Probability method

Another way to find formulae for option's with payoffs that depend *only* on geometric averages is a follows.

The continuous geometric average also may be defined as

$$\widehat{G}_t = \exp\left(\frac{1}{t}\int_0^t \log(S_u)\,du\right).$$

In the Black–Scholes framework, the risk-neutral process for S_t is

$$\frac{dS_t}{S_t} = (r-y)\,dt + \sigma\,dW_t$$

and hence

$$S_t = S_0 \exp\left((r-y-\tfrac{1}{2}\sigma^2)t + \sigma\,W_t\right).$$

Thus

$$\begin{aligned}\log(\widehat{G}_t) &= \log(S_0) + \frac{1}{t}\int_0^t \left((r-y-\tfrac{1}{2}\sigma^2)\,u + \sigma\,W_u\right)\,du \\ &= \log(S_0) + \tfrac{1}{2}(r-y-\tfrac{1}{2}\sigma^2)\,t + \frac{\sigma}{t}\int_0^t W_u\,du\end{aligned}$$

Integration by parts establishes that

$$\int_0^t W_u\,du = \int_0^t (t-u)\,dW_u$$

is a zero-mean normally distributed variable with variance

$$\int_0^t (t-\tau)^2\,d\tau = \tfrac{1}{3}t^3.$$

This shows that

$$\frac{\sigma}{t}\int_0^t W_u\,du \sim \mathcal{N}\left(0,\,\tfrac{1}{3}\sigma^2\,t\right).$$

Therefore the risk-neutral process for $\widehat{G}_t$ can be written as

$$\log(\widehat{G}_t) = \log(S_0) + \tfrac{1}{2}\left(r-y-\tfrac{1}{2}\sigma^2\right)t + \frac{1}{\sqrt{3}}\,\sigma\,\widehat{W}_t$$

which implies that*

$$\frac{d\widehat{G}_t}{\mathcal{G}_t} = \tfrac{1}{2}\left(r - y - \tfrac{1}{6}\sigma^2\right)dt + \tfrac{1}{\sqrt{3}}\,\sigma\,d\widehat{W}_t. \tag{15}$$

If we have a payoff of the form $V_T = P_o(\widehat{G}_T)$ then we can regard the option price as a function only of $\widehat{G}_t$ and t, i.e., $V_t = V(\widehat{G}_t, t)$, and use the GBM (15) as the risk-neutral SDE to obtain a Black-Scholes equation for $V(\widehat{G}, t)$ in the usual way.

*If averaging starts at $t = 0$ we also know that $\widehat{G}_0 = S_0$.

More general Asian options

It is very easy to vary the framework described above. For example, the averages may be weighted; they need not be arithmetic or geometric, but can be other functions of the asset price, and so on. This is true whether the average is sampled continuously or discretely.

The **Asian Tail** describes an option where the Asian feature is only active for part of the option's life—often the last part (it is found in pension plans and 'guaranteed equity' products). This insures the holder against last-minute fluctuations in the asset price. They are easy to value; just solve as an Asian option while the Asian feature is active and as a normal European option when it is not (in much the same way that a forward-start option is valued).

Other approaches to average rate options

- Geman–Yor for the continuous case
  - formulate the problem for the Laplace transform of the price;
  - perform a contour integration of a hypergeometric function (to invert a Laplace transform);
  - best done in an environment that can cope with complicated complex functions (and that can handle branch cuts).
- Vecer PDE formulation for continuous and discrete cases; J. Comp Fin. **4**, 2001, 105–113 and RISK, June 2002.
- Asymptotic expansion approximations; Dewynne & Shaw, EJAM **19**, 2008, 353–391 and Siyanko, EJAM **23**, 2012, 395–415.

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Appendix: Geman & Yor simplified

This is essentially a way of obtaining the Geman-Yor result, amongst others, for arithmetic Asians. These all satisfy

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S\, \frac{\partial V}{\partial S} + S\,\frac{\partial V}{\partial A} - r\,V = 0.$$

First, the $-r\,V$ term can be eliminated by setting

$$V(S,A,t) = e^{-r(T-t)}\,U(S,A,t).$$

This preserves the payoff and U satisfies

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r-y)\, S\, \frac{\partial U}{\partial S} + S\,\frac{\partial U}{\partial A} = 0.$$

Autonomy

Both of the equations above are autonomous in A, that is, for any constant A_0 the equation is the same in terms of

$$A' = A - A_0$$

as it is in terms of A.

How we exploit this depend on the particular payoff. There are two important cases:

Average *rate* options, for example, $U(S,A,T) = \max(A/T - K, 0)$

Average *strike* options, for example, $U(S,A,T) = \max(S - A/T, 0)$

Scaling, rediscounting and time reversal

It is convenient to use $S > 0$ as our basic measure of value and write

$$U(S, A', t) = S\,\phi(\xi, t), \quad \xi = A'/S\,T.$$

In terms of these variables, (16) reduces to

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\sigma^2\,\xi^2\,\frac{\partial^2 \phi}{\partial \xi^2} + \frac{1}{T}\frac{\partial \phi}{\partial \xi} + (r - y)\left(\phi - \xi\,\frac{\partial \phi}{\partial \xi}\right) = 0.$$

The payoffs become:

Average *rate:* $\phi(\xi, T) = \max(\xi, 0)$

Average *strike:* $\phi(\xi, T) = \max(1 - \xi, 0)$

In the case of average *rate* options we set $A' = A - K\,T$.

In the case of average *strike* options we leave A unchanged, $A' = A$.

As the PDE is autonomous in A, in both cases we have

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 U}{\partial S^2} + (r - y)S\,\frac{\partial U}{\partial S} + S\,\frac{\partial U}{\partial A'} = 0 \qquad (16)$$

but, with the examples above, the terminal conditions become

Average *rate* option: $U(S, A', T) = \max(A'/T, 0)$

Average *strike* option: $U(S, A', T) = \max(S - A'/T, 0)$

We can perform a further simplification by "re-discounting" with

$$\phi = e^{(r-y)(T-t)}\psi.$$

and then non-dimensionalising time (to get a forward equation) by

$$\tau = \tfrac{1}{2}\sigma^2 (T - t).$$

Doing so gives

$$\frac{\partial \psi}{\partial \tau} = \xi^2 \frac{\partial^2 \psi}{\partial \xi^2} + \left(\frac{2}{\sigma^2 T} - \frac{2(r-y)}{\sigma^2} \xi \right) \frac{\partial \psi}{\partial \xi}, \qquad (17)$$

with the initial conditions

$\psi(\xi, 0) = \max(\xi, 0)$ for the *rate* option,

$\psi(\xi, 0) = \max(1 - \xi, 0)$ for the *strike* option.

The reduced domain for the rate option

For *rate* options, it appears that the domain on which we need to solve (17) is $\tau > 0$ and $-\infty < \xi < \infty$. This, however, is not the case.

To see this note that $S_t > 0$ and hence $dA_t = S_t\,dt > 0$. If A_t attains the value $K\,T$, it can never drop below this value.

This means that, in terms of

$$\xi_t = (A_t - K\,T)/T\,S_t,$$

if $\xi_t > 0$ then a *rate* option's moniness is fixed. So, we can find the solution for $\xi > 0$ by noting the payoff is linear in ξ and seeking a linear solution of (17). This is $\psi(\xi, \tau) = e^{(r-y)\tau}\xi + z(\tau)$, for some function $z(\tau)$.

Along the at-the-money boundary, $\xi = 0$, we have

$$\psi(\xi, \tau) = z(\tau).$$

In summary, the problem for an average rate call is then just (17) in the domain $\tau > 0$, $\xi < 0$ with the boundary and initial conditions

$$\psi(0,\tau) = z(\tau), \quad \tau > 0$$

and

$$\psi(\xi,0) = 0, \quad \xi < 0.$$

These reduced domain conditions and the boundary condition at $\xi = 0$ are important for understanding the structure of rate options.*

*Although our rate option is a call, there are similar results for a rate put.

Unwinding the transforms

We have performed a series of transformations to reduce the problem (for average rate options) to its most basic form. At this stage it is helpful to unwind the transformations and see the form of the solution in terms of the original, dimensional, financial variables.

After some algebra we find

$$V(S,A,t) = S\,e^{-y(T-t)}\,\psi\left(\frac{A}{T\,S},\,T-t\right),$$

for an average *strike* option, and

$$V(S,A,t) = S\,e^{-y(T-t)}\,\psi\left(\frac{A-K\,T}{T\,S},T-t\right),$$

for an average *rate* option.

A simplified derivation of Geman-Yor's result

Geman and Yor essentially take the Laplace transform of (17) and find the Laplace transform of ψ.

We start with the PDE (17) for ψ, which is most closely related to Geman & Yor's dependent variable C.

Their variables, in terms of the variables above, are

$$\nu = 2(r-y)/\sigma^2 - 1, \qquad \tau_2 = \tfrac{1}{4}\sigma^2\tau,$$

$$\alpha = -\tfrac{1}{4}\sigma^2 T\xi, \qquad C = \tfrac{1}{4}\sigma^2 T\psi.$$

On the reduced domain $\alpha > 0$, $\tau_2 > 0$, their PDE for C is

$$\frac{\partial C}{\partial \tau_2} = 2\alpha^2\frac{\partial^2 C}{\partial \alpha^2} - (1+2(1+\nu)\alpha)\frac{\partial C}{\partial \alpha} + 2(1+\nu)C = 0,$$

with the boundary and initial conditions

$$C(0,\tau_2) = \frac{1}{2(1+\nu)}\left(e^{2(1+\nu)\tau_2} - 1\right), \quad C(\alpha,0) = 0.$$

Now introduce that Laplace transform

$$\widehat{C}(\alpha,p) = \int_0^\infty e^{-p\tau_2}C(\alpha,\tau_2)\,d\tau_2$$

for $\mathrm{Re}(p) > \max(0, 2(1+\nu))$ — this ensures that the transform of the boundary condition exists.

From the partial differential equation above we obtain

$$2\alpha^2\frac{\partial^2\widehat{C}}{\partial\alpha^2} - (1+2(1+\nu)\alpha)\frac{\partial\widehat{C}}{\partial\alpha} + (2(1+\nu)-p)\widehat{C} = 0$$

with one boundary condition

$$\widehat{C}(0,p) = \frac{1}{(p-2(1+\nu))p}.$$

The second boundary condition will be dealt with shortly.

The solution to this ODE can be given in terms of a pair of confluent hypergeometric functions.

It is

$$\widehat{C}(\alpha, p) = C_1(p)\, A_1(\alpha, p) + C_2(p)\, A_2(\alpha, p),$$

where, in terms of

$$\mu = \sqrt{\nu^2 + 2p} \quad \text{and} \quad \beta_{\pm} = -\tfrac{1}{2}(\nu + 2 \pm \mu),$$

we have

$$A_1(\alpha, p) \;=\; {}_1F_1\left(\beta_+, 1 - \mu; -\tfrac{1}{2\alpha}\right) / (2\alpha)^{\beta_+},$$

$$A_2(\alpha, p) \;=\; {}_1F_1\left(\beta_-, 1 + \mu; -\tfrac{1}{2\alpha}\right) / (2\alpha)^{\beta_-}.$$

To have a valid Laplace transform we need to pick the solution that is holomorphic in a right half-plane $p > \gamma$, this excludes A_1.

We note that if $\mathrm{Re}(z) > 0$ then, as $|z| \to \infty$,

$${}_1F_1(a; b; z) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the gamma function. Applying this to the transformed boundary condition fixes $C_2(p)$ (this is our 2nd boundary condition) and we obtain

$$\widehat{C}(\alpha, p) = \frac{\Gamma(2+(\mu+\nu)/2)\,(2\alpha)^{-\beta_-}}{\Gamma(1+\mu)\, p\,(p-2(1+\nu))}\; {}_1F_1\left(\beta_-, 1 + \mu; -1/2\alpha\right).$$

This is the GY model for the transformed price (in fact, GY gave their original result in terms of an integral which can easily be shown to be the hypergeometric function given above –