American Options

An American option can be exercised at *any* time up to and including expiry. This implies that:

- it must be worth at least as much as its European equivalent;
- it can never be worth less than its payoff (which must be defined for the whole life of the option and may depend on time).

It is an integral part of the problem to decide *when* to exercise an American option and it is also important to note that it is *the holder, not the writer, who chooses when to exercise an American option.*

An American option's price, V_{am}, can't be worth less than an otherwise equivalent European option's price, V_{eu}, because the holder is free to choose when to exercise the American option. This includes the freedom to not exercise the American option prior to expiry, in which case it is equivalent to a European option.

More formally, if we find that $V_{\text{am}} < V_{\text{eu}}$ then buy the American option and write or short-sell the European one, giving an immediate profit of $V_{\text{eu}} - V_{\text{am}} > 0$. Then hold the American option to expiry, when it has the same payoff as the European one. As the options are identical except for their exercise rights, any intermediate cash-flows cancel and so this is an arbitrage.

The reason an American option's price, V_{am}, can't be worth less than its payoff, P_{O}, is that if it were then there would be a very obvious and highly exploitable arbitrage:

- buy the option for $V_{\text{am}} < P_{\text{O}}$;

- exercise it immediately for P_{O};
- pocket the profit of $P_{\text{O}} - V_{\text{am}} > 0$.

If we are able to buy the option for $V_{\text{am}} < P_{\text{O}}$ then we can, and should, execute this arbitrage.

Basic results

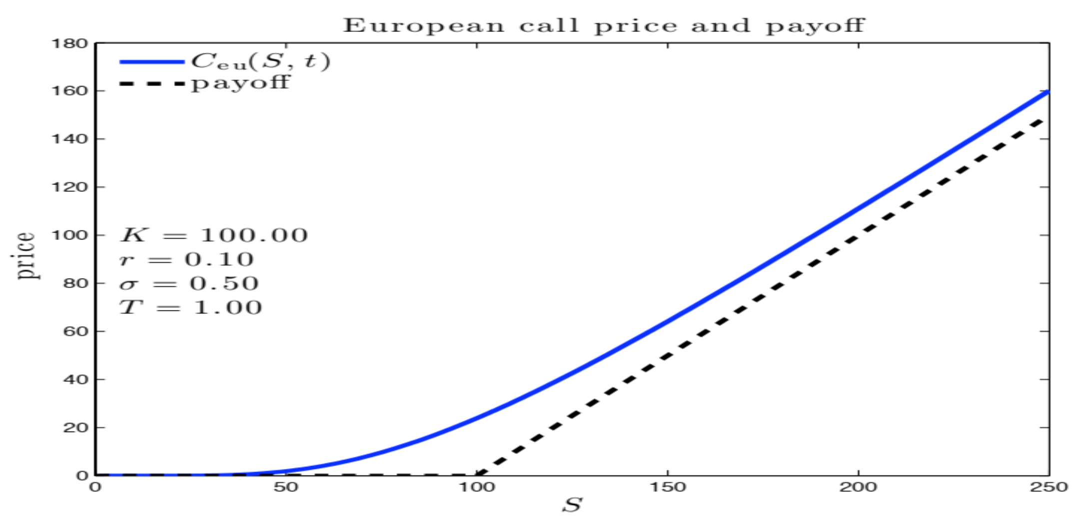
Calls:

In the absence of dividends, the value of a European call always lies above its payoff, P_{O}. As $C_{\text{am}} \geq C_{\text{eu}} > P_{\text{O}}$, it can't be optimal to *exercise* an American call early, it is always better to *sell* it.

A formal argument involves looking at two portfolios, Π_1 with an American call C_{am} and a bond worth $K\, e^{-r(T-t)}$, and Π_2 with the underlying asset S. The values of these two portfolios are

$$\Pi_1 = C_{\text{am}} + K\, e^{-r(T-t)},$$

$$\Pi_2 = S.$$



The price of a European call option on an asset that pays no dividend

If we hold Π_1 and exercise the option early, at $t < T$, we get

$$S - K + K\,e^{-r(T-t)} < S = \Pi_2.$$

If we hold Π_1 to expiry we get

$$\begin{aligned} \Pi_1 &= \max(S-K,0)+K \\ &= \max(S,K) \\ &\geq \Pi_2. \end{aligned}$$

Thus it is never optimal to exercise the call option early.

This shows that, with no dividends, American and European calls have the same value.*

*It also shows that there are options for which the early-exercise right is worthless, because it is never optimal to use the right.

The argument above may be summarized by saying that there is no point in exercising a call option early as the only effect of doing so is to pay the strike earlier, which implies that you lose interest payments you would otherwise have received on the strike.

Even if *you* don't want to hold the option until expiry, *someone else* will and you'd get more by *selling the option* to them than you'd get by exercising it early.

Note: this result is *not* necessarily true if the underlying asset pays (positive) dividends because the owner of the asset gets the dividends, while the holder of option does not. If the dividends are large enough, they may be worth more than the interest you lose on the strike by exercising early.

Puts:

For sufficiently small asset prices S, it may be optimal to exercise an American put option early.* If not, American and European puts must have the same prices. For $t < T$ and at $S = 0$, the value of a European put is

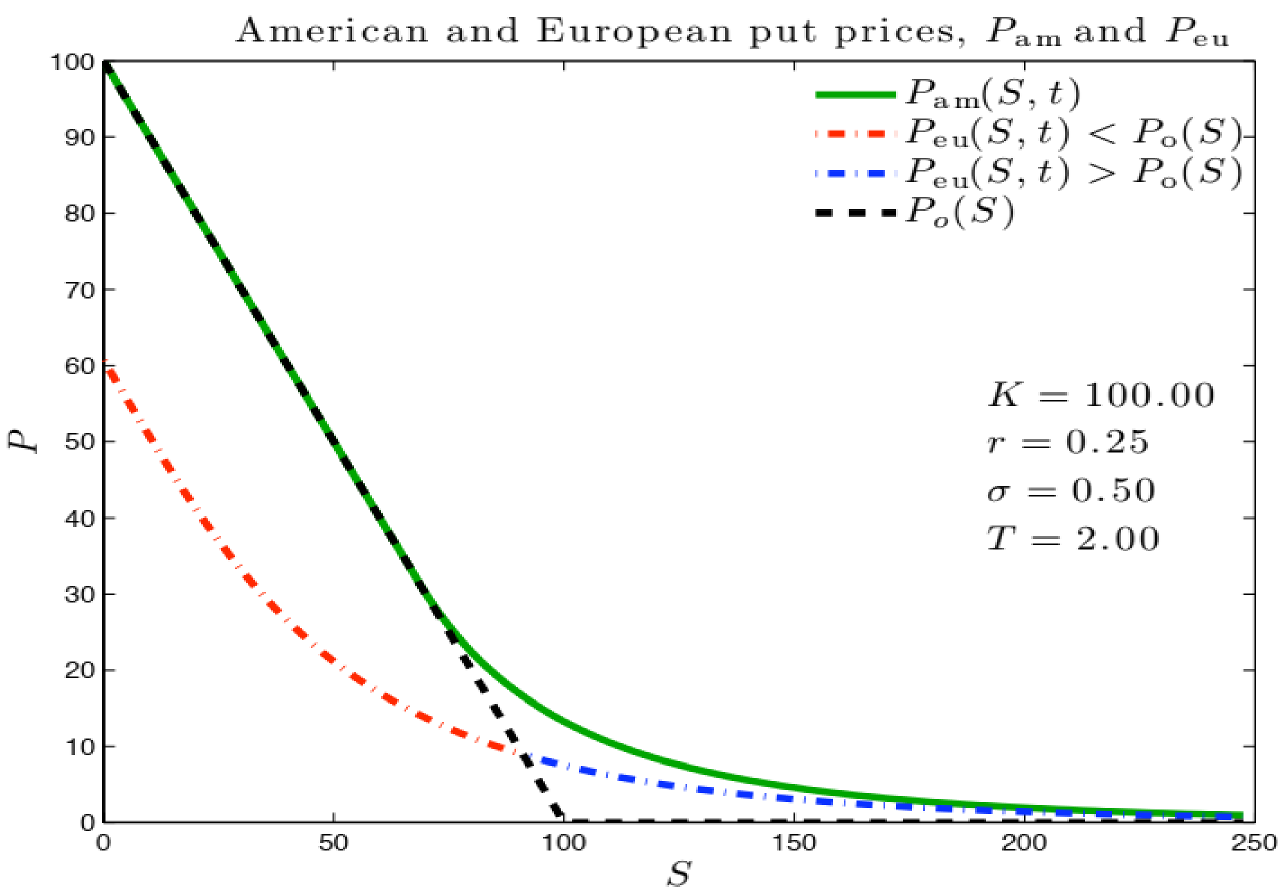
$$P(0,t) = K\,e^{-r(T-t)} < K,$$

and hence, by continuity, for small enough S

$$P(S,t) < K - S.$$

That is, the European put's value is *less* than the payoff for $t < T$.

*Although we are assuming that there are no dividends here, the argument remains true when there are positive dividends.



The parameters are chosen to exaggerate the difference in prices.

If this put was American and so could be exercised at any time, we'd have an obvious arbitrage opportunity — buy the put and then exercise immediately; there is no doubt that this would be exploited.

A formal argument is to look at two portfolios, this time Π_1 consisting of an American put and its underlying asset, and Π_2 containing a bond worth $K\,e^{-r(T-t)}$,

$$\Pi_1 = P_{\text{am}} + S, \quad \Pi_2 = K\,e^{-r(T-t)}.$$

If we exercise the put at time $t < T$ then

$$\Pi_1 = K > \Pi_2 = Ke^{-r(T-t)},$$

whereas if we hold the put until expiry then

$$\Pi_1 = \max(S, K) \geq \Pi_2 = K.$$

Thus there is some chance that at expiry $\Pi_1 = \Pi_2$ whereas prior to expiry $\Pi_1 > \Pi_2$. Therefore, there must be situations where it is optional to exercise the put early (namely when the option is well in-the-money).

If the put is out-of-the-money, we would be foolish to exercise early because doing so would only remove the right to exercise later, for a zero payoff. The spot price will change in the future, if it rises further it makes no difference to the payoff (of zero) but if it falls enough the payoff will become positive.

So there are also situations in which we wouldn't exercise early, namely when the option is out-of-the-money.

A slightly more intuitive explanation is as follows.

In the case of the call option, we have to *pay* money (the strike, K) if we exercise the option. But if we pay the strike, we cease to earn interest on it. Whenever we exercise, we still get the asset. So, we hold off exercising the call option for as long as possible.

In the case of the put option, we *receive* money (again, the strike) and we'd like to get this money as soon as possible in order to earn interest on it. Against this is the possibility that the spot price, S, might decrease further in the future, but we don't receive the benefits from any further decreases in the spot price if we've already exercised early, which is why we don't *always* exercise an in-the-money put early.

Clearly if we introduce (positive) dividends for the underlying asset, this makes us:

- more likely to exercise a call early — the trade off for a call is interest lost on the strike against dividends gained from holding the underlying asset;

- less likely to exercise a put early — the trade off for a put is interest gained on the strike against dividends lost from the underlying asset.

On top of these considerations is the effect of volatility; if we exercise early it is always possible that subsequently the spot price will move in a way that means we would have been better off exercising later.

For small enough S, an American put's value is (strictly) greater than an equivalent European put's, so its price *can't* satisfy *exactly* the same problem as the European put's, which involves only the Black–Scholes equation and the payoff. If it did, it would have the same price as the European put and that would imply an arbitrage.

Both options have exactly the same payoff, so the American put *can't* satisfy the Black–Scholes equation for all $S > 0$.* We have to find a new pricing relation for the American put in particular and American options in general. In view of the American call with no dividends, this must also include the Black–Scholes equation as a special case.

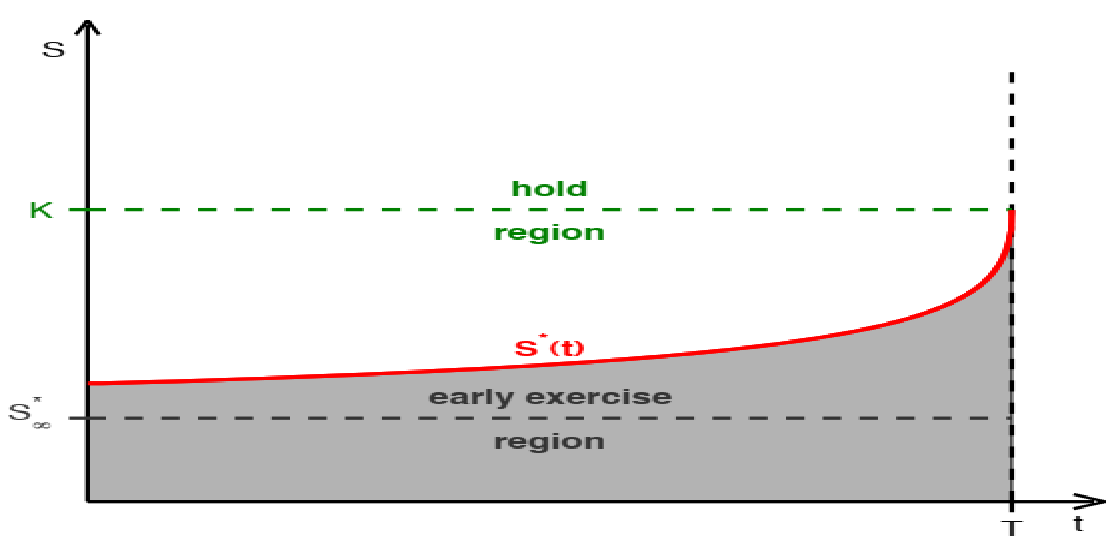
*At this point, we can't exclude the possibility that it doesn't satisfy the Black–Scholes equation at all, but it turns out that it does satisfy it for large enough values of S.

At this point, it is worth re-emphasising that there are *two* issues involved in pricing and hedging an American option:

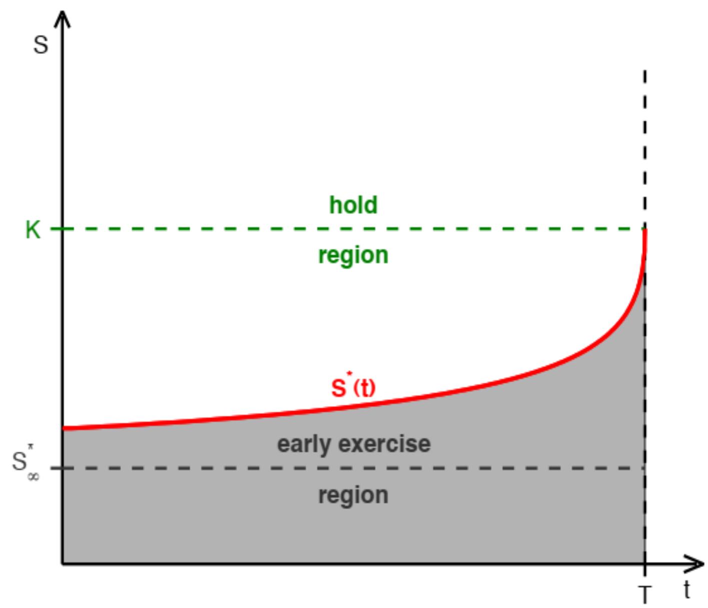
- Determining the hedging strategy and option price; and
- Determining when one should hold the option (and hence hedge it) and when one should exercise it immediately.

The two aspects are usually, but not always, coupled.

Put another way, it is part of the problem to determine under what conditions it is optimal to hold an American option and under what conditions it is optimal to exercise the option early. This may be though of in terms of finding the hold and early-exercise regions in (t, S)-space.



The hold and early-exercise regions in the (t, S)-plane for an American put with constant r and σ. The line dividing the two regions, $S^*(t)$, is the optimal exercise boundary. The grey dashed line indicates the asymptote of the optimal exercise boundary as $T - t \to \infty$. The optimal exercise boundary is computed numerically, from the linear complementarity formulation of the problem.



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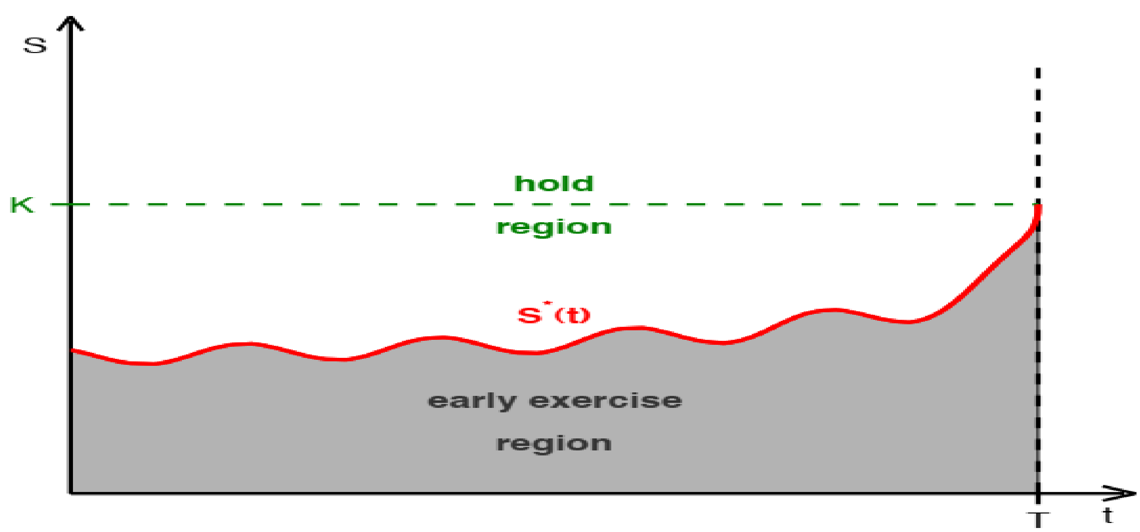


The hold and early-exercise regions in the (t, S)-plane for an American put with constant risk-free rate r and time-periodic volatility, $\sigma(t) = \frac{1}{2}\sigma_0(1 + \sin(2\pi t)^2)$. The optimal exercise boundary, $S^*(t)$, is not a monotonic function of time. The optimal exercise boundary is computed numerically, from the linear complementarity formulation of the problem.

There are a number of ways of formulating the American option problem. The most common of these are as:

- an optimal stopping time problem;
- a free boundary problem* (FBP) for the Black–Scholes equation;
- a linear complementarity problem (LCP);
- a parabolic variational inequality (VI).

*In the context of parabolic partial differential equations, such as the Black–Scholes equation, free boundary problems are sometimes also known as *moving-boundary problems.*

The basic ideas behind the linear complementarity formulation are:

- An American option can't be less valuable than its payoff;
- For any given state of the world, either it is optimal to exercise the option or it isn't. *To decide which, we consider the return on the delta-hedged portfolio.* If this return is equal to the risk-free return on the value of the portfolio, we continue holding the portfolio, but if it is less then we wind up the portfolio.*

For simplicity, we ignore dividends, and assume the underlying asset's price process is described by the usual stochastic differential equation,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$

*For a European option, no-arbitrage implies the two returns must be the same. For American options, however, they need not be.

We write the option's value as*

$$V(S,t).$$

Then, during an infinitessimal timestep dt, Itô's lemma tells us that

$$dV = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\, dS.$$

We set up the usual delta-hedged portfolio with market price

$$\Pi = V - \Delta\, S,$$

where Δ stays fixed during an infinitesimal time step of length dt.

*The argument given here follows the Δ-hedging argument. As an exercise, you might find it instructive to modify the self-financing replication argument to allow for early exercise.

Then our hedging strategy * implies that

$$d\Pi = dV - \Delta\, dS$$

and choosing $\Delta = \partial V/\partial S$ gives a *risk-free* instantaneous increment of

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

At this stage in the process, for a European option, we argue that

$$d\Pi = r\, \Pi\, dt,$$

because the alternatives, $d\Pi > r\,\Pi\, dt$ or $d\Pi < r\,\Pi\, dt$, both lead to arbitrages.

*Set Δ at the start of each infinitesimal time step and leave it fixed until the end of the time step.

Aside: The Black–Scholes operator, $\mathcal{L}_{\text{bs}}$, measures the difference between the instantaneous changes in a risk-free, Δ-hedged portfolio and an equivalent amount invested in risk-free bonds;

$$\mathcal{L}_{\rm bs}[V]\,dt \;=\; \underbrace{\Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)\,dt}_{\substack{d\Pi \\ \left(\substack{\Delta-\text{hedged} \\ \text{portfolio}}\right)}} \;-\; \underbrace{\Big(r\,V - r\,S\,\frac{\partial V}{\partial S}\Big)\,dt}_{\substack{r\,\Pi\,dt \\ \left(\substack{\text{risk}-\text{free} \\ \text{account.}}\right)}}$$

The Black-Scholes equation,

$$\mathcal{L}_{\rm bs}[V] = 0,$$

asserts that these two risk-free changes are the same.

Now, consider an American option. We can't have

$$d\Pi > r\,\Pi\,dt$$

as this represents an easily exploitable arbitrage and the law of supply and demand would act to rapidly eliminate the inequality.*

So, for an American option with price $V_{\rm am}$, no exploitable arbitrage opportunities implies

$$\mathcal{L}_{\rm bs}[V_{\rm am}]\,dt = (d\Pi - r\,\Pi\,dt) \le 0,$$

or, in our particular case,

$$\frac{\partial V_{\rm am}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{\rm am}}{\partial S^2} + r\,S\,\frac{\partial V_{\rm am}}{\partial S} - r\,V_{\rm am} \le 0.$$

**If* it is possible to borrow at the risk-free rate, the arbitrage is to borrow as much as possible and buy the option for its Black–Scholes price, then close out the position at a later date. This may not work during a credit crunch, however.

If, however, we try to show that

$$d\Pi < r\,\Pi\,dt$$

implies an exploitable arbitrage opportunity we may fail.

The argument which works in the European case involves *selling* the delta-hedged portfolio. In particular, it involves short-selling or writing the option.

If the option is European, it can't be exercised until expiry, and there is no immediate risk in doing so.*

An American option, however, can be exercised at *any* time and this makes the situation rather different.

*Furthermore, the risk at expiry can be Δ-hedged away.

Due to early exercise, there *might* be times when it is not sensible to write, sell or short-sell an American option. At these times it *is* possible to have

$$d\Pi < r\,\Pi\,dt.$$

Indeed, this is precisely why it *is* optimal to exericse the option early; the delta-hedged portfolio is under-performing a risk-free deposit on the value of the portfolio, so we unwind the portfolio and put the money in the bank.

Thus for an American option we can only conclude, in general, that

$$\mathcal{L}_{\text{bs}}[V_{\text{am}}] = d\Pi - r\,\Pi\,dt \leq 0$$

which in our case, with a a geometric Brownian motion, becomes

$$\frac{\partial V_{\text{am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{\text{am}}}{\partial S^2} + r\,S\,\frac{\partial V_{\text{am}}}{\partial S} - r\,V_{\text{am}} \leq 0.$$

If we know that it is not optimal to exercise the option immediately then we can safely short sell the option, in which case

$$d\Pi < r\,\Pi\,dt$$

does represent an exploitable arbitrage.

Therefore, *if* we know that it is not optimal to exercise the option immediately, we can also conclude that

$$\mathcal{L}_{\mathsf{bs}}[V_{\mathsf{am}}] = d\Pi - r\,\Pi\,dt = 0.$$

In our particular case, with geometric Brownian motion, this means that *if* it is not optimal to exercise the option immediately then the Black–Scholes equation *is* satisfied,

$$\frac{\partial V_{\mathsf{am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{\mathsf{am}}}{\partial S^2} + r\,S\,\frac{\partial V_{\mathsf{am}}}{\partial S} - r\,V_{\mathsf{am}} = 0.$$

In itself this is not yet much use as we don't know *when* it is or isn't optimal to exercise the option immediately; part of the problem for an American option is to determine *when* it is optimal to exercise and when it is optimal to hold (or sell) the option.

To proceed further, we note that in addition to the inequality

$$\mathcal{L}_{\mathsf{bs}}[V_{\mathsf{am}}] \leq 0,$$

we also have the inequality *

$$V_{\mathsf{am}}(S,t) \geq P_{\mathsf{o}}(S,t),$$

where $P_{\mathsf{o}}(S,t)$ is the option's payoff.

*Note that for an American option, the payoff P_{o} could depend on t.

Suppose that the option's price is greater than the payoff,

$$V_{\mathsf{am}}(S,t) > P_{\mathsf{o}}(S,t).$$

It would be absurd to exercise the option immediately, because you could *sell* it for more.

The option would not be exercised immediately,* it is safe to short sell it, and $\mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0$ *is* an exploitable arbitrage opportunity.

Hence, we can conclude that

$$\textit{if } V_{\text{am}}(S,t) > P_{\text{o}}(S,t) \textit{ then } \mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0.$$

*At least it would not be exercised by a rational investor. It turns out that the strategy of exercising early to spite the writer backfires in this case and the only person who loses out is the early-exerciser.

Now suppose that

$$\mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0.$$

In this case, the risk-free, Δ-hedged portfolio's return is less than the risk-free return available by selling the portfolio and putting the money into the bank (or bonds).*

The rational thing to do in this circumstance is to sell off the portfolio and put the funds into the bank.

It is now helpful to consider the conditions under which it possible for $\mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0$ to occur.

*Risk-free bonds, of course.

The absence of any exploitable arbitrage opportunities implies that we can *never* have

$$V_{\text{am}}(S,t) < P_{\text{o}}(S,t)$$

and we've just shown that

$$\text{if} \quad V_{\text{am}}(S,t) > P_{\text{o}}(S,t) \quad \text{then} \quad \mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0.$$

So, the only remaining possibility in this case is that *

$$V_{\text{am}} = P_{\text{o}},$$

and therefore

$$\textit{if } \mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0 \textit{ then } V_{\text{am}}(S,t) = P_{\text{o}}(S,t).$$

*This implies that we *exercise* the option, because no-one else would want to buy it for precisely the same reason we want to get rid of it — under these circumstances Δ-hedging it is a *risk-free means of losing money.*

In summary, we have shown that:

- if $V_{\text{am}} > P_{\text{o}}$ then $\mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0$ and the option is not exercised;
- if $\mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0$ then $V_{\text{am}} = P_{\text{o}}$ and the option is exercised;
- $V_{\text{am}} < P_{\text{o}}$ is impossible as it is an obvious arbitrage.

The first two cases are mutually exclusive and cover all possibilities. They may be summarized more succinctly by saying that *either:*

- $V_{\text{am}} > P_{\text{o}}$ and $\mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0$ (the option is not exercised early);
- $V_{\text{am}} = P_{\text{o}}$ and $\mathcal{L}_{\text{bs}}[V_{\text{am}}] < 0$ (the option is exercised early).

In a compact form, this can then be written

$$\max\{\mathcal{L}_{\text{bs}}[V_{\text{am}}], P_{\text{o}} - V_{\text{am}}\} = 0$$

The key fact is that solving this equation does not require explicit calculation of the exercise boundary.

Begin aside

In general, an American option does not satisfy the European pricing equation, rather it satisfies an inequality involving the same pricing operator as its European equivalent. In our case this is

$$\frac{\partial V_{\text{am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{\text{am}}}{\partial S^2} + r\,S\,\frac{\partial V_{\text{am}}}{\partial S} - r\,V_{\text{am}} \le 0.$$

In the language of stochastic processes this may be expressed by stating that while the discounted price process of a European option is a martingale under the risk-neutral measure, the discounted price process of its American equivalent is, in general, a *super-martingale* under that measure.

End aside

It is helpful to look at an American put option at this point.

From the above we can conclude that if

$$P_{\text{am}}(S,t) > \max(K-S,0)$$

we should hold (and delta-hedge) the option and that its value satisfies the Black–Scholes equation.

When the option's value is the same as the payoff we have

$$P_{\text{am}}(S,t) = K-S;$$

no rational investor would exercise the put early if it was out-of-the-money, so we may assume that $\max(K-S,0) = K-S > 0$.

It is a simple calculation to show that if $P_{\text{am}} = K-S$ then *

$$\mathcal{L}_{\text{bs}}[P_{\text{am}}] = -r\,K < 0.$$

The expression $-r\,K$ here is *not* a coincidence. It is the rate at which we are losing interest on the strike.

It arises because, under these circumstances, the Δ-hedged portfolio has a risk-free return of *zero.* That is,

$$\frac{\partial P_{\text{am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{\text{am}}}{\partial S^2} = 0.$$

*We are assuming that $r > 0$. The argument above still works when $r < 0$, but this particular example does not because you'd never exercise an in-the-money put early if interest rates were negative — if you don't see why then think about it!

The market value of the delta-hedged porfolio is

$$\Pi \;=\; P_{\text{am}} - S\,\frac{\partial P_{\text{am}}}{\partial S}$$

$$= (K - S) - (-S)$$
$$= K.$$

If we wound up the delta-hedged portfolio and invested the proceeds at the risk-free rate then we would earn money at the rate $r\,K$. Winding up the portfolio involves exercising the put immediately.*

By not exercising the put we are losing the interest that we could otherwise be getting on the strike, at the rate $-r\,K$. This is precisely why we should exercise the put immediately and place the strike in the bank where it will earn interest.

*No one is likely to want to buy it, for the same reason we want to get rid of it.

We may identify the optimal exercise boundary, $S^*(t)$, as the spot value at time t where the option's price switches

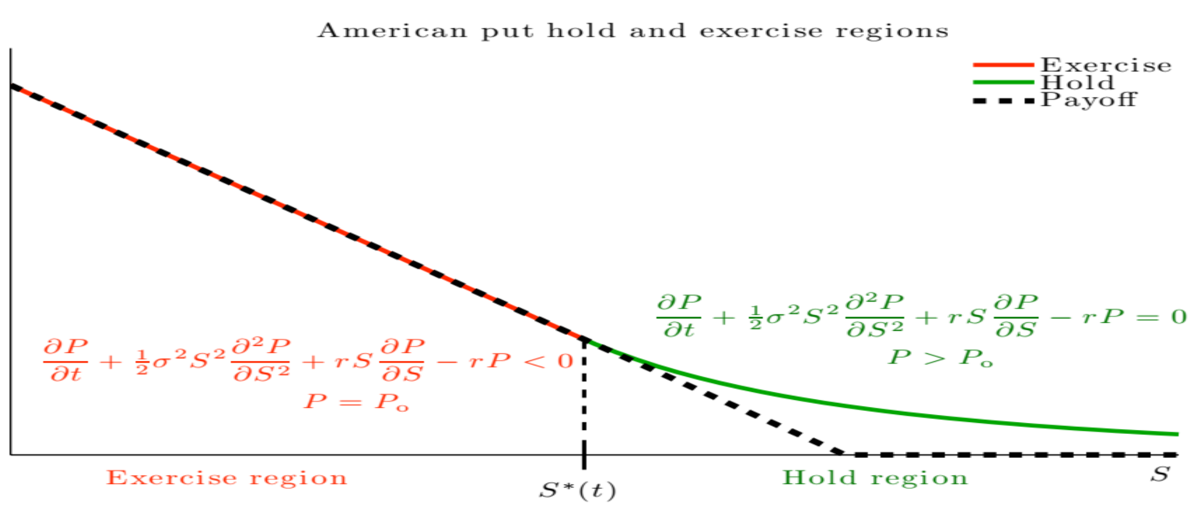
$$\text{from} \quad \mathcal{L}_{\mathrm{bs}}[P_{\mathrm{am}}] = 0 \quad \text{and} \quad P_{\mathrm{am}} > \max(K - S, 0)$$
$$\text{to} \quad \mathcal{L}_{\mathrm{bs}}[P_{\mathrm{am}}] < 0 \quad \text{and} \quad P_{\mathrm{am}} = \max(K - S, 0).$$

In the case of American put we find that

$$\mathcal{L}_{\mathrm{bs}}[P_{\mathrm{am}}] < 0, \quad P_{\mathrm{am}} = \max(K - S, 0) \quad \text{for} \quad S < S^*(t),$$
$$\mathcal{L}_{\mathrm{bs}}[P_{\mathrm{am}}] = 0, \quad P_{\mathrm{am}} > \max(K - S, 0) \quad \text{for} \quad S > S^*(t).$$

In general, an optimal-exercise boundary separates a pair of adjacent regions in which $\mathcal{L}_{\mathrm{bs}}[V_{\mathrm{am}}] < 0$ and $\mathcal{L}_{\mathrm{bs}}[V_{\mathrm{am}}] = 0$, respectively.*

*Such a boundary might only exist for a limited part of the option's life and, for complex payoffs, there may be several such boundaries.



We can write the general American option problem in the compact form

$$\mathcal{L}_{\text{bs}}[V_{\text{am}}] \le 0, \quad V_{\text{am}} - P_{\text{o}} \ge 0,$$

$$(V_{\text{am}} - P_{\text{o}}) \cdot \mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0,$$

which is the essence of the LCP formulation.

This is consistent with the observation that if the option is always more valuable than the payoff then the option's price satisfies the Black-Scholes equation. That is, if

$$V_{\text{am}}(S,t) > P_{\text{o}}(S,t) \quad \text{for all} \quad S > 0, \; t < T,$$

then we must have

$$\mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0 \;\; \text{for all} \;\; S > 0, \; t < T.$$

Of course, the partial differential equation

$$\mathcal{L}_{\text{bs}}[V_{\text{am}}] = 0 \;\; \text{for all} \;\; S > 0, \; t < T$$

is exactly the same equation an equivalent European option's price, V_{eu}, satisfies. As both have identical payoffs at expiry, this implies that the American and European versions of the option are identical,

$$V_{\text{am}} = V_{\text{eu}}.$$

A form of the converse also holds, namely that if

$$V_{\text{eu}}(S,t) > P_{\text{o}}(S) \;\; \text{for all} \;\; S > 0, \; t < T,$$

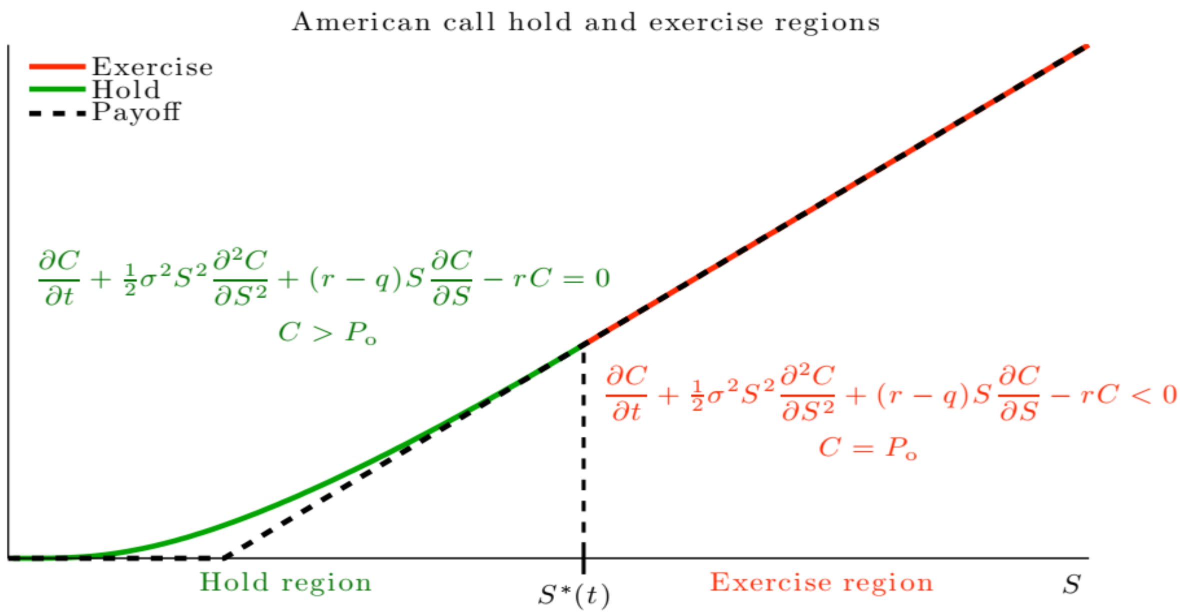
where $P_{\text{o}}(S)$ does *not* depend on time, then we must have

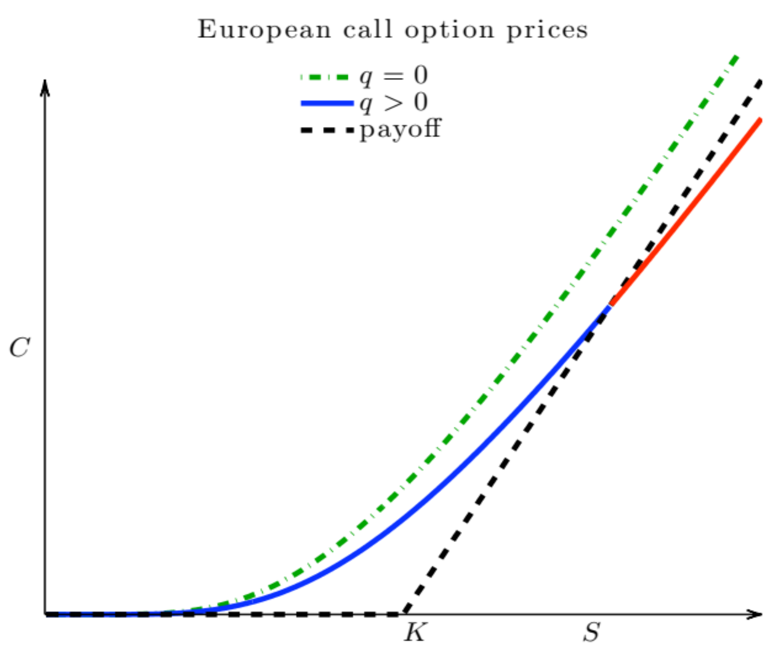
$$V_{\text{am}} = V_{\text{eu}}.$$

Another way to look at this is to note that if $V_{\text{am}} > P_{\text{o}}$ then it would be stupid to exercise early. Therefore the right to exercise early has no value. The only difference between otherwise equivalent American and European options is the right to exercise early and so, if we know this right is worthless, we must have $V_{\text{am}} = V_{\text{eu}}$.

This is why, in the absence of dividends, an American call option has the same value as an otherwise equivalent European call.

Note that if the underlying asset pays (positive) dividends, then the European call price falls below the payoff, the right to exercise early does have a positive value and the American call is more valuable than the otherwise equivalent European one.





This is usually still not sufficient to uniquely determine V_{am} and yet another condition is needed to obtain a unique price.* The most general statement of this condition is that

the option holder *should choose the exercise strategy that maximises the value of the option.*

We call this particular exercise strategy the *optimal-exercise strategy.* To date, it has proved impossible to state this condition in any more helpful way for an option with an arbitrary payoff. When the payoff is locally both convex and twice differentiable, it is equivalent to what is known as a *smooth pasting* condition.

*There are examples where continuity in S, together with the no-arbitrage condition that $V_{\text{am}} \geq P_{\text{o}}$, do give a unique price. We will see examples where this is true, but we will also see examples where it isn't.

If the optimal-exercise strategy is obvious *a priori,* and not implicitly determined as part of the solution, continuity of $V_{\text{am}}(S,t)$ as a function of S is enough.*

A simple example which illustrates this point is an American digital call option. The payoff is

$$P_{\text{o}}(S) = \begin{cases} 0 & \text{if } S < K, \\ 1 & S \geq K, \end{cases}$$

where $K > 0$ is the strike and the option can be exercised at any time up to and including expiry, T. See appendix.

*In these cases it is actually *impossible* to require more than continuity.

The American put and smooth pasting

If optimal exercise doesn't occur at a discontinuity in the payoff, then the optimal-exercise strategy is usually determined by a *smooth-pasting* condition, which asserts that

$$V_{\text{am}}(S,t) \text{ and } \frac{\partial V_{\text{am}}}{\partial S}(S,t) \text{are a continuous in } S.$$

This is illustrated in the following figure, which shows the price of an American put. The unknown optimal exercise boundary $S^*(t)$ is determined *implicitly* by the smooth pasting conditions, namely

$$P_{\text{am}}\Big(S^*(t),t\Big) = K - S^*(t) \quad \text{and} \quad \frac{\partial P_{\text{am}}}{\partial S}\Big(S^*(t),t\Big) = -1.$$

For the American put, the pricing problem in terms of the smooth-pasting conditions is, for $S > S^*(t)$ and $t < T$,

$$\frac{\partial P_{\text{am}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{\text{am}}}{\partial S^2} + r\,S\,\frac{\partial P_{\text{am}}}{\partial S} - r\,P_{\text{am}} = 0,$$

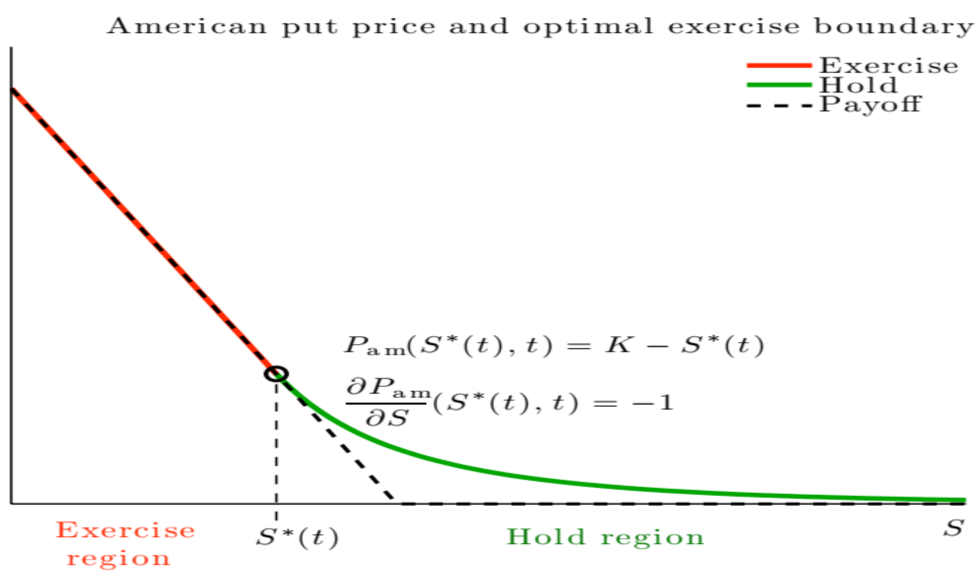
$$P_{\text{am}}(S,T) = \max(K-S)$$

$$P_{\text{am}}\Big(S^*(t),t\Big) = K - S^*(t), \quad \frac{\partial P_{\text{am}}}{\partial S}\Big(S^*(t),t\Big) = -1,$$

$$\lim_{S\to\infty} P_{\text{am}}(S,t) \to 0.$$

For $S \le S^*(t)$, $t < T$, $P_{\text{am}}(S,t) = K - S$.

Note that $S^*(t)$ is not given in advance, if it were there would be too many boundary conditions, but rather it is found as part of the solution.



In general, a smooth-pasting condition means that there is some point, $S^*(t)$, where $V_{\text{am}}(S,t)$ meets $P_{\text{o}}(S,t)$,

$$V_{\text{am}}\left(S^*(t),t\right) = P_{\text{o}}\left(S^*(t),t\right),$$

and *either*

$$V_{\text{am}}\Big(S,t\Big) > P_{\text{o}}\Big(S,t\Big) \quad \text{for} \quad S > S^*(t), \quad \textit{or}$$

$$V_{\text{am}}(S,t) > P_{\text{o}}(S,t) \quad \text{for} \quad S < S^*(t), \quad (\textit{but not both})$$

and the two curves, $V_{\text{am}}(S,t)$ and $P_{\text{o}}(S,t)$ thought of as functions of S, meet tangentially at $S^*(t)$,

$$\frac{\partial V_{\text{am}}}{\partial S}\left(S^*(t),t\right) = \frac{\partial P_{\text{o}}}{\partial S}\left(S^*(t),t\right).$$

Clearly the payoff must be differentiable at $S^*(t)$ for this to work.

It is worth emphasising that

the smooth-pasting condition does not apply to all American options.

Although it applies to the American put, it *doesn't* apply to the American digital call, for example.

Browsing the literature might give the false impression that it is a universal law. It is not. Indeed, it is known that it doesn't hold at all if the underlying price is driven by certain Levy processes.

Under certain, fairly common, circumstances smooth pasting gives the optimal-exercise strategy, but it is *not* equivalent to it in general.

Supposing for the moment, however, that it is the correct condition to apply, it is clear that if $S^*(t)$ is prescribed in advance, the smooth-pasting condition gives us two boundary conditions at the the optimal-exercise boundary and this, it turns out, is one too many for a Black–Scholes problem.

If smooth pasting applies it *determines* $S^*(t)$. In this case, we *don't* prescribe the optimal-exercise boundary, $S^*(t)$, we find it as part of the solution of the pricing problem.

This is an example of what is known as a *free boundary problem*, where the boundary is *not* prescribed in advance. Rather, it is given implicitly by an extra boundary condition and has to *found* as part of the solution of the problem.