Reflection principle, lookbacks and barriers

Lookback options I

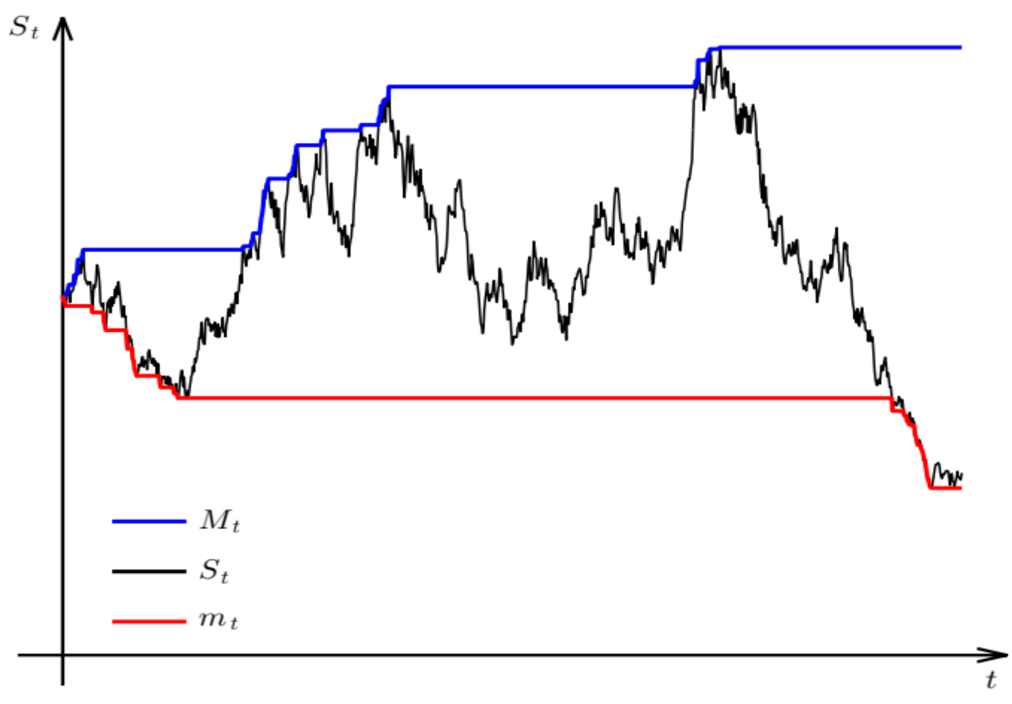
Lookback options are options whose payoffs depend on the realized maximum or minimum of the asset price over the life of the option. We assume the risky asset's price process, S_t, evolves as

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t^{\mathbb{P}}.$$

For the moment, assume the minimum, m_t, and maximum, M_t, are sampled continuously in time

$$m_t = \min_{0\le\tau\le t} S_\tau; \quad \text{minimum},$$

$$M_t = \max_{0\le\tau\le t} S_\tau; \quad \text{maximum}.$$



Realisations of S_t, M_t and m_t

Some examples of lookbacks are:

1. lookback strike put, with payoff $M_T - S_T$, (sell at the high);
2. lookback rate put, with payoff $\max(K - m_T, 0)$;
3. lookback strike call, with payoff $S_T - m_T$, (buy at the low);
4. lookback rate call, with payoff $\max(M_T - K, 0)$;
5. lookback straddle, with payoff $M_T - m_T$, (maximum gain).

In what follows we concentrate mainly on payoffs which depend on the maximum, M_T, but these results easily generalise to payoffs dependent on the minimum m_T.

With the standard assumptions, the value of a (European) lookback with payoff

$$V_T = P_o(S_T, M_T)$$

may be expressed as

$$V_t = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\,P_o(S_T, M_T)\,\right], \tag{1}$$

where as usual the risky asset's price process is

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t^{\mathbb{Q}}$$

under the risk-neutral measure $\mathbb{Q}$ and

$$M_t = \max_{0 \le \tau \le t} S_\tau.$$

The pair (S_t, M_t) is a Markov process and so there is some function $V(S, M, t)$ such that

$$V_t = V(S_t, M_t, t) \quad \text{for} \quad t \leq T.$$

By definition of M_t, this function is only of interest in the region

$$0 \leq S_t \leq M_t$$

and so with continuous sampling the domain of definition of V is

$$0 \leq S \leq M, \quad 0 \leq t \leq T.$$

To obtain the pricing equation, we may proceed as follows.

Note that if $S_t < M_t$ then

$$dM_t = 0.$$

Note also that this result is *not* true at $S_t = M_t$, but it the case that all $S_t \leq M_t$, $dM_t \, dS_t = 0$.*

The usual assumptions mean that $e^{-rt}V_t$ is a $\mathbb{Q}$-martingale for $t < T$, and Itô's lemma gives

$$\begin{aligned} d\left(e^{-rt}V_t\right) &= e^{-rt}\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r\, S_t \frac{\partial V}{\partial S} - r\, V\right) dt \\ &+ e^{-rt}\frac{\partial V}{\partial M}\, dM_t + e^{-rt}\sigma\, S\, \frac{\partial V}{\partial S}\, dW_t^{\mathbb{Q}}. \end{aligned}$$

*Except on a set of measure zero.

For $t < T$ we must have

$$\mathbb{E}_t^{\mathbb{Q}}\left[d\left(e^{-rt}V_t\right)\right] = 0.$$

Setting the coefficient of dt to zero gives

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r\, S_t \frac{\partial V}{\partial S} - r\, V = 0, \quad 0 < S_t < M_t,$$

and setting the coefficient of dM_t to zero shows that*

$$\frac{\partial V}{\partial M} = 0 \quad \text{at} \quad S_t = M_t.$$

The payoff gives the terminal condition

$$V(S_T, M_T, T) = P_o(S_T, M_T), \quad 0 \leq S_T \leq M_T.$$

*Recall that $dM_t = 0$ if $S_t < M_t$. If $S_t = M_t$ then $dM_t \geq 0$.

In total, we obtain

$$\begin{gathered} \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad 0 < S < M, \\ \frac{\partial V}{\partial M}(M, M, t) = 0, \quad 0 < M, \\ V(S, M, T) = P_o(S, M), \quad 0 \leq S \leq M. \end{gathered} \tag{2}$$

The analogous problem for an option which depends on m_t is

$$\begin{gathered} \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad 0 < m < S, \\ \frac{\partial V}{\partial m}(m, m, t) = 0, \quad 0 < m, \\ V(S, m, T) = P_o(S, m), \quad 0 \leq m \leq S. \end{gathered} \tag{3}$$

Similarity reduction

For strike style payoffs, such as $M - S$, there is a similarity reduction. All of V, S and M are prices and the problem should not (and does not) depend on the unit of price we choose, that is, for any $\lambda > 0$ it is invariant under the stretching group

$$S \mapsto \lambda S, \quad M \mapsto \lambda M, \quad V \mapsto \lambda V.$$

Two invariants of this group are $\phi = V/M$ and $\xi = S/M$, which suggest we use M as our numeraire and write

$$V(S, M, t) = M\,\phi(\xi, t), \quad \xi = S/M.$$

If we do so, the problem simplifies to

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \tfrac{1}{2}\sigma^2\,\xi^2\,\frac{\partial^2 \phi}{\partial \xi^2} + r\,\xi\,\frac{\partial \phi}{\partial \xi} - r\,\phi = 0, \quad 0 < \xi < 1, \\ \frac{\partial \phi}{\partial \xi}(1, t) = \phi(1, t), \\ \phi(\xi, T) = 1 - \xi, \quad 0 \le \xi \le 1. \end{aligned} \tag{4}$$

In principle, this and similar problems may be solved by reducing the problem to the heat equation on a semi-infinite interval subject to a 'radiation' boundary condition at one end.*

Some formulæ are given in the Appendix.

*In the case of rate lookbacks, it is usually easiest to obtain formulæ by probability methods.

Stop-loss options

These are perpetual options* which implement the following strategy.

- The underlying asset is bought at time $t = 0$, from which point both S_t and M_t are tracked.

- If at any time the asset price reaches

  $$S_t = \lambda M_t,$$

  where $0 < \lambda < 1$ is a fixed constant, the asset is immediately sold and the proceeds given to the option holder.

*A time-dependent version of the problem is possible and may be solved numerically or in terms of an infinite Fourier series.

The option's price function, $V(S, M)$, is defined only for

$$\lambda M \leq S \leq M.$$

For $\lambda M < S < M$, it satisfies the steady Black-Scholes equation

$$\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0.$$

The boundary conditions at λM and M are

$$V(\lambda M, M) = \lambda M, \quad \frac{\partial V}{\partial M}(M, M) = 0.$$

There is a similarity reduction of the form

$$V(S, M) = M\, \phi(\xi), \quad \xi = S/M.$$

In terms of the similarity variables the problem becomes

$$\tfrac{1}{2}\sigma^2 \xi^2 \phi'' + (r - y)\, \xi\, \phi' - r\, \phi = 0,$$

$$\phi(\lambda) = \lambda, \quad \phi'(1) = \phi(1).$$

The solution is

$$\phi(\xi) = \lambda \left(\frac{(1 - \alpha_-)\, \xi^{\alpha_+} + (\alpha_+ - 1)\, \xi^{\alpha_-}}{(1 - \alpha_-)\, \lambda^{\alpha_+} + (\alpha_+ - 1)\, \lambda^{\alpha_-}} \right),$$

where $\alpha_+ > 0$ and $\alpha_- < 0$ are the roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2 \alpha^2 + (r - y - \tfrac{1}{2}\sigma^2)\, \alpha - r = 0.$$

If $y = 0$ then $\alpha_+ = 1$, $\alpha_- = -2r/\sigma^2$ and the solution reduces to $\phi = \xi$, which is equivalent to $V = S$.