1.5. Common Systems of Linear Equations

– *Kroneker-Capelli's Theorem*
– *Gauss-Jordan Method*
– *Principal and Free Variables*
– *Fundamental Set of Solutions of Homogeneous Linear System*
– *General Solution of Homogeneous Linear System*
– *General Solution of Non-Homogeneous Linear System*

1.5.1. Solving of Linear Systems

Let us consider a linear system of m equations in n unknowns $x_1, x_2, \dots, x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m. \end{cases} \quad (1.8)$$

To solve a system it means to investigate if the system is consistent. And if the system is consistent, it is important to know, how many solutions this system has. The rank tells us everything about the number of solutions to a given system of linear equations.

Kroneker-Capelli's Theorem

Theorem 1.4.

(*Kroneker — Capelli*). *A system of linear algebraic equations* (1.8) *is consistent if and only if the rank of the coefficient matrix* A *of a system is equal to the rank of the augmented matrix* $\bar{\mathrm{A}}$ *of this system*:

$$\mathrm{r}(\mathrm{A}) = \mathrm{r}(\bar{\mathrm{A}}).$$

We recall that

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad \bar{A} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right).$$

The proof of this theorem is omitted here.

Gauss – Jordan Method

Let us rewrite the linear system (1.8) in matrix notation

$$AX = B.$$

Then we apply Gaussian elimination to the augmented matrix of this system:

$$\left(\begin{array}{ccccccccccc|c} \tilde{a}_{1k_1} & \dots & \dots & & & & & & & & & \tilde{b}_1 \\ 0 & \dots & 0 & \tilde{a}_{2k_2} & \dots & & & & & & & \tilde{b}_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & & \tilde{a}_{rk_r} & \dots & \dots & & \tilde{b}_r \\ 0 & \dots & 0 & 0 & \dots & 0 & & 0 & & 0 & & \tilde{b}_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & & 0 & \dots & 0 & & \tilde{b}_m \end{array}\right).$$

This last matrix is said to be in *reduced row-echelon form*. Every matrix has a unique reduced row-echelon form.

A linear system (1.8) is inconsistent provided that one of the $\tilde{b}_i, i = \overline{r+1, m}$ does not equal zero, since

$$r(A) = r < r(\bar{A}) = r + 1.$$

If all $\tilde{b}_i, i = \overline{r+1, m}$ are equal to zero then the system (1.8) is consistent.

If the system is consistent we apply the back Gaussian elimination to the obtained row-echelon matrix. And we get reduced matrix

$$\left(\begin{array}{cccccccc|c} 1 & 0 & \dots & 0 & l_{1(r+1)} & \dots & l_{1n} & d_1 \\ 0 & 1 & \dots & 0 & l_{2(r+1)} & \dots & l_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & l_{r(r+1)} & \dots & l_{rn} & d_r \end{array}\right).$$

That of reducing a matrix to a reduced row-echelon form is known *Gauss-Jordan elimination*.

This matrix is associated with the following system

$$\begin{cases} x_1 + l_{1(r+1)}x_{r+1} + \dots + l_{1n}x_n = d_1, \\ x_2 + l_{2(r+1)}x_{r+1} + \dots + l_{2n}x_n = d_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_r + l_{r(r+1)}x_{r+1} + \dots + l_{rn}x_n = d_r. \end{cases} \tag{1.9}$$

Variables which correspond to the leaders in the matrix — $x_1, \dots, x_r$ — are called *the principal variables*, other variables — $x_{r+1}, \dots, x_n$— are called *free* (*secondary*) *variables*.

Let's consider two cases:

1. *The number of unknowns* n *is equal to the number* r *of equations of the system* (1.9) $(n = r)$, *but then the system is of the following form*

$$\begin{cases} x_1 = d_1, \\ x_2 = d_2, \\ \dots\dots\dots \\ x_r = d_r. \end{cases}$$

The system in this case has the unique solution.

2. *The number of unknowns* n *is greater than the number* r of equations of the system (1.9) $(n > r)$.

In this case the system has the infinite number of solutions.

Let's assign arbitrary values $C_1, ..., C_{n-r}$ to the free variables $x_{r+1}, ..., x_n$ and express the principal variables $x_1, ..., x_r$ in terms of these arbitrary values:

$$\begin{cases} x_{r+i} = C_i, i = \overline{1, n-r}, \\ x_1 = d_1 - l_{1(r+1)}C_1 - ... - l_{1n}C_{n-r}, \\ x_2 = d_2 - l_{2(r+1)}C_1 - ... - l_{2n}C_{n-r}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_r = d_r - l_{r(r+1)}C_1 - ... - l_{rn}C_{n-r}. \end{cases} \tag{1.10}$$

The solution (1.10) is *the general solution of the linear system*.

Fig. 1.22 outlines the connection between the concept of rank of a matrix and the solution of a linear system

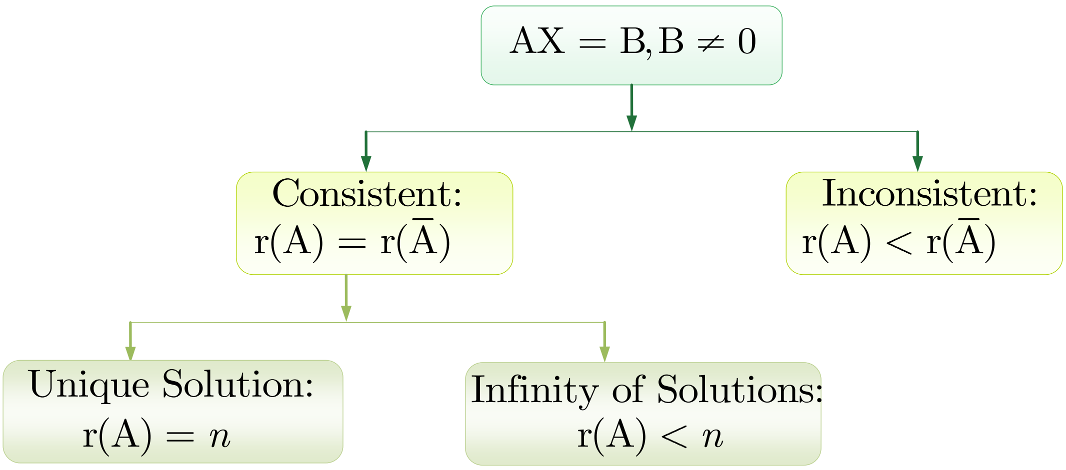

Fig. 1.22

We now show realization of Gauss-Jordan elimination on an example.

Example

Let us find the general solution of the system:

$$\begin{cases} 5x_1 - x_2 + 2x_3 + x_4 = 7, \\ 2x_1 + x_2 + 4x_3 - 2x_4 = 1, \\ x_1 - 3x_2 - 6x_3 + 5x_4 = 0. \end{cases}$$

❍ The solution of this system we start from finding the rank of the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & 0 \\ 2 & 1 & 4 & -2 & 1 \\ 5 & -1 & 2 & 1 & 7 \end{array}\right) \begin{array}{c} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = \overleftarrow{a}_2 - 2\overleftarrow{a}_1 \\ \overleftarrow{b}_3 = \overleftarrow{a}_3 - 5\overleftarrow{a}_1 \end{array} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & 0 \\ 0 & 7 & 16 & -12 & 1 \\ 0 & 14 & 32 & -24 & 7 \end{array}\right) \begin{array}{c} \overleftarrow{c}_1 = \overleftarrow{b}_1 \\ \overleftarrow{c}_2 = \overleftarrow{b}_2 \\ \overleftarrow{c}_3 = \overleftarrow{b}_3 - 2\overleftarrow{b}_2 \end{array} \sim \left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & 0 \\ 0 & 7 & 16 & -12 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right)$$

We obtain that the rank of the coefficient matrix equals 2, and the rank of the augmented matrix equals 3:

$$\mathrm{r(A)} \neq \mathrm{r}(\bar{\mathrm{A}}).$$

It means that our system is inconsistent, the system has no solution. ●

Let us consider the next example.

Example

Let us find the general solution of the system:

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + x_4 + 2x_5 = 6, \\ 3x_1 + 6x_2 + 2x_3 + 2x_4 + x_5 = 4, \\ 9x_1 + 18x_2 + x_3 + 7x_4 - x_5 = 2. \end{cases}$$

❍We first find the rank of the augmented matrix:

$$\left(\begin{array}{ccccc|c} 2 & 4 & 3 & 1 & 2 & 6 \\ 3 & 6 & 2 & 2 & 1 & 4 \\ 9 & 18 & 1 & 7 & -1 & 2 \end{array}\right) \begin{array}{c} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = -2\overleftarrow{a}_2 + 3\overleftarrow{a}_1 \\ \overleftarrow{b}_3 = -\overleftarrow{a}_3 + 3\overleftarrow{a}_2 \end{array} \sim$$

$$\sim \left(\begin{array}{ccccc|c} 2 & 4 & 3 & 1 & 2 & 6 \\ 0 & 0 & 5 & -1 & 4 & 10 \\ 0 & 0 & 5 & -1 & 4 & 10 \end{array}\right) \begin{array}{c} \overleftarrow{c}_1 = \overleftarrow{b}_1 \\ \overleftarrow{c}_2 = \overleftarrow{b}_2 \\ \overleftarrow{c}_3 = \overleftarrow{b}_3 - \overleftarrow{b}_2 \end{array} \sim \left(\begin{array}{ccccc|c} 2 & 4 & 3 & 1 & 2 & 6 \\ 0 & 0 & 5 & -1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The result shows, that the rank of the coefficient matrix equals 2, and the rank of the augmented matrix also equals 2:

$$\mathrm{r}(\mathrm{A}) = \mathrm{r}(\overline{\mathrm{A}}) = r = 2\,.$$

Hence, the system is consistent.

Since $\mathrm{r} < n$, the system has infinite number of solutions. In order to find the general solution let us continue transformations:

$$\left(\begin{array}{ccccc|c} 2 & 4 & 3 & 1 & 2 & 6 \\ 0 & 0 & 5 & -1 & 4 & 10 \end{array}\right) \begin{array}{c} \overleftarrow{d}_1 = (1/2)\overleftarrow{c}_1 \\ \overleftarrow{d}_2 = -\overleftarrow{c}_2 \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1,5 & 0,5 & 1 & 3 \\ 0 & 0 & -5 & 1 & -4 & -10 \end{array}\right).$$

As we see, x_1, x_4 are principal variables, and

$$\begin{aligned} x_2 &= C_1, \\ x_3 &= C_2, \\ x_5 &= C_3, \end{aligned}$$

$C_1, C_2, C_3 \in \mathbb{R}$, are free variables.

Solving for principal variables, we obtain

$$x_4 = -10 + 5C_2 + 4C_3$$

$$x_1 = 3 - 2C_1 - 1,5C_2 - 0,5x_4 - C_3 = 8 - 2C_1 - 4C_2 - 3C_3$$

Thus, the general solution of given system is of the form

$$\begin{cases} x_1 = 8 - 2C_1 - 4C_2 - 2C_3, \\ x_2 = C_1, \\ x_3 = C_2, \\ x_4 = -10 + 5C_2 + 4C_3, \\ x_5 = C_3. \end{cases} \qquad \bullet$$

1.5.2. Homogeneous Systems of Linear Equations

Let us consider a homogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0, \end{cases} \tag{1.11}$$

or in a matrix notation

$$\mathrm{AX} = 0 \tag{1.12}$$

Notice immediately that a homogeneous system is always consistent, it has at least the particular solution

$$X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

which is usually called *the trivial solution*.

For homogeneous systems the following theorem is true.

Theorem 1.5.

1. *Sum of the solutions of a homogeneous system is also the solution of this system.*

2. *Product of the solution of a homogeneous system by any arbitrary nonzero scalar is also the solution of this system.*

► Let X_1 and X_2 are the solutions of a homogeneous system (1.12):

$$\mathrm{AX}_1 = 0, \quad \mathrm{AX}_2 = 0.$$

We have

$$\mathrm{AX}_1 + \mathrm{AX}_2 = 0,$$

whence,

$$\mathrm{A}(\mathrm{X}_1 + \mathrm{X}_2) = 0.$$

The first statement is proved.

Similarly, if X is the solution of a homogeneous system, then we get:

$$\mathrm{AX} = 0 \Rightarrow \alpha\mathrm{AX} = 0 \Rightarrow \mathrm{A}(\alpha\mathrm{X}) = 0. \blacktriangleleft$$

Corollary

A linear combination of the solutions of a homogeneous system is the solution of this system too.

A homogeneous system has either the trivial solution only, or infinite number of solutions, including the trivial solution.

Claim 1.3.

A homogeneous linear system has a non-trivial solution if and only if rank r of the coefficient matrix of a system is less than number of unknowns n of this system:

$$r(\mathrm{A}) = r < n.$$

If a matrix A *is square, this condition means that the determinant of this matrix* A *is equal to zero* ($\det \mathrm{A} = 0$).

Fig. 1.23 outlines the connection between the concept of rank of a coefficient matrix and the solution of a homogeneous system

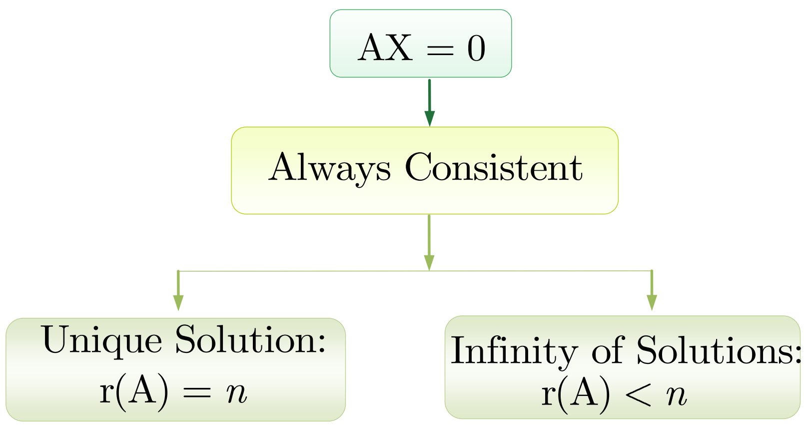

Fig.1.23

If the rank of the coefficient matrix $A_{m\times n}$ of a homogeneous system equals r, then a system has $n-r$ linearly independent solutions.

Definition 1.16.

The set of $(n-r)$ linearly independent solutions of a homogeneous system is called the fundamental set of solutions.

Let $\{X_1, \ldots, X_{n-r}\}$ is the fundamental set of solutions of a homogeneous system, then the general solution of a homogeneous system X_{gh} is of the form:

$$X_{\text{gh}} = C_1X_1 + C_2X_2 + \ldots + C_{n-r}X_{n-r} = \sum_{j=1}^{n-r} C_jX_j . \qquad (1.13)$$

Formula (1.13) defines *a structure of the general solution of a homogeneous system*.

Example

Let us find the fundamental system of the solutions and the general solution of the homogeneous system:

$$\begin{cases} x_1 + 2x_2 + 4x_3 - 3x_4 = 0, \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0, \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0, \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 = 0. \end{cases}$$

❍ We are using the *Gauss-Jordan method*. Let's transform the coefficient matrix of this system:

$$A=\begin{pmatrix}1&2&4&-3\\3&5&6&-4\\4&5&-2&3\\3&8&24&-19\end{pmatrix}\begin{matrix}\overleftarrow{b_1}=\overleftarrow{a_1}\\\overleftarrow{b_2}=\overleftarrow{a_2}-3\overleftarrow{a_1}\\\overleftarrow{b_3}=\overleftarrow{a_3}-4\overleftarrow{a_1}\\\overleftarrow{b_4}=\overleftarrow{a_4}-\overleftarrow{a_2}\end{matrix}\sim$$

$$\sim\begin{pmatrix}1&2&4&-3\\0&-1&-6&5\\0&-3&-18&15\\0&3&18&-15\end{pmatrix}\begin{matrix}\overleftarrow{c_1}=\overleftarrow{b_1}\\\overleftarrow{c_2}=-\overleftarrow{b_2}\\\overleftarrow{c_3}=\overleftarrow{b_3}-3\overleftarrow{b_2}\\\overleftarrow{c_4}=\overleftarrow{b_4}+\overleftarrow{b_3}\end{matrix}\sim$$

$$\begin{matrix}x_1&x_2&C_1&C_2\end{matrix}$$

$$\sim\begin{pmatrix}1&2&4&-3\\0&1&6&-5\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Since $r(A)=2$, the system with four unknown variables $(n=4)$ has infinite number of solutions.

Let's choose principal and free variables. The leaders of the matrix are coefficients of x_1, x_2. That is why these variables are principal variables. Hence, $x_3=C_1, x_4=C_2$

$(C_1, C_2\in\mathbb{R})$ are free variables.

And the general solution is of the form:

$$X_{gh}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}8C_1-7C_2\\5C_2-6C_1\\C_1\\C_2\end{pmatrix}.$$

And now we are finding the fundamental system of the solutions of this system.

Let us suppose that

$$C_1 = 1, C_2 = 0.$$

Then we get

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 1 \\ 0 \end{pmatrix}.$$

Putting

$$C_1 = 0, C_2 = 1,$$

we obtain

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 0 \\ 1 \end{pmatrix}.$$

The solutions X_1, X_2 form the fundamental system of the solutions of given system, therefore

$$X_{gh} = C_1 X_1 + C_2 X_2. \ \bullet$$

Remark

If the rank of linear homogeneous system is one less then the number of unknowns: $r = n - 1$, then the fundamental system consists of only one solution, and any non-zero solution forms the fundamental system. In this case, any two solutions differ only by constant.

Example

Let us find the fundamental system of the solutions and the general solution of the homogeneous system:

$$\begin{cases} x - y + 2z = 0, \\ 2x - 3y + 5z = 0. \end{cases}$$

❍ Transform the coefficient matrix of this system:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 5 \end{pmatrix} \begin{array}{c} \overleftarrow{b}_1 = \overleftarrow{a}_1 \\ \overleftarrow{b}_2 = \overleftarrow{a}_2 - 2\overleftarrow{a}_1 \end{array} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

We have: $r = 2$, $n = 3$.

The solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

is a non-zero partial solution of given system. That's why it forms the fundamental system of solutions. All other solutions of this system are propotional to it, and therefore, general solution is of the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -c \\ c \\ c \end{pmatrix}. \quad \bullet$$

1.5.3. Structure of General Solution of Non-Homogeneous System

General solution of a non-homogeneous system (1.9)

$$\begin{cases} x_{r+i} = C_i, i = \overline{1, n-r}, \\ x_1 = d_1 - l_{1(r+1)}C_1 - ... - l_{1n}C_{n-r}, \\ x_2 = d_2 - l_{2(r+1)}C_1 - ... - l_{2n}C_{n-r}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_r = d_r - l_{r(r+1)}C_1 - ... - l_{rn}C_{n-r} \end{cases}$$

we may write in the form

$$\mathrm{X}_{\text{gnh}} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 - \sum\limits_{i=1}^{n-r} l_{1(r+i)} \\ \vdots \\ d_r - \sum\limits_{i=1}^{n-r} l_{r(r+i)} \\ C_1 \\ \vdots \\ C_{n-r} \end{pmatrix},$$

or

$$\mathrm{X}_{\text{gnh}} = \begin{pmatrix} d_1 \\ \vdots \\ d_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} l_{1(r+1)} \\ \vdots \\ l_{r(r+1)} \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + C_{n-r} \begin{pmatrix} l_{1n} \\ \vdots \\ l_{rn} \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

We also may write the general solution of the non-homogeneous system as:

$$X_{gnh} = X_{pnh} + C_1 X_1 + C_2 X_2 + \ldots + C_{n-r} X_{n-r}.$$

It is not so difficult to see that X_{pnh} is some *partial solution of the non-homogeneous system*, $X_1, X_2, \ldots, X_{n-r}$ form *the fundamental system of solutions of the corresponding homogeneous system*.

Therefore, the following theorem is true.

Theorem 1.6.

The general solution X_{gnh} *of a non-homogeneous linear system*

$$AX = B$$

is equal to the sum of the general solution X_{gh} *of the corresponding homogeneous system*

$$AX = 0$$

and the partial solution X_{pnh} *of given non-homogeneous system:*

$$X_{gnh} = X_{pnh} + X_{gh}.$$

► Indeed,

$$AX_{gnh} = A(X_{pnh} + X_{gh}) = AX_{pnh} + AX_{gh} = B + 0 = B.$$

It means that X_{gnh} is the solution of a *a non-homogeneous linear system*

$$AX = B. ◄$$

Questions to Chapter 1

I.

1. What is meant by the order of a matrix?
2. What are equal matrices?
3. Describe how to perform scalar multiplication.
4. How are matrices added?
5. Describe how to multiply matrices. Describe, when the multiplication of two matrices is not defined.
6. What is a transpose matrix?
7. What is an inverse matrix?
8. How the determinants of the second and third orders are calculated?
9. What is a Minor of the element? a Cofactor of the element?
10. Formulate the basic properties of determinants.
11. What is the determinant of the n-th order?
12. What is a minor of the second order?
13. What is the solution of a linear system?
14. What types of solutions of a linear system do you know?
15. When we may use Cramer's rule?
16. What does Gauss method lye in?
17. State the Kroneker-Capelli's theorem.
18. What is the Fundamental System of solutions? May we define it for non-homogeneous system?
19. State the structure of the general solution of the homogeneous system.
20. State the structure of the general solution of the non-homogeneous system.

II.

1. Is it possible to multiply the matrix of the size 3×5 by the matrix of the same size?
2. Is it possible to multiply the matrices that can be added?
3. Is it possible to add the matrices that can be multiplied?
4. What conditions do the matrices A and B satisfy if:
   a) there exists the product AB;
   b) there exists the product BA;
   c) there exist matrix products AB and BA?
5. Is it possible to multiply the square matrix by a non-square matrix?
6. Is it possible to get a number when multiplying matrices?
7. Does it follow from the equality $AB = 0$ that one of the matrices is a Zero matrix?
8. Which matrix you need to multiply the matrix A in order to get:
   a) the first column of the matrix A;
   b) the first row of the matrix A as a result?
9. We know that the determinant of a matrix A (matrix A is a matrix of the third order) equals 5 ($\det A_3 = 5$).

Compute $\det A^T$; $\det(3A)$.

10. Determine the rank of a matrix and specify the basic minor of a matrix:

$$1)\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ 2)\ \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix};\ 3)\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ 4)\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ 5)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$6)\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix};\ 7)\ \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 3 \\ 4 & 2 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

11. Describe matrices with the rank 0; 1.