The continuous arithmetic-average strike call

This option has payoff $\max\left(S_T - \frac{1}{T}\int_0^T S_u\,du,\, 0\right)$, so here

$$A_t = \int_0^t S_u\,du \quad \text{and} \quad F(S_t, t) = S_t.$$

The problem for $V(S, A, t)$ is thus

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\,S\,\frac{\partial V}{\partial S} + S\,\frac{\partial V}{\partial A} - r\,V = 0, \qquad (11)$$

$$V(S, A, T) = \max\left(S - A/T, 0\right).$$

Similarity reduction

There is a similarity reduction* which simplifies problem (11). This is because each of V, S and A/T must scale in the same way; the problem must be independent of which unit of currency we choose to measure prices in. Thus, (11) must be (and is) invariant under the scaling

$$V' = \lambda\,V, \quad S' = \lambda\,S, \quad A' = \lambda\,A,$$

for *any* $\lambda > 0$, in the sense that

$$\frac{\partial V'}{\partial t} + \tfrac{1}{2}\sigma^2 S'^2 \frac{\partial^2 V'}{\partial S'^2} + (r - y)\,S'\,\frac{\partial V'}{\partial S'} + S'\,\frac{\partial V'}{\partial A'} - r\,V' = 0,$$

$$V'(S', A'T) = \max(S' - A'/T, 0).$$

*It is equivalent to a wise choice of numeraire.

Since the problem is invariant under these transformations, we may look for solutions in terms of other invariants of the transformation. A set of obvious candidates are

$$\phi = V/S, \quad \xi = A/S.$$

We know that $S > 0$, so we are in no danger of dividing by zero.

The payoff condition is

$$V(S, A, T) = S\,\phi(\xi, T) = \max\left(S - A/T, 0\right),$$

so at expiry ϕ is a function of ξ only,

$$\phi(\xi, T) = \max\left(1 - \xi/T, 0\right).$$

So we introduce these new variables

$$\phi = \frac{V}{S}, \quad \xi = \frac{A}{S},$$

and assume* that the problem has a solution of the form

$$V(S, A, t) = S\,\phi(\xi, t).$$

This is simply a change of numeraire; we are now measuring all prices relative to S_t rather than a risk-free bond.

*We show below this form is self-consistent and, in principle, we could establish that $\phi(\xi, t)$ exists. Then we could, in principle, establish uniqueness of solutions to the original problem to show that this is the *only* solution.

The chain rule establishes that

$$\frac{\partial V}{\partial t} = S\,\frac{\partial \phi}{\partial t}, \quad \frac{\partial V}{\partial S} = \phi - \xi\,\frac{\partial \phi}{\partial \xi},$$

and

$$\frac{\partial V}{\partial A} = \frac{\partial \phi}{\partial \xi}, \quad \frac{\partial^2 V}{\partial S^2} = \frac{\xi^2}{S}\,\frac{\partial^2 \phi}{\partial \xi^2}.$$

When we substitute these expressions into the Black-Scholes equation in (11) we find that $S > 0$ drops out as it occurs only as a common factor multiplying each of

$$\frac{\partial \phi}{\partial t}, \quad \frac{\partial \phi}{\partial \xi} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial \xi^2}.$$

The net result is a *lower* dimensional problem, namely

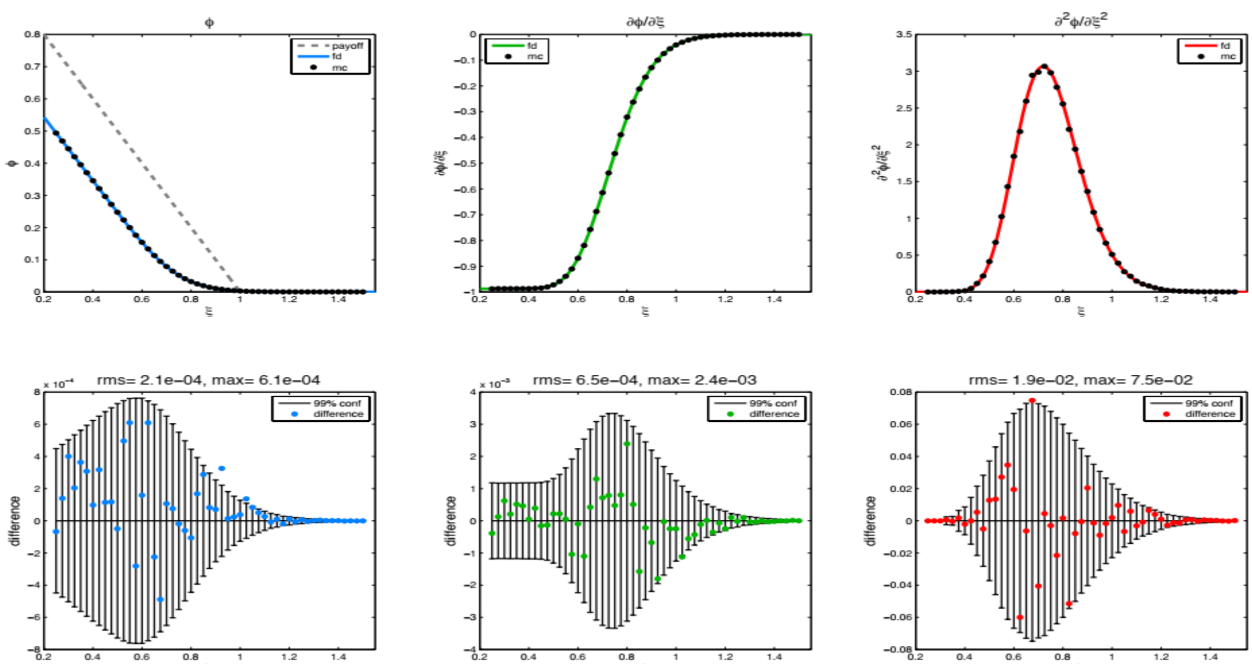
$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 \phi}{\partial \xi^2} + (1 + (y-r)\,\xi)\frac{\partial \phi}{\partial \xi} - y\,\phi = 0, \qquad \phi(\xi, T) = \max(1 - \xi/T). \tag{12}$$

This may be solved numerically either by a finite-difference scheme or by Monte Carlo. Monte Carlo is possible as the solution may be written in the form

$$\phi(\xi, t) = e^{-y(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\,\max(1 - \xi_T/T, 0)\,|\,\xi_t = \xi\,\right],$$

where the expectation $\mathbb{E}_t^{\mathbb{Q}}$ is with respect to the process

$$d\xi_t = (\,1 + (y-r)\,\xi_t\,)\,dt + \sigma\,\xi_t\,dW_t^{\mathbb{Q}}.$$



Numerical solutions of (12); $r = 0.05$, $y = 0.05$, $\sigma = 0.30$ and $T = 0.25$

Average rate options with geometric averaging

Suppose that the option payoff depends on the continuously sampled geometric average of S,

$$\exp\left(\frac{1}{T}\int_0^T \log(S_u)\,du\right).$$

To be specific, consider a European geometric average rate call with payoff

$$\max\left(e^{G_T/T}-K,0\right) \quad \text{where} \quad G_t=\int_0^t \log(S_u)du;$$

here it is convenient to replace the generic I_t with G_t.

The problem for $V(S,G,t)$ is

$$\begin{gathered}\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2\,S^2\frac{\partial^2 V}{\partial S^2}+(r-y)\,S\,\frac{\partial V}{\partial S}+\log S\,\frac{\partial V}{\partial G}-r\,V=0,\\ V(S,G,T)=\max\left(e^{G/T}-K,0\right).\end{gathered} \tag{13}$$

As we know, $\log S_t$ has a special role in a Geometric Brownian Motion; in terms of $x_t=\log(S_t)$, GBM becomes a brownian motion with drift and the Black-Scholes equation in S becomes a constant coefficient equation in x. It also means there is a (different) similarity reduction for this case.*

*If $S\to\lambda S$ then $G\to G+t\log\lambda$, so G/S is not an invariant. However, $G-t\,\log S$ *is* an invariant and this is essentially how the similarity variables below arise.

If we first set

$$x=\log(S)$$

then the partial differential equation in (13) reduces to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - y - \tfrac{1}{2}\sigma^2)\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial G} - r\,V = 0.$$

Now, if we note that at expiry our payoff depends only on G and K, but *not* on x, then we can reduce the terms

$$\frac{\partial V}{\partial t} + x\frac{\partial V}{\partial G}$$

to a single derivative by introducing the new variables

$$x' = G + (T - t)\,x, \quad t' = t.$$

In terms of x' and t' we find the problem reduces to

$$\frac{\partial V}{\partial t'} + \tfrac{1}{2}\,\sigma^2\,(T - t')^2\,\frac{\partial^2 V}{\partial x'^2} + (r - y - \tfrac{1}{2}\sigma^2)(T - t')\,\frac{\partial V}{\partial x'} - r\,V = 0,$$

$$V(x', T) = \max\left(e^{x'/T} - K, 0\right).$$

The form of the payoff here suggests setting

$$y = \frac{x'}{T} = \frac{G + (T - t)\,\log(S)}{T}.$$

This gives

$$\frac{\partial V}{\partial t'} + \tfrac{1}{2}\,\sigma^2\left(\frac{T - t'}{T}\right)^2 \frac{\partial^2 V}{\partial y^2} + (r - y - \tfrac{1}{2}\sigma^2)\left(\frac{T - t'}{T}\right)\frac{\partial V}{\partial y} - r\,V = 0,$$

$$V(y, T) = \max\left(e^y - K, 0\right).$$

Finally, to see that this is simply the usual Black–Scholes equation with time dependent parameters, put

$$S' = e^y = e^{G/T}\,S^{(1 - t/T)}.$$

This turns the problem into

$$\frac{\partial V}{\partial t'} + \frac{\sigma^2}{2}\left(\frac{T-t'}{T}\right)^2 S'^2 \frac{\partial^2 V}{\partial S'^2} + \left(r - y - \frac{\sigma^2 t'}{2\,T}\right)\left(\frac{T-t'}{T}\right) S' \frac{\partial V}{\partial S'} - r\,V = 0,$$

$$V(S', T) = \max(S' - K, 0).$$

In terms of S' and $t' = t$, the problem reduces to a Black–Scholes call but with time dependent parameters.

Recall that to solve a Black-Scholes problem with time dependent parameters, we need the $T - t$ averaged values of the parameters.

These are (relatively) easily computed and the net result is the price for a continuous geometric average rate call;

$$C_{\mathsf{gr}}(S, I, t) = S'\,e^{-\hat{y}(T-t)}\,\mathsf{N}\left(\hat{d}_+\right) - K\,e^{-r(T-t)}\,\mathsf{N}\left(\hat{d}_-\right),$$

where

$$S' = e^{G/T}\,S^{(1-t/T)}, \quad \hat{d}_\pm = \frac{\log\left(S'/K\right) + (r - \hat{y} \pm \frac{1}{2}\hat{\sigma}^2)(T-t)}{\hat{\sigma}\sqrt{T-t}},$$

$$\hat{y} = \tfrac{1}{2}r(1 + t/T) + \tfrac{1}{2}y(1 - t/T) + \tfrac{1}{12}\sigma^2(1 - t/T)(1 + 2t/T), \qquad (14)$$

$$\hat{\sigma}^2 = \tfrac{1}{3}\,\sigma^2(1 - t/T)^2.$$

The trick works with *any* payoff which depends on $G_T = \int_0^T \log(S_u)\,du$, but *not* on S_T. We simply need to express the payoff in terms of S' and make the above substitutions into the Black–Scholes formula for the price of the option with that payoff.

A geometric-average rate put has the payoff

$$P_{\mathsf{gr}}(S,G,T) \;=\; \max(K - e^{G/T}, 0) \;=\; \max(K - S', 0)$$

and so the Black-Scholes' value is

$$P_{\mathsf{gr}}(S,G,t) \;=\; K e^{-r(T-t)}\,\mathsf{N}(-\widehat{d}_-) \;-\; S' e^{-\widehat{y}(T-t)}\,\mathsf{N}(-\widehat{d}_+),$$

where the symbols are as defined in (14).

A geometric-average rate digital call's payoff is

$$C^{\mathsf{d}}_{\mathsf{gr}}(S,G,T) \;=\; \mathcal{H}(e^{G/T} - K) \;=\; \mathcal{H}(S' - K)$$

and its Black-Scholes' value is

$$C^{\mathsf{d}}_{\mathsf{gr}}(S,G,t) = e^{-r(T-t)}\,\mathsf{N}\left(\widehat{d}_-\right).$$

A geometric-average rate digital put's payoff is

$$P^{\mathsf{d}}_{\mathsf{gr}}(S,G,T) \;=\; \mathcal{H}(K - e^{G/T}) \;=\; \mathcal{H}(K - S')$$

and the Black-Scholes' value is

$$P^{\mathsf{d}}_{\mathsf{gr}}(S,G,t) = e^{-r(T-t)}\,\mathsf{N}\left(-\widehat{d}_-\right).$$

Put-call parity formulæ

The formulæ above imply that *in the Black-Scholes model* there is a put-call parity result,

$$C_{\mathsf{gr}} - P_{\mathsf{gr}} = e^{-\widehat{y}(T-t)}\,S' - e^{-r(T-t)}\,K.$$

This is *not* model independent as there is no model independent way of replicating a *geometric* average.

The digital put-call parity result is simply

$$C^{\mathsf{d}}_{\mathsf{gr}} + P^{\mathsf{d}}_{\mathsf{gr}} = e^{-r(T-t)},$$

which clearly *is* model independent.