The reflection principle for Brownian Motion

We begin by looking at the joint distribution of a Brownian motion and its running maximum. We use W_t for the Brownian motion and the notion

$$z_t = \min_{0 \le \tau \le t} W_\tau, \quad Z_t = \max_{0 \le \tau \le t} W_\tau.$$

Here we only consider the joint distribution of (W_t, Z_t) in detail.

Consider the (nonstandard) distribution function

$$F_{[W_t,Z_t]}(x,y) = \text{prob}(W_t < x,\, Z_t > y).$$

Clearly if $y \le 0$ then $F_{[W_t,Z_t]}(x,y) = 0$ and if $x \ge y$ then the condition $W_t < x$ is redundant. Therefore in what follows $y > 0$ and $x < y$.

Choose $y > 0$ and $x < y$ and define a new process, $\widehat{W}_t$, as follows:

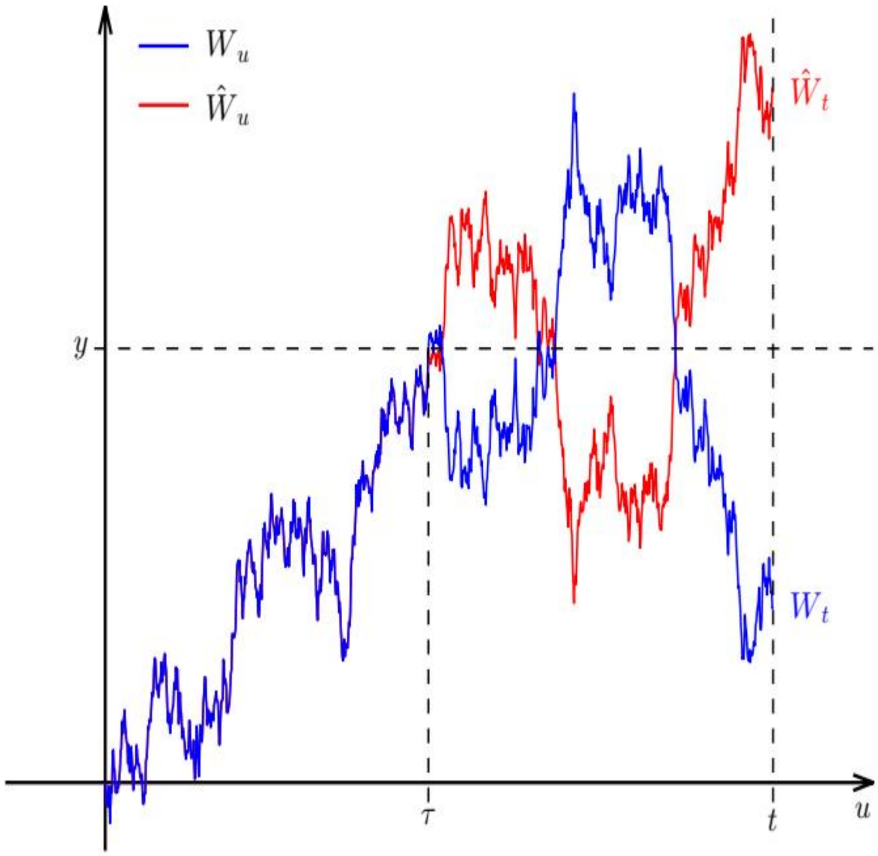
- If there is some $\tau < t$ such that* $W_\tau = y$ and $W_u < y$ for all $u < \tau$ then set

$$\widehat{W}_u = \begin{cases} W_u & \text{for } 0 \le u < \tau, \\ 2\,y - W_u & \text{for } \tau \le u \le t. \end{cases}$$

- Otherwise, set $\widehat{W}_u = W_u$ for all $0 \le u \le t$.

That is, $\widehat{W}_u = W_u$ until W_u first hits y, at which point it turns into $\widehat{W}_u = 2y - W_u$, the reflection of W_u about y. It is easy to see that $\widehat{W}_u$ is also a Brownian motion and that the distributions of W_t and $\widehat{W}_t$ are identical.

*Clearly τ is a stopping time.



If such a τ exists then the statement "($W_t < x$) and ($Z_t > y$)" is equivalent to the statement "$\widehat{W}_t > 2\,y - x$".

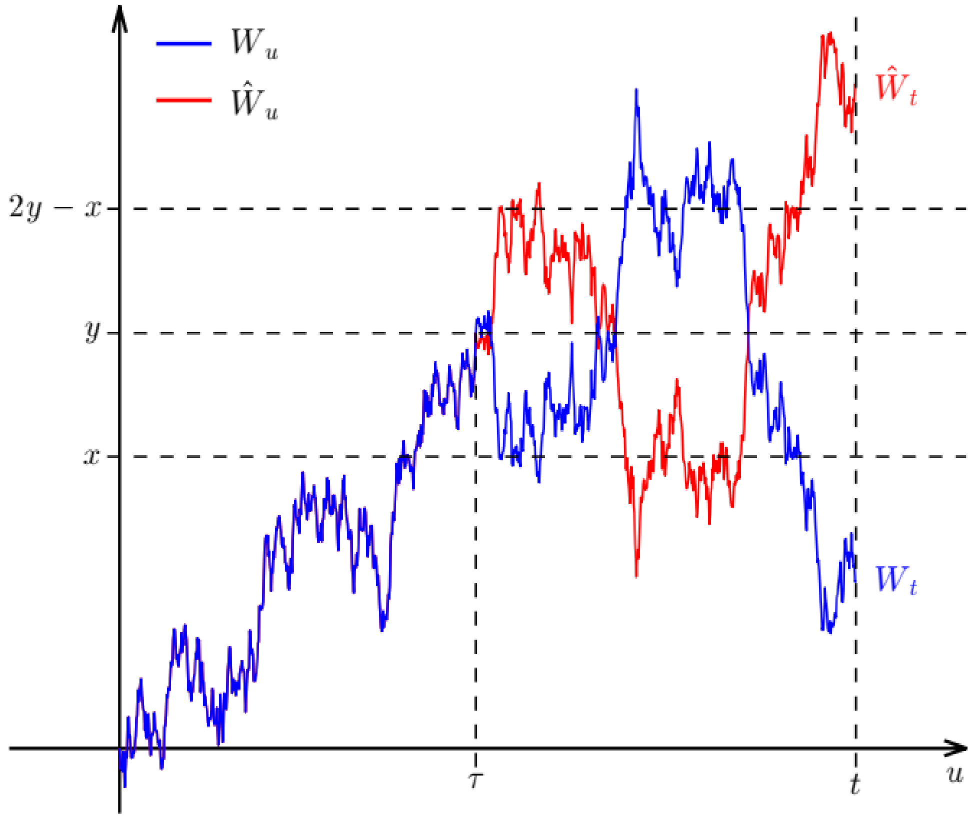
If there is no such τ, both statements are false.

Thus, these two statements are always equivalent. As

$$\text{prob}(\widehat{W}_t > 0) = \text{prob}(-\widehat{W}_t > 0),$$

this means

$$\begin{aligned}\text{prob}(W_t < x,\, Z_t > y) &= \text{prob}(\widehat{W}_t > 2\,y - x)\\ &= \text{prob}(\widehat{W}_t < x - 2\,y)\\ &= \text{N}\left(\frac{x - 2\,y}{\sqrt{t}}\right).\end{aligned}$$



Thus, we find that for $y > 0$ and $x < y$ we have

$$F_{[W_t,Z_t]}(x,y) = \mathrm{N}\left(\frac{x-2\,y}{\sqrt{t}}\right).$$

If $f_{[W_t,Z_t]}(x,y)$ is the joint density function then

$$F_{[W_t,Z_t]}(x,y) = \int_{-\infty}^{x}\int_{y}^{\infty} f_{[W_t,Z_t]}(\xi,\eta)\,d\eta\,d\xi.$$

Differentiating with respect to x and y shows

$$f_{[W_t,Z_t]}(x,y) = (2\,y-x)\sqrt{\frac{2}{\pi\,t^3}}\,\exp\left(-\frac{(2\,y-x)^2}{2\,t}\right) \qquad (5)$$

for $y > 0$ and $x \le y$.

If $y \le 0$ or $x > y$ then $f_{[W_t,Z_t]}(x,y) = 0$.

The marginal distribution for Z_t is

$$\begin{aligned} f_{Z_t}(y) &= \int_{-\infty}^{y} f_{[W_t,Z_t]}(x,y)\,dx \\ &= \sqrt{\frac{2}{\pi\,t}}\,e^{-y^2/2t} \end{aligned} \tag{6}$$

for $y > 0$ and zero otherwise.

The marginal distribution for W_t is, of course,

$$\begin{aligned} f_{W_t}(x) &= \int_{\max(x,0)}^{\infty} f_{[W_t,Z_t]}(x,y)\,dy \\ &= \frac{1}{\sqrt{2\pi\,t}}\,e^{-x^2/2t}. \end{aligned} \tag{7}$$

The *conditional* density function for Z_t, given that

$$W_t = x,$$

is given by

$$\begin{aligned} f_{[Z_t\,|\,W_t=x]}(y,t) &= \frac{f_{[W_t,Z_t]}(x,y)}{f_{W_t}(x)} \\ &= (2/t)\,(2\,y-x)\,e^{-2y(y-x)/t} \end{aligned}$$

if $y > 0$, $x < y$ and zero otherwise.

This form of distribution is known as a Rayleigh distribution, first used by J.W. Strutt (Lord Rayleigh) circa 1880.*

*Lord Rayleigh, *On the resultant of a large number amplitudes of the same pitch and arbitrary phase,* Phil. Mag., S.5., Vol. 10, No.60, Aug. 1880.

Exercise: Show that the *conditional* distribution function for Z_t, given that

$$W_t = x,$$

is given by

$$F_{[Z_t \,|\, W_t = x]}(y) = 1 - e^{-2y(y-x)/t}$$

for $y > \max(0, x)$ and zero otherwise.

Use this to construct a method of sampling from the distribution of the maximum of a Brownian motion given its terminal value.

Exercise: Show that the conditional distribution function for W_t, given that $Z_t = y > 0$, is

$$f_{[W_t \,|\, Z_t = y]}(x) = \left(\frac{2\,y - x}{t}\right) \exp\left(\frac{(x-y)\,(3\,y-x)}{2\,t}\right)$$

for $x < y$ and zero otherwise.

Exercise: Note that if W_t is a Brownian motion then so too is $-W_t$ and then use the fact that

$$\begin{aligned} z_t &= \min_{0 \le \tau \le t} (W_\tau) \\ &= -\max_{0 \le \tau \le t} (-W_\tau) \end{aligned}$$

to find the analogous distribution and density functions for z_t.

Lookback options II

The aim of this section is to outline how one may compute a lookback's price directly from the representation

$$V_t = e^{-r(T-t)}\, \mathbb{E}_t^{\mathbb{Q}} \left[\, P_o(S_T, M_T) \,\right]. \qquad (1)$$

We will consider the case of a lookback put with payoff

$$P_o(S_T, M_T) = M_T - S_T,$$

where the maximum, M_t, is sampled continuously.

In order to achieve this end, we work in small steps starting with the simplest version of the problem.

Under the usual Black–Scholes assumptions, there is a unique risk-neutral measure, $\mathbb{Q}$, under which the underlying asset's price, S_t, evolves as

$$S_t = S \exp\Big((r - \tfrac{1}{2}\sigma^2)\, t + \sigma W_t^{\mathbb{Q}}\Big),$$

where $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion and $S_0 = S$. Taking logs gives

$$X_t = x + (r - \tfrac{1}{2}\sigma^2)\, t + \sigma\, W_t^{\mathbb{Q}},$$

where $X_t = \log(S_t)$ and $x = \log(S)$, and hence

$$dX_t = (r - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t^{\mathbb{Q}}, \quad X_0 = x.$$

For reasons which will become apparent shortly, it is convenient to consider first the case where $r = \sigma^2/2$, i.e., the case where X_t has no drift (and is therefore a martingale).

The simple case

Consider the simplest case where $r = \sigma^2/2$ and

$$X_t = x + \sigma\, W_t^{\mathbb{Q}}.$$

Define Y_t as the running maximum of Y_t,

$$Y_t \;=\; \max_{0\le\tau\le t} X_t \;=\; x + \max_{0\le\tau\le t} \sigma\, W_t^{\mathbb{Q}}.$$

As x is constant, it's clear that

$$Y_t - X_t = \sigma(Z_t - W_t^{\mathbb{Q}}) \equiv (Z_{\sigma^2 t} - W_{\sigma^2 t}^{\mathbb{Q}}),$$

where, as before and also in what follows,

$$Z_t = \max_{0\le\tau\le t} W_\tau^{\mathbb{Q}}.$$

This allows us to calculate, for example,

$$\begin{aligned} U_0 &= \mathbb{E}_0^{\mathbb{Q}}[\,Y_t - X_t\,] \\ &= \int_0^{\infty}\int_{-\infty}^{\eta} (\eta - \xi)\, f_{[W_{\sigma^2 t}, Z_{\sigma^2 t}]}(\xi, \eta)\, d\xi\, d\eta \\ &= \sqrt{\frac{2\,\sigma^2\,t}{\pi}} \end{aligned}$$

using the joint density (5).

Alternatively, as $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale, we may write

$$U_0 \;=\; \sigma\,\mathbb{E}_0^{\mathbb{Q}}[\,Z_t\,] - \sigma\,\mathbb{E}_0^{\mathbb{Q}}[\,W_t^{\mathbb{Q}}\,] \;=\; \sigma\,\mathbb{E}_0^{\mathbb{Q}}[\,Z_t\,],$$

then use the marginal density (6) to find $\mathbb{E}_0^{\mathbb{Q}}[\,\sigma\,Z_t\,] = \mathbb{E}_0^{\mathbb{Q}}[\,Z_{\sigma^2 t}\,]$.

In the original variables, i.e.,

$$S_t = e^{X_t} = S\,e^{\sigma W_t^{\mathbb{Q}}}, \quad M_t = \max_{0\le\tau\le t}(S_\tau),$$

we find that since $z \mapsto e^z$ is monotonically increasing

$$M_t = e^{Y_t} = S\,e^{\sigma\,Z_t}.$$

Therefore we may evaluate

$$V_0 \;=\; \mathbb{E}_0^{\mathbb{Q}}[\,M_t - S_t\,] \;=\; S\,\mathbb{E}_0^{\mathbb{Q}}\Big[e^{\sigma Z_t} - e^{\sigma W_t^{\mathbb{Q}}}\Big]$$

using the joint density (5), or we may write it as

$$V_0 = S\left(\mathbb{E}_0^{\mathbb{Q}}\left[e^{\sigma Z_t}\right] - \mathbb{E}_0^{\mathbb{Q}}\left[e^{\sigma W_t^{\mathbb{Q}}}\right]\right),$$

and then use the marginal densities (6) and (7).

Either way, we find that*

$$V_0 = S\, e^{\sigma^2 t/2} \left(2\,\mathsf{N}(\sigma\sqrt{t}\,) - 1\right).$$

So, *provided that* $S_t = S e^{\sigma W_t}$, i.e., *if*

$$\frac{dS_t}{S_t} = \tfrac{1}{2}\sigma^2\, dt + \sigma\, dW_t^{\mathbb{Q}}, \quad S_0 = S, \tag{8}$$

we can compute

$$\mathbb{E}_0^{\mathbb{Q}}\left[M_t - S_t\right] = S\, e^{\sigma^2 t/2} \left(2\,\mathsf{N}(\sigma\sqrt{t}\,) - 1\right). \tag{9}$$

This is precisely the expectation we need to find the price of a look-back put, *provided* the risk-neutral price process is given by (8).

*Here, as always, $\mathsf{N}(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} e^{-u^2/2}\, du.$

The general case

There is a problem if S_t has drift other than $\frac{1}{2}\sigma^2$. To see this, consider

$$S_t = S\, e^{X_t}, \quad M_t = S\, e^{Y_t},$$

with X_t and Y_t defined by

$$X_t = x + \nu\, t + \sigma\, W_t^{\mathbb{Q}}, \quad Y_t = \max_{0 \le \tau \le t} X_\tau.$$

If $\nu \neq 0$ then it is *not* true that

$$Y_t = (x + \nu\, t) + (\sigma\, Z_t),$$

because, in general, the best we can say is

$$\max_{0 \le \tau \le t} (a_\tau + b_\tau) \le \max_{0 \le \tau \le t} (a_\tau) + \max_{0 \le \tau \le t} (b_\tau),$$

usually with strict inequality; the maximum values of two processes usually occur at *different* times.

We get around this problem by introducing an equivalent measure, $\mathbb{R}$, under which X_t becomes a martingale, thereby eliminating the drift.

Girsanov's theorem asserts that if $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion and α_t is a well-behaved process* then there is a measure, $\mathbb{R} \sim \mathbb{Q}$, with

1. $\left.\frac{d\mathbb{R}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\Big(-\frac{1}{2}\int_0^t \alpha_\tau^2\, d\tau - \int_0^t \alpha\, dW_\tau\Big)$; and

2. $W_t^{\mathbb{R}} = \Big(\int_0^t \alpha_\tau\, d\tau\Big) + W_t^{\mathbb{Q}}$ is an $\mathbb{R}$-Brownian motion.

*I.e., it satisfies the Novikov condition $\mathbb{E}^{\mathbb{Q}}\big[\exp\big(\frac{1}{2}\int_0^t \alpha_\tau^2\, d\tau\big)\big] < \infty$.

In order to transform

$$X_t = x + \nu\, t + \sigma\, W_t^{\mathbb{Q}} \quad \text{to} \quad X_t = x + \sigma\, W_t^{\mathbb{R}}$$

we need $\sigma\, W_t^{\mathbb{R}} = \nu\, t + \sigma\, W_t^{\mathbb{Q}}$, so the process α_t should satisfy

$$\sigma \int_0^t \alpha_\tau\, d\tau = \nu\, t.$$

Therefore we choose $\alpha_t = \nu/\sigma = \theta$, which gives us

$$\left.\frac{d\mathbb{R}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\Big(-\tfrac{1}{2}\,\theta^2 t - \theta\, W_t^{\mathbb{Q}}\Big), \quad \theta = \nu/\sigma.$$

As $W_t^{\mathbb{R}} = \theta\, t + W_t^{\mathbb{Q}}$, the inverse Radon-Nikodym derivative is

$$\begin{aligned} \mathcal{R}_t \doteq \left.\frac{d\mathbb{Q}}{d\mathbb{R}}\right|_{\mathcal{F}_t} &= \exp\Big(\tfrac{1}{2}\,\theta^2 t + \theta\, W_t^{\mathbb{Q}}\Big) \\ &= \exp\Big(-\tfrac{1}{2}\,\theta^2 t + \theta\, W_t^{\mathbb{R}}\Big). \end{aligned} \tag{10}$$

Expectations with respect to the $\mathbb{Q}$ and $\mathbb{R}$ measures are related by

$$\mathbb{E}_s^{\mathbb{Q}}[\,\beta_t\,] = \mathbb{E}_s^{\mathbb{R}}[\,\beta_t \times (\mathcal{R}_t/\mathcal{R}_s)\,]$$

for any suitable process β_t.

Under the $\mathbb{R}$-measure, X_t is a driftless Brownian motion,

$$X_t = x + \sigma\, W_t^{\mathbb{R}},$$

which means that under $\mathbb{R}$ we *can* write

$$Y_t = x + \sigma\, Z_t$$

and proceed as above, except that we must apply the factor $\mathcal{R}_t/\mathcal{R}_s$ in order to convert back to the $\mathbb{Q}$-measure.

In a Black–Scholes framework the risk-neutral price process is

$$S_t = S\,\exp\Big((r - \tfrac{1}{2}\sigma^2)\,t + \sigma\, W_t^{\mathbb{Q}}\Big).$$

Taking logarithms gives

$$\begin{aligned} X_t \;&\doteq\; \log(S_t) \;=\; \log(S) + \Big(r - \tfrac{1}{2}\sigma^2\Big)\,t + \sigma\, W_t^{\mathbb{Q}} \\ &=\; x + \Big(r - \tfrac{1}{2}\sigma^2\Big)\,t + \sigma\, W_t^{\mathbb{Q}}. \end{aligned}$$

To find the $\mathbb{R}$-measure under which this process becomes driftless, i.e. $X_t = x + \sigma\, W_t^{\mathbb{R}}$, the appropriate choice of ν in (10) is

$$\nu = r - \tfrac{1}{2}\sigma^2,$$

which gives the relevant Radon Nikodym derivative as

$$\mathcal{R}_t = \exp\Big(-\tfrac{1}{2}\theta^2\,t + \theta\, W_t^{\mathbb{R}}\Big), \quad \theta = (r - \tfrac{1}{2}\sigma^2)/\sigma. \tag{11}$$

The value of a lookback put is then computed as

$$\begin{aligned} V_0 &= V(S_t, M_t, t) \\ &= e^{-r(T-t)}\, \mathbb{E}_0^{\mathbb{Q}}[\, M_T - S_T\,] \qquad (12) \\ &= e^{-r(T-t)}\, \mathbb{E}_0^{\mathbb{R}}[\,(M_T - S_T) \times \mathcal{R}_T\,], \end{aligned}$$

where $\mathcal{R}_T$ is given by (11),

$$\begin{aligned} S_T &= e^{X_T} = S\, e^{\sigma W_T^{\mathbb{R}}}, \\ M_T &= e^{Y_T} = S\, e^{\sigma Z_T} \end{aligned}$$

and the joint density (5) is used to evaluate the $\mathbb{R}$-expection above.

The details are gruesome . . .