwise \end{cases} \qquad (36)$$

Rectangular window

$$w_R(n) = \begin{cases} 1 \;\; ; \;\; 0 \le n < N \\ 0 \;\; ; \;\; Otherwise \end{cases} \qquad (37)$$

Non Parametric Spectrum Estimation: Periodogram Method

- The Performance of the periodogram is valuated by its bias and variance
- Periodogram Bias and Variance:

  - **Asymptotical Unbiased** ➡

$$\lim_{N\to\infty} E\;\hat{P}_{per}(e^{j\omega}) = P_x(e^{j\omega}) \qquad (38)$$

  - **Non Consistent Estimate** ➡

$$\lim_{N\to\infty} \mathbf{Var}\;\hat{P}_{per}(e^{j\omega}) \approx P_x^2(e^{j\omega}) \qquad (39)$$

- **Resolution** of Periodogram:

$$\mathbf{Res}\{\hat{P}_{per}(e^{j\omega})\} = \Delta\omega = 0.89\frac{2\pi}{N} \qquad (40)$$

- **Spectral Masking**: power leakage through the side lobs creates confusion that as if the signal posses spectral component at $\omega_0 \pm \frac{2\pi}{N}k$ frequencies

Non Parametric Estimation: Modified Periodogram

- The periodogram of a signal after applying a general window $w(n)$ on the process is referred to as the **Modified Periodogram**
- The modified periodogram is given by:

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w(n)e^{-jn\omega}\right|^2 \qquad (41)$$

Where:

$$U = \frac{1}{N}\sum_{n=0}^{N-1}|w(n)|^2 \qquad (42)$$

- the modified periodogram provides a trade of between::
  - Spectral Resolution or Main Lobe Width and,
  - Spectral Masking or Side Lobe Amplitude

Non Parametric Estimation: Modified Periodogram

- The bias and variance of Modified Periodogram

- **Asymptotical Unbiased** ➡ $$\lim_{N\to\infty} E\ \hat{P}_M(e^{j\omega}) = P_x(e^{j\omega}) \qquad (43)$$

- **Non Consistent Estimate** ➡ $$\lim_{N\to\infty} \mathbf{Var}\ \hat{P}_M(e^{j\omega}) \approx P_x^2(e^{j\omega}) \qquad (44)$$

Table: Spectral characteristics of different windows [2]

Window Type	Side Lob Amplitude (Db)	3dB Bandwidth $\rightarrow (\Delta\omega)_{3dB}$
Rectangular	-13	$0.89(2\pi/N)$
Bartlett	-27	$1.28(2\pi/N)$
Hanning	-32	$1.44(2\pi/N)$
Hamming	-43	$1.30(2\pi/N)$
Blackman	-58	$1.68(2\pi/N)$

Non Parametric Estimation: Bartlett's method

- A method using Periodogram averaging and the periodograms are computed partitioning the signal data in to non-overlapping sequences
- The Bartlett estimate is given by

$$\hat{P}_B(e^{j\omega}) = \frac{1}{K}\sum_{i=0}^{K-1}\frac{1}{L}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 \qquad (45)$$

- The bias and variance of Bartlett estimate

❖ **Asymptotical Unbiased** ➡

$$\lim_{L\to\infty} E\ \hat{P}_B(e^{j\omega}) = P_x(e^{j\omega}) \qquad (46)$$

❖ **Consistent Estimate** ➡

$$\lim_{L,K\to\infty} \mathbf{Var}\ \hat{P}_M(e^{j\omega}) \approx \lim_{L,K\to\infty}\frac{1}{K}P_x^2(e^{j\omega}) \approx 0 \qquad (47)$$

Non Parametric Estimation: Bartlett's method

- A method using Periodogram averaging and the periodograms are computed by partitioning the signal data in to non-overlapping sequences
- The Bartlett estimate is given by

$$\hat{P}_B(e^{j\omega}) = \frac{1}{K}\sum_{i=0}^{K-1}\frac{1}{L}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-jn\omega}\right|^2 \quad (48)$$

- The bias and variance of Bartlett estimate

❖ **Asymptotical Unbiased**

$$\lim_{L\to\infty} E\left\{\hat{P}_B(e^{j\omega})\right\} = P_x(e^{j\omega}) \quad (49)$$

❖ **Consistent Estimate**

$$\lim_{L,K\to\infty} \mathbf{Var}\left\{\hat{P}_M(e^{j\omega})\right\} \approx \lim_{L,K\to\infty}\frac{1}{K}P_x^2(e^{j\omega}) \approx 0 \quad (50)$$

Non Parametric Estimation: Welch method

- A method using Modified Periodogram averaging and the modified periodograms are computed by partitioning the signal data in to overlapping sequences
- The Welch estimate is given by

$$\hat{P}_W^{(i)}(e^{j\omega}) = \frac{1}{LUK}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega}\right|^2 \qquad (51)$$

- The bias and variance (considering 50% overlap) of Welch estimate

❖ **Asymptotical Unbiased**

$$\lim_{L\to\infty} E\left\{\hat{P}_W(e^{j\omega})\right\} = P_x(e^{j\omega}) \qquad (52)$$

❖ **Consistent Estimate**

$$\lim_{L,K\to\infty} \mathbf{Var}\left\{\hat{P}_W(e^{j\omega})\right\} \approx \lim_{L,K\to\infty} \frac{9}{8K} P_x^2(e^{j\omega}) \approx 0 \qquad (53)$$

Non Parametric Estimation: Blackman Tukey

- Blackman Tukey method is based on windowing the autocorrelation estimates
- The Blackman Tukey estimate is given by

$$\hat{P}_{BT}(e^{j\omega}) = \sum_{k=-M}^{M} \hat{r}_x(k)w(k)e^{-jk\omega} \qquad (54)$$

- The bias of Blackman Tukey estimate

$$E\left\{\hat{P}_{BT}(e^{j\omega})\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{ju})\, W(e^{j(\omega-u)})\, du \qquad (55)$$

- The variance of Blackman Tukey estimate

$$Var\left\{\hat{P}_{BT}(e^{j\omega})\right\} \approx \frac{1}{2\pi N} P_x^2(e^{j\omega}) \int_{-\pi}^{\pi} W^2(e^{j(\omega-u)})\, du \qquad (56)$$

Parametric Spectrum Estimation

- Incorporate the process model into spectrum estimation to:
  - Improve accuracy
  - Provide higher-resolution
- Autoregressive Spectrum Estimation:

$$\hat{P}_{AR}(e^{j\omega}) = \frac{|\hat{b}(0)|^2}{\left|1+\sum_{k=1}^{p}\hat{a}_p(k)e^{-jk\omega}\right|^2} \quad (57)$$

- Moving Average Spectrum Estimation:

$$\hat{P}_{MA}(e^{j\omega}) = \left|\sum_{k=0}^{q}\hat{b}_q(k)e^{-jk\omega}\right|^2 \quad (58)$$

- Autoregressive Moving Average Spectrum Estimation:

$$\hat{P}_{ARMA}(e^{j\omega}) = \frac{\left|\sum_{k=0}^{q}\hat{b}_q(k)e^{-jk\omega}\right|^2}{\left|1+\sum_{k=1}^{p}\hat{a}_p(k)e^{-jk\omega}\right|^2} \quad (59)$$

Adaptive Filtering

- For the case of non stationary signal, Wiener filter is not appropriate and Adaptive filter becomes necessary

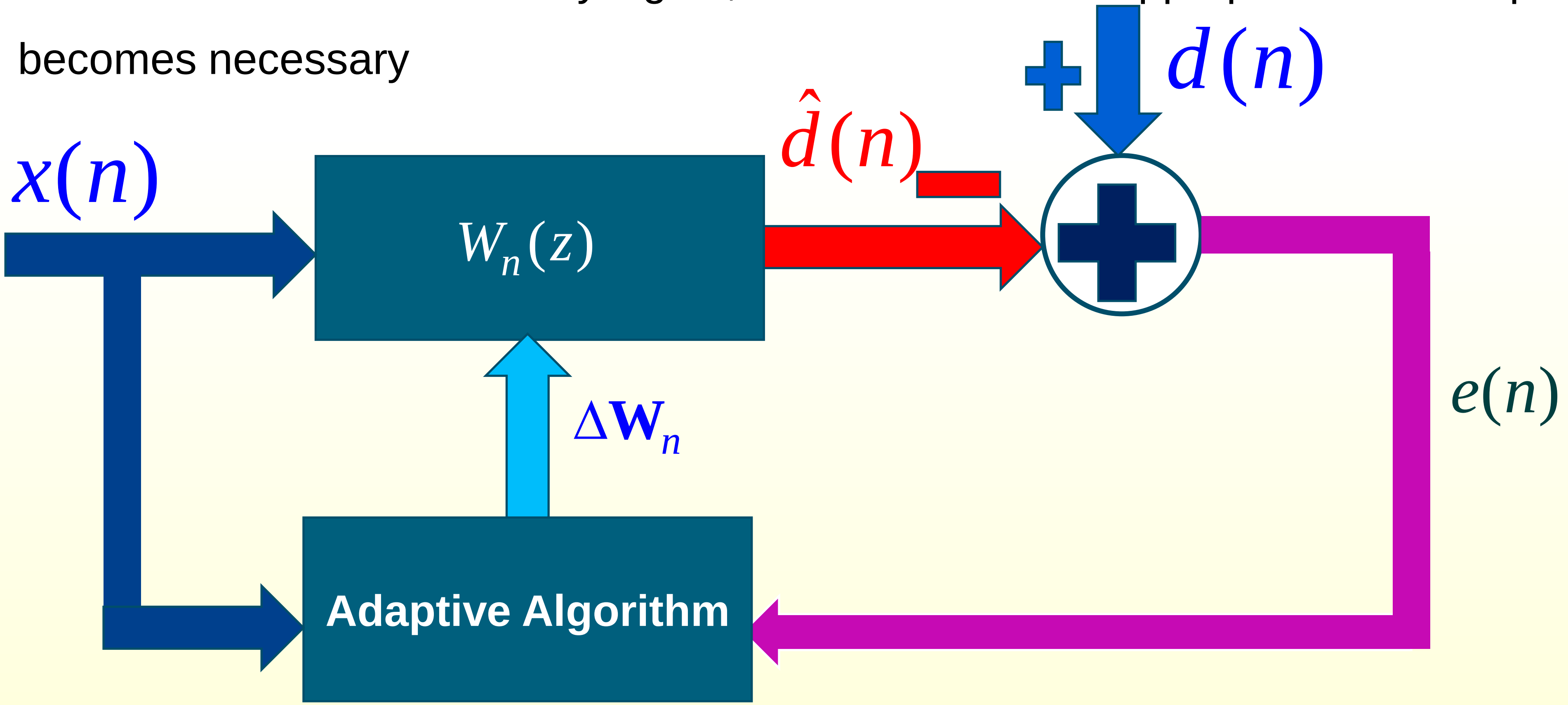

Figure 8: Illustration of adaptive filtering

FIR Adaptive Filter

- General weight update equation of adaptive filter

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \Delta\mathbf{w}_n \qquad (60)$$

- FIR adaptive filter is non-recursive and feed-forward type of adaptive filter
- In FIR adaptive filtering, Steepest decent and LMS algorithm are widely used to search the optimal solution
- The weight update equation of the steepest decent algorithm:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu E\{e(n)\mathbf{x}^*(n)\} \qquad (61)$$

- The weight update equation of LMS algorithm:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e(n)\mathbf{x}^*(n) \qquad (62)$$

IIR Adaptive Filter

- Recursive type of adaptive filter
- The weight update equation of the steepest decent algorithm:

$$\mathbf{\Theta}_{n+1} = \mathbf{\Theta}_n + \mu E \; e(n) \nabla y^*(n) \qquad (63)$$

- The weight update equation of LMS algorithm:

$$\mathbf{\Theta}_{n+1} = \mathbf{\Theta}_n + \mu e(n) \nabla y^*(n) \qquad (64)$$

Summary

- Review of Signal modeling
- Review of Levinson-Durbin Recursion and Lattice Filters
- Review of Optimum filters
- Review of Spectrum estimation
- Review of Adaptive filtering

References

[1] Charles W. Therrien,‘’ Discrete Random Signals and Statistical Signal Processing’’, Prentice Hall, Pp.356, 1992.

[2] Monson H. Hayes, *“Statistical Digital Signal Processing and Modeling”,* John Wiley and sons, Pp.411, 1996.

Contents Here

Thank You!