Asian Options

These options have payoffs which depend on some average property of the underlying asset's price over the life, or some part of the life, of the option.

The average is less volatile than the asset itself (but more volatile than a constant such as a strike), so options may be cheaper and less subject to manipulation.

Asian options may be found embedded in structured products. Many other contracts have Asian features, for example, volatility or variance swaps.

Many Asians use a simple arithmetic average of the asset price, and have a payoff which depends upon the variable

$$\frac{1}{T}\int_0^T S_t\,dt.$$

It is usually more convenient to work with the cumulative sum*

$$A_t = \int_0^t S_u\,du, \qquad dA_t = S_t\,dt.$$

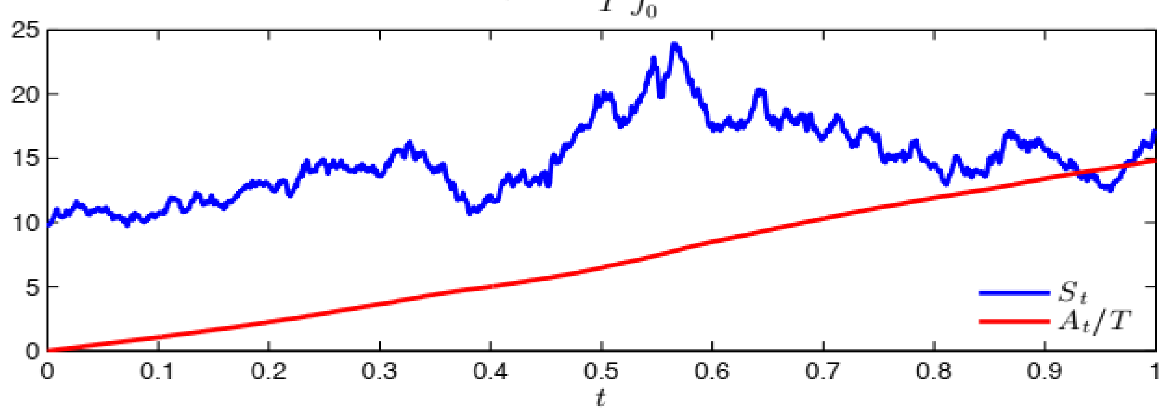
and write

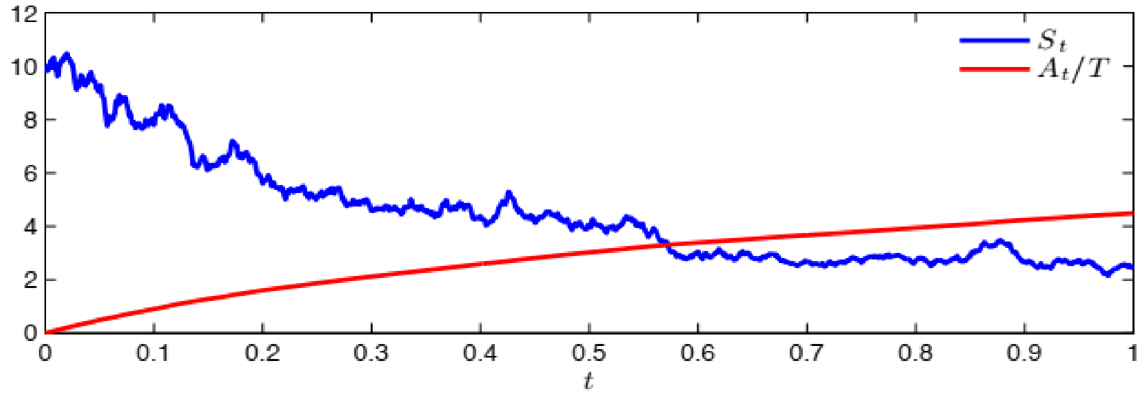
$$\frac{1}{T}\int_0^T S_t\,dt = A_T/T.$$

The following figure shows two simulated $(S_t, A_t/T)$ paths.

*As opposed to working with, say, the running average $\frac{1}{t}\int_0^t S_\tau\,d\tau$.

$$A_t/T = \frac{1}{T}\int_0^t S_u\,du$$





There are very few formulæ for options involving arithmetic averages. There are formulæ for options based on a geometric average which, at expiry, is

$$\exp\left(\frac{1}{T}\int_0^T \log(S_t)\,dt\right).$$

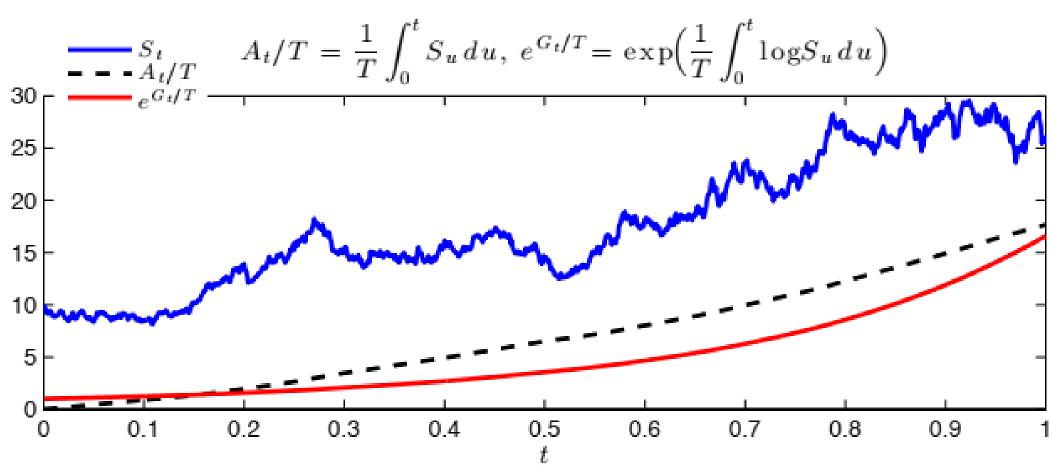
In numerical work, it is often more convenient to work*

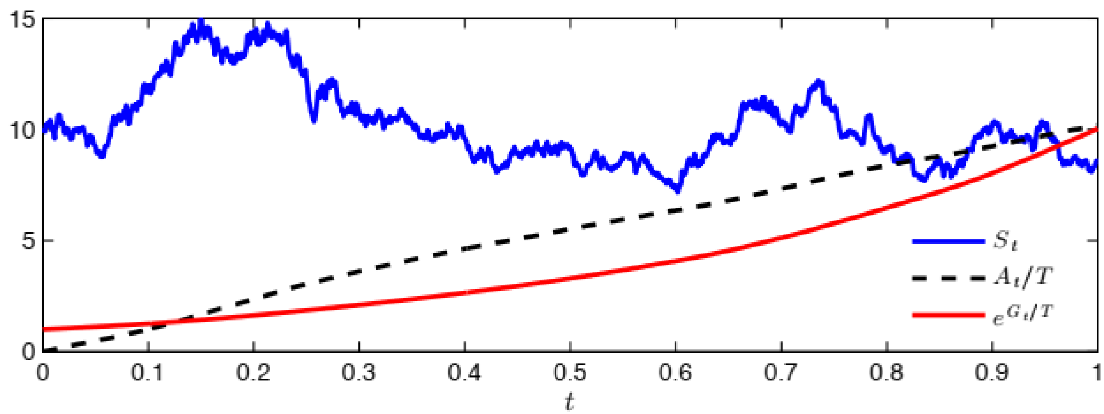
$$G_t = \int_0^t \log(S_\tau)\,d\tau, \quad dG_t = \log(S_t)\,dt$$

and write

$$\exp\left(\frac{1}{T}\int_0^T \log(S_t)\,dt\right) = e^{G_T/T}.$$

*As opposed to working with the running geometric average $\exp\left(\frac{1}{t}\int_0^t \log(S_\tau)\,d\tau\right)$.





Rate and strike options

The adjectives *rate* and *strike* are used fairly widely to classify certain types of Asian options:*

- A *rate* option has the same payoff except the spot price S_T is replaced by the average $\mathcal{A}_T$,

$$\begin{array}{ccc} \text{European} & & \text{Asian} \\ P_o(S_T, K) & \longrightarrow & P_o(\mathcal{A}_T, K); \end{array}$$

- A *strike* option has the same payoff except the fixed strike K is replaced by the average $\mathcal{A}_T$,

$$\begin{array}{ccc} \text{European} & & \text{Asian} \\ P_o(S_T, K) & \longrightarrow & P_o(S_T, \mathcal{A}_T). \end{array}$$

*Here $\mathcal{A}_t$ is a generic average, it may be continuously or discretely sampled and it may be an arithmetic, geometric or any other form of average.

A continuously sampled arithmetic-average *strike* call has payoff

$$\max\left(S_T - \frac{1}{T}\int_0^T S_u\,du,\, 0\right).$$

It is more likely to pay out if the asset price is volatile, as this increases the chances that S_T is a long way from the average at expiry.

A continuously sampled arithmetic-average *rate* call has payoff

$$\max\left(\frac{1}{T}\int_0^T S_u\,du - K,\, 0\right),$$

where K is the option's fixed strike. It is similar to a vanilla call, except the asset price is replaced by the average price, which reduces the effects of volatility.

A continuously sampled geometric-average *strike* call has payoff

$$\max\left(S_T - \exp\Big(\frac{1}{T}\int_0^T \log(S_u)\,du\Big),\, 0\right),$$

while a continuously sampled geometric-average *rate* call has payoff

$$\max\left(\exp\Big(\frac{1}{T}\int_0^T \log(S_u)\,du\Big) - K,\, 0\right).$$

In these cases, the continuously sampled average involves a term of the form

$$\frac{1}{T}\int_0^T F(S_u, u)\,du,$$

In the first two cases $F(S_u, u) = S_u$ and $F(S_u, u) = \log(S_u)$ in the last two.

Other possibilities are an exponentially weighted arithmetic average,

$$\frac{\lambda}{1 - e^{-\lambda T}}\int_0^T e^{-\lambda(T-u)} S_u\,du,$$

where $\lambda > 0$, or even a harmonic average*

$$\left(\frac{1}{T}\int_0^T \frac{du}{S_u}\right)^{-1},$$

both of which also can be expressed in terms of

$$\int_0^T F(S_u, u)\,du$$

for suitable functions $F(S, u)$.

*So far as I know, options involving harmonic averages are purely of academic interest; I've not encountered one in the wild.

The running sum

The options above are path-dependent. Their payoffs depend not only on S_T but also on an average. So, in addition to S_t and t, we need another process to encapsulate what we know (at time t) about the average.

For a payoff involving $\frac{1}{T}\int_0^T F(S_u, u)\,du$ it is convenient to define a generic running sum,

$$I_t = \int_0^t F(S_u, u)\,du$$

which encapsulates what we know about this integral at time t.

If we make the usual Black-Scholes assumption that

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t \tag{1}$$

and note that*

$$dI_t = F(S_t, t)\,dt \tag{2}$$

it is clear that the pair (S_t, I_t) is a Markov process.

It follows that the price process V_t of an option dependent on S_t, I_t and t may be expressed in terms of a value function, $V(S, I, t)$, as

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[\,V_T\,] = V(S_t, I_t, t).$$

*Strictly speaking, we require $F(S, t)$ to be continuous in S and t to do this.

Itô's lemma

Assuming (1) and (2), Itô's formula for $V_t = V(S_t, I_t, t)$ is

$$
\begin{aligned}
dV_t &= \frac{\partial V}{\partial t}\,dt + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2}\,dt + \frac{\partial V}{\partial S}\,dS_t + \frac{\partial V}{\partial I}\,dI_t \\
&= \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2} + F(S_t, t)\,\frac{\partial V}{\partial I}\right) dt + \frac{\partial V}{\partial S} dS_t.
\end{aligned}
$$

Note that the new variable I only occurs as a first derivative and that dS_t is the only random term here; $dI_t = F(S_t, t)\,dt$ does not introduce any *new* source of randomness.

The Black–Scholes equation

Construct the usual portfolio, with market price

$$\Pi_t = V_t - \Delta_t\, S_t$$

and self-financing hedging strategy

$$d\Pi_t = dV_t - \Delta_t\, dS_t.$$

As always, choose

$$\Delta_t = \frac{\partial V}{\partial S}(S_t, t)$$

to instantaneously eliminate risk.

Assuming there are no exploitable arbitrage opportunities, with this choice of Δ_t we must have

$$d\Pi_t = r\, \Pi_t\, dt$$

which when written out in full becomes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r\, S_t \frac{\partial V}{\partial S} + F(S_t, t) \frac{\partial V}{\partial I} - r\, V_t = 0.$$

As this must hold for all attainable* values of (S_t, I_t, t), we have

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S \frac{\partial V}{\partial S} + F(S, t) \frac{\partial V}{\partial I} - r\, V = 0. \qquad (3)$$

*Here $0 \le t < T$ and any $S_t > 0$ is attainable so (3) holds for all $0 \le t < T$ and $S > 0$. In order to determine the range for I in this equation, however, we need to know $F(S, t)$ in order to determine the values of I_t that are attainable.

With the standard modifications to this argument to accommodate a continuous dividend yield, y, we obtain the slightly more general equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S \frac{\partial V}{\partial S} + F(S, t) \frac{\partial V}{\partial I} - r\, V = 0. \qquad (4)$$

In both cases the payoff gives us the terminal condition

$$V(S, I, T) = P_o(S, I). \qquad (5)$$

Thus, the general pricing problem is either (3) or (4) subject to the terminal condition (5). In all cases the problem is posed for all $S > 0$ and all $t < T$, and the range of I is determined by the attainable values of I_t.

Continuous arithmetic-average options

In this case it is convenient to replace the generic running sum I_t with the more specific A_t,

$$A_t = \int_0^t S_u \, du$$

corresponding to $F(S, t) = S$. This gives the Black-Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y) S \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - r\, V = 0, \qquad (6)$$

subject to a payoff of the form*

$$V(S, A, T) = P_o(S, A). \qquad (7)$$

*In both (6) and (7) we have $S > 0$, $A > 0$ and in (6) we also have $t < T$

There is no, useful, general solution to (6), but for the linear payoff

$$V(S, A, T) = a + b\,A + c\,S.$$

there is a solution of the form

$$V(S, A, t) = \alpha(t) + \beta(t)\,A + \gamma(t)\,S \tag{8}$$

Evidently

$$\frac{\partial V}{\partial t} = \dot{\alpha}(t) + \dot{\beta}(t)\,A + \dot{\gamma}(t)\,S, \quad \frac{\partial V}{\partial A} = \beta(t), \quad \frac{\partial V}{\partial S} = \gamma(t), \quad \frac{\partial^2 V}{\partial S^2} = 0$$

Thus, if we substitute (8) into (6) and equate the coefficients of S, A and the term independent of both of these to zero we obtain ordinary differential equations for $\alpha(t)$, $\beta(t)$ and $\gamma(t)$.

These are

$$\begin{aligned} \dot{\alpha} &= r\,\alpha, & \alpha(T) &= a, \\ \dot{\beta} &= r\,\beta, & \beta(T) &= b, \\ \dot{\gamma} &= y\,\gamma - \beta, & \gamma(T) &= c. \end{aligned}$$

The solutions are*

$$\begin{aligned} \alpha(t) &= a\,e^{-r(T-t)}, \\ \beta(t) &= b\,e^{-r(T-t)}, \\ \gamma(t) &= c\,e^{-y(T-t)} + b\,(e^{-y(T-t)} - e^{-r(T-t)})/(r-y). \end{aligned} \tag{9}$$

*In the case that $r = y$, take limits to get $\gamma(t) = (c + b(T-t))\,e^{-r(T-t)}$.

We may interpret these in the obvious fashion:

- $\alpha(t)$ is cash growing at the risk-free rate;

- $\beta(t)$ is the cash needed to cover the known value of $b\int_0^t S_u\,du$ when this value is projected forward to expiry;

- The first term in $\gamma(t)$ is the number of underlying we need so that if all dividends are reinvested we have c underlying at expiry;

- The second term in $\gamma(t)$ is the number of underlying we need in order to replicate $b\int_t^T S_u\,du$, by selling $b\,e^{-r(T-t)}\,dt$ assets during each interval $[t, t+dt)$. The terms involving y arise because we also reinvest dividends from this component of our holding.

Arithmetic average put-call parity

For arithmetic-average strike puts and calls we have

$$C_{\mathsf{as}}(S,A,T) - P_{\mathsf{as}}(S,A,T) = S - A/T.$$

Hence, from (8) and (9),

$$\begin{aligned} C_{\mathsf{as}}(S,A,t) - P_{\mathsf{as}}(S,A,t) \;&=\; S\,e^{-y(T-t)} - e^{-r(T-t)}\,A/T \\ &+\; \left(\frac{e^{-r(T-t)} - e^{-y(T-t)}}{(r-y)\,T}\right) S \end{aligned}$$

For arithmetic-average rate puts and calls

$$C_{\mathsf{ar}}(S,A,T) - P_{\mathsf{ar}}(S,A,T) = A/T - K.$$

Hence, again from (8) and (9),

$$\begin{aligned} C_{\mathsf{ar}}(S,A,t) - P_{\mathsf{ar}}(S,A,t) \;&=\; (A/T - K)\,e^{-r(T-t)} \\ &-\; S\left(e^{-r(T-t)} - e^{-y(T-t)}\right)/(r-y)T. \end{aligned}$$

All these formulæ for linear payoffs, including the parity results, are *model independent*. This is because there is a model independent way of replicating

$$A_T - A_t = \int_t^T S_u \, du.$$

To replicate the running sum's value at expiry is a relatively simple thing to do. At time $t < T$ we need to have cash equal to the present value of the current running sum at expiry,

$$A_t \, e^{-r(T-t)}$$

and a strategy to generate, at expiry, cash equal to*

$$\int_t^T S_u \, du.$$

If, during each interval $[u, u + du)$ between t and T, we invest

$$e^{-r(T-u)} \, S_u \, du$$

at the risk free rate, at expiry we will have exactly the required sum.

*There is an analogous strategy to replicate an arithmetic average when it computed using samples from a discrete set of times.

Suppose that at time $t \leq u < T$ we have $n(u)$ underlying set aside for this purpose. During $[u, u + du)$ we receive

$$y \, n(u) \, S_u \, du$$

in cash as dividends, but we need to generate a total of

$$e^{-r(T-u)} \, S_u \, du.$$

We obtain the difference by selling

$$\left(e^{-r(T-u)} - y \, n(u)\right) du$$

underlying at price S_u. Thus

$$\frac{dn}{du} = y \, n - e^{-r(T-u)}. \tag{10}$$

As we need only replicate the running sum, we need no underlying assets at expiry, hence

$$n(T) = 0.$$

Solving (10) subject to this terminal condition gives

$$n(t) = \frac{e^{-y(T-t)} - e^{-r(T-t)}}{r - y}.$$

The strategy to replicate $\int_t^T S_u \, du$ is, at each in $u \in [t, T)$, take the dividend and add to this the cash generated by selling $dn(u)$ underlying, for a total of $e^{-r(T-u)} S_u \, du$, then invest this sum at the risk-free rate until expiry, when it will be worth $S_u \, du$.