The free boundary formulation

We have already seen this formulation for the perpetual American put.

In this formulation, we say that the option's value satisfies the Black–Scholes equation if it is optimal to hold the option and the smooth-pasting conditions where it meets the payoff. It therefore presupposes that the smooth-pasting conditions are valid.

It is implicit in the formulation that when the option value is above the payoff it is optimal to hold the option and that when it is optimal to exercise the option its value must equal the payoff.

In general, it is part of the problem is to *find* the optimal-exercise boundary — we do not prescribe it in advance.

The simplest proper case is the free boundary formulation for th American put, in which $S^*(t)$ is the optimal-exercise boundary. In th hold region, $S > S^*(t)$,

$$\frac{\partial P_{\text{am}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{\text{am}}}{\partial S^2} + r\,S\,\frac{\partial P_{\text{am}}}{\partial S} - r\,P_{\text{am}} = 0$$

with the smooth-pasting conditions

$$P_{\text{am}}\Big(S^*(t), t\Big) = K - S^*(t), \qquad \frac{\partial P_{\text{am}}}{\partial S}\Big(S^*(t), t\Big) = -1,$$

at $S = S^*(t)$. At expiry we have the usual payoff condition

$$P_{\text{am}}(S, T) = \max(K - S, 0)$$

and sometimes we also include the (redundant) condition that

$$P_{\text{am}}(S, t) \to 0 \ \text{ as } \ S \to \infty.$$

Just as for the perpetual version of this problem, if we attempt to prescribe $S^*(t)$ in advance we have too many boundary conditions.

We can not, in general, prescribe $S^*(t)$; we have to *find* it so that the smooth-pasting conditions are satisfied.

This is why the optimal-exercise boundary, $S^*(t)$, is also known as a *free boundary** and why the formulation is referred to as a free boundary formulation.

There are a number of reasons why this is *not* a particularly useful way to formulate the American option problem in general, and most of these also apply to the formulation for an American put given above.

That said, it is some times very useful. For example, it is useful for finding the behaviour of $S^*(t)$ close to expiry.

*Or, sometimes, as a *moving boundary*.

Firstly, it is almost impossible to find a global analytic solution to such a formulation. None of the standard techniques, except similarity methods, work — the smooth-pasting, or free boundary, conditions make the problem inherently non-linear and this formulation is no easier to solve analytically than any other formulation.*

If it known that there is only one free boundary then it is possible to use the Green's function† for Black–Scholes equation to reformulate the free boundary problem as a non-linear integral equation for $S^*(t)$. This integral equation can be solved numerically and, at present, this is the only way of obtaining $S^*(t)$ with very high precision.

*Local solutions, valid only near $S^*(t)$, can be found and they may be useful.

†Or the conditional risk-neutral density, which is the same thing.

Secondly, the free boundary formulation is an appalling problem to tackle numerically.

The problem lies in the fact that the S-domain on which the Black–Scholes equation is posed *changes* with t. For example, the American put problem the Black–Scholes equation is only valid on

$$S^*(t) < S < \infty,$$

and $S^*(t)$ varies with t.

To compound this problem, the free boundary conditions do not really help us when it comes to tracking $S^*(t)$ in time:* even if we know the value of $S^*(t + dt)$, how do we find $S^*(t)$ in order to know *where* to solve the Black-Scholes equation at t?

*Remember, we solve the Black–Scholes equation *backwards* in time.

Thirdly, the formulation as a free boundary problem becomes very complicated if there is more than one free boundary.

It becomes an absolute nightmare if a new free boundary suddenly appears, even in the unlikely event that we know when and where it is going to appear. A free boundary might disappear at some time, or two might move toward each other and then merge. All of these effects can easily happen, particularly if the payoff is time dependent.

A simple example of a time dependent payoff arises if we have a Bermudan option — one way to consider such an option is to think of it as an American option with a payoff that switches off, i.e. it becomes zero, when the option can't be exercised and switches back on when the option can be exercised. In this case free boundaries will disappear and then re-appear, quite possibly all over the place.

Finally, and possibly most importantly, the free boundary formulation borders on the useless for an American option written on more than a single asset.

The smooth-pasting condition, which is then the condition that the option price and its gradient* are continuous across the optimal-exercise boundary, is virtually useless for any practical purposes — it is of no help either analytically or numerically. Moreover, it is not clear that the smooth-pasting conditions are even valid at all times, at least for the payoffs we are typically interested in, where the gradients are often discontinuous across an entire surface.

*Assuming the payoff itself is smooth, this is equivalent to requiring the option value and its normal derivative to be continuous across the free boundary. If the payoff isn't smooth, then it is very likely that the smooth-pasting conditions don't apply anyway.

Optimal stopping formulation

Another way to formulate an American option problem is as explicit optimal stopping time problem. Specifically, if $\mathbb{Q}$ is the risk-neutral measure under which the price process evolves as

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t$$

and $\mathcal{T}$ is the set of all $\mathbb{Q}$-stopping times $\tau \le T$ then the price of an American option may be expressed as

$$V(S,t) = \max_{\tau\in\mathcal{T}} \mathbb{E}_t^{\mathbb{Q}}\Big[\,e^{-r(\tau-t)}P_{\mathrm{O}}(S_\tau,\tau)\,|\,S_t = S\,\Big].$$

This simply says that we choose to exercise the option at a time which maximises its value.

The optimal exercise strategy is simply the stopping time which achieves this maximum,

$$\tau^* = \arg\max_{\tau\in\mathcal{T}} \mathbb{E}_t^{\mathbb{Q}}\Big[\,e^{-r(\tau-t)}P_{\mathrm{O}}(S_\tau,\tau)\,|\,S_t = S\,\Big].$$

If there is an optimal exercise boundary, S^*, then the optimal stopping time τ^* is simply the *first* time the price process hits this boundary,

$$\tau^* = \inf_{\tau\le T}\{\tau : S_\tau = S^*(\tau)\}.$$

If we knew $S^*(t)$ in advance then the American option would be equivalent to a barrier option which immediately paid out the payoff the first time the price hit the “barrier” $S^*(t)$.

Comments about numerical solutions

In general, it is not possible to find an explicit analytic formula for the price of a time dependent American option. So, in practice, we have to resort to a *numerical* procedure in order to price and hedge an American option.

As noted above, there are three common formulations for an American style option:

- as an optimal stopping time problem;
- as a free boundary problem for the Black–Scholes equation;
- as a linear complementarity problem.

As a practical means of computation, the first formulation is useful to compute bounds on the price. This involves taking an educated guess at the optimal strategy and then performing Monte Carlo simulations.*

This approach works as well for options on many assets as it does in options on one, so we can at least find price bounds in high dimensions. *This is more than we can do with other methods in moderately or very high dimensions.*

*In this formulation, the option price is defined as the maximum expected value of the discounted payoff over all exercise strategies. By definition, an arbitrary strategy gives a lower bound on this price. There is also a dual formulation, involving a minimisation over all exercise strategies. In this version of the problem an arbitrary exercise strategy gives an upper bound.

The free boundary formulation is helpful for local analysis, but as a practical method for computing prices and deltas it is almost useless. The free boundaries involved (or not involved, as the case may be) have to be determined numerically along with the solution of the Black-Scholes equation. This is highly non-trivial, even in the simplest case of a single free boundary and only one underlying asset.

With two underlying assets it is just about possible, but very cumbersome and has to be implemented on a case-by-case basis.

With more than two underlying assets, it is utterly impractical.

The linear complementarity formulation is to find $V(S,t)$ which is at least continuous in S and such that

$$\mathcal{L}_{\mathrm{bs}}[V] \leq 0, \quad V(S,t) \geq P_{\mathrm{O}}(S,t),$$

$$\Big(V(S,t) - P_{\mathrm{O}}(S,t)\Big) \cdot \mathcal{L}_{\mathrm{bs}}[V] = 0,$$

with

$$V(S,T) = P_{\mathrm{O}}(S,T),$$

and, from the numerical point of view, we can ignore the smoothness conditions and simply find the numerical solution that has the greatest value consistent with these requirements.

Its advantages from a numerical point of view are:

- The free boundary or boundaries do not explicitly occur in the formulation;
- The free boundary or boundaries are recovered from the solution *a posteriori;*
- The method is more easily implemented numerically than any of the alternatives.

The *big* disadvantage is that it is impractical to implement it for high dimensional problems.

Appendix I: Smooth-pasting in general

Provided the payoff is continuous and at least piecewise differentiable, the optimal-exercise strategy is often determined by a smooth-pasting condition.*

Suppose our exercise strategy is to exercise the option whenever

$$S = \bar{S}(t)$$

and that the option value corresponding to this strategy is

$$V\left(S, t; \bar{S}(t)\right).$$

*Many assume that the smooth-pasting condition is universally valid even though, as we have seen, there is an abundance of simple counter examples.

We will assume that

$$P_{\text{o}}\left(\bar{S}(t), t\right) > 0$$

and that P_{o} is differentiable in S for S close enough to $\bar{S}(t)$. The exercise strategy determined by $\bar{S}(t)$ need not be optimal, but as we exercise the option when $S = \bar{S}(t)$ we must have

$$V\left(\bar{S}(t), t; \bar{S}(t)\right) = P_{\text{o}}\left(\bar{S}(t), t\right).$$

We will use $S^*(t)$ for the optimal strategy,* that is,

$$V\left(S, t; S^*(t)\right) \geq V\left(S, t; \bar{S}(t)\right) \tag{1}$$

for all relevant S and exercise strategies $\bar{S}(t)$.

*We are assuming that it exists. One of the main difficulties lies in showing that the inequality is valid for *all* S.

It is by not obvious that such an optimal strategy even exists. We will assume that it does exist and present heuristic, and not very rigorous, arguments to show that smooth pasting is the appropriate condition to determine $S^*(t)$, given smoothness conditions on the payoff and, more importantly, V as a function of $\bar{S}(t)$.*

If (1) is to hold for all $t < T$, then it certainly must hold for any particular $t < T$. Therefore, we now fix some $t < T$.

In this case S remains a variable, with either $S < \bar{S}(t)$ or $S > \bar{S}(t)$ depending on the payoff, but $\bar{S}(t)$ is effectively a parameter which we may vary and t is a parameter that does not vary.

[*] Another difficulty is in showing that the smoothness conditions on $V(S,t;\bar{S}(t))$, in $\bar{S}(t)$, are consistent with the Black–Scholes problem, in S and t.

To simplify notation write

$$x = S, \quad c = \bar{S}(t), \quad c^* = S^*(t) \quad \text{and} \quad f(x,c) = V\left(S,t;\bar{S}(t)\right).$$

The exercise condition becomes[*]

$$f(c,c) = P_{\mathrm{O}}(c)$$

and the optimisation problem is to find c^* such that

$$f(x,c^*) \geq f(x,c).$$

In the original problem, it is safe to assume that if $S^*(t)$ exists at all then it is an internal maximum since, in principle, $0 < \bar{S}(t) < \infty$.

[*] We suppress explicit dependence of the payoff on t as we are holding t fixed.

Thus, we can assume that c^* is an interior maximum, and so

$$f(x,c^*) \geq f(x,c^*+\epsilon)$$

for all small ϵ. Assuming[*] that f is at least twice differentiable in c, this implies that

$$f(x,c^*) \geq f(x,c^*) + \epsilon\,\frac{\partial f}{\partial c}(x,c^*) + \tfrac{1}{2}\,\epsilon^2\,\frac{\partial^2 f}{\partial c^2}(x,c^*) + \cdots .$$

Therefore, we must have[†]

$$\frac{\partial f}{\partial c}(x,c^*) = 0 \ \text{ and } \ \frac{\partial^2 f}{\partial c^2}(x,c^*) < 0.$$

[*] This is the same as assuming that $V(S,t;\bar{S})$ is twice differentiable in its third argument.

[†] We can, in fact, have $(\partial^2 f/\partial c^2)(x,c^*) = 0$ provided that the first non-zero partial derivative at c^* is an even one and has a negative value.

We now write the exercise condition in the form

$$f(x,x) = P_{\text{o}}(x)$$

and take its total derivative,*

$$\frac{dP_{\text{o}}}{dx}(x) = \frac{d}{dx}\,f(x,x) = \frac{\partial f}{\partial x}(x,x) + \frac{\partial f}{\partial c}(x,x)$$

At our optimal-exercise boundary, $x = c^*$,

$$\frac{\partial f}{\partial c}(x,c^*) = 0$$

for all x, so it follows that

$$\frac{\partial f}{\partial x}(c^*,c^*) = \frac{dP_{\text{o}}}{dx}(c^*).$$

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*Here $\partial/\partial x$ denotes the derivative with respect to the first argument, $\partial/\partial c$ with respect to the second.

In our original variables this becomes

$$\frac{\partial V}{\partial S}\left(S^*(t),t;S^*(t)\right) = \frac{\partial P_{\text{o}}}{\partial S}\left(S^*(t),t\right),$$

which is the usual smooth-pasting condition.

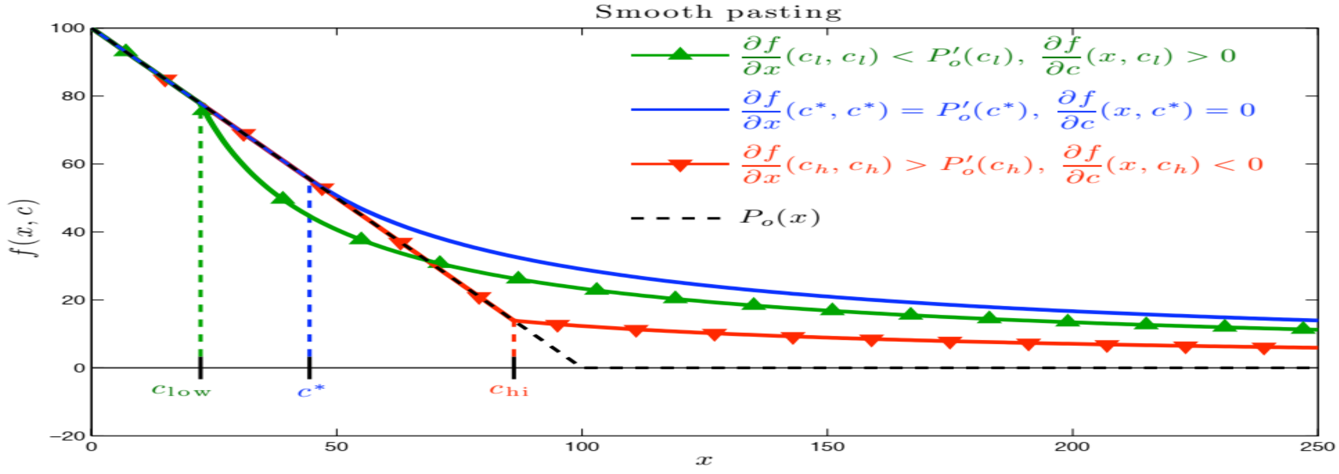
Furthermore, it is clear that if we choose c such that

$$\frac{\partial f}{\partial x}(c,c) \neq \frac{dP_{\text{o}}}{dx}(c)$$

then

$$\frac{\partial f}{\partial c} \neq 0,$$

and so c can not be an optimal value, because we can make the value of $f(x,c)$ greater by either increasing or decreasing c, according to the sign of $\partial f/\partial c$, as illustrated in the following figure.



For the argument above to be made rigorous, we have to show that it is possible to find a function

$$V = V(S, t; \bar{S})$$

which satisfies the Black–Scholes equation in S and t and

$$V(\bar{S}, t; \bar{S}) = P_{\text{o}}(\bar{S}, t),$$

in $\bar{S}$, together with a function $S^*(t)$ such that

$$\frac{\partial V}{\partial \bar{S}}\Big(S, t; S^*(t)\Big) = 0$$

for all S and $t < T$,

$$\frac{\partial^2 V}{\partial \bar{S}^2}\Big(S, t; S^*(t)\Big) < 0$$

for all S and $t < T$. These are *sufficient* conditions for smooth pasting to hold.

It is relatively easy to show that they are satisfied if the payoff is a twice-differentiable concave function of S,

$$P_{\text{o}}(S) \;\; \text{with } P_{\text{o}}''(S) \geq 0,$$

by solving the Black–Scholes equation subject to $V(S, T) = P_{\text{o}}(S)$ and $V(\bar{S}, t) = P_{\text{o}}(\bar{S})$, then differentiating the solution with respect to $\bar{S}$ — this is simply a more complicated version of what we did for the steady-state American put.*

The case where $P_{\text{o}}(S)$ is not concave is more difficult, because there may be more than one optimal-exercise point or smooth pasting may not even be valid.

*Determining the smoothness of $S^*(t)$, as a function of time, is a truly nasty problem and we will say no more about it.

Normally we are not particularly interested in the value of an American option with a suboptimal-exercise strategy $\bar{S}(t)$, only the value with the optimal strategy $S^*(t)$, so we suppress the dependence on $S^*(t)$ and write

$$V_{\text{am}}(S, t) = V\Big(S, t; S^*(t)\Big).$$

The smooth-pasting conditions become

$$\begin{aligned} V_{\text{am}}\left(S^*(t),t\right) &= P_{\text{o}}\left(S^*(t),t\right), \\ \frac{\partial V_{\text{am}}}{\partial S}\left(S^*(t),t\ \right) &= \frac{\partial P_{\text{o}}}{\partial S}\left(S^*(t),t\right), \end{aligned}$$

where it is understood that, in conjunction with the underlying Black-Scholes problem for the option's price, these two equations implicitly determine the optimal-exercise boundary, $S^*(t)$.

Sufficient conditions for these to be valid are listed above.

In particular, if the payoff* is a convex C^2-function of only S, then these conditions are valid.†

Recall, however, that we have already seen one example where they are *not* valid and, in fact, could not possibly be valid because the smooth-pasting condition simply doesn't hold.

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[*] To see how restrictive these conditions are, note that the payoffs for puts and calls don't satisfy them as these are not twice differentiable.

[†] Strictly speaking, we also need the condition that the underlying process for S has continuous paths.