Barriers II

Consider, for example, an up-and-out barrier option with barrier level B and payoff $P_o(S_T)$. The payoff is received if and only if

$$S_t < B \quad \forall\ 0 \le t \le T,$$

which is the same thing as saying it is received if and only if

$$M_T = \max_{0 \le t \le T}(S_t) < B.$$

Under the usual assumptions, this means we may write

$$V_0 = e^{-r(T-t)}\,\mathbb{E}_0^{\mathbb{Q}}\left[P_o(S_T)\,\mathbb{1}_{\{M_T<B\}}\right].$$

In order to evaluate the risk-neutral expectation we change to measure $\mathbb{R}$ under which

$$S_t = S\,e^{\sigma W_t^{\mathbb{R}}}, \quad M_t = S\,e^{\sigma Z_t}$$

and then use

$$\mathbb{E}_0^{\mathbb{Q}}\left[P_o(S_T)\,\mathbb{1}_{\{M_T<B\}}\right] = \mathbb{E}_0^{\mathbb{R}}\left[P_o(S_T)\,\mathbb{1}_{\{M_T<B\}} \times \mathcal{R}_T,\right],$$

where

$$\mathcal{R}_t = \left.\frac{d\mathbb{Q}}{d\mathbb{R}}\right|_{\mathcal{F}_t} = \left.\frac{d\mathbb{R}}{d\mathbb{Q}}\right|_{\mathcal{F}_t}^{-1},$$

then use the joint density function in (5) to evaluate the $\mathbb{R}$-expectation.

The details are again gruesome...

Appendix A: the pricing equation

For $n > 1$, define

$$I_n(t) = \left(\int_0^t S_\tau^n \, d\tau\right)^{1/n}, \quad J_n(t) = \left(\int_0^t S_\tau^{-n} \, d\tau\right)^{-1/n}.$$

Provided that S_u is continuous for $0 \le u \le t$ then

$$\lim_{n\to\infty} I_n(t) = M_t, \quad \lim_{n\to\infty} J_n(t) = m_t.$$

Consider now an option whose payoff depends on $I_n(T)$.* We have

$$dI_n(t) = \frac{S_t^n}{n\, I_n(t)^{n-1}}.$$

*Similar arguments to the case where the payoff depends on $J_n(T)$.

Following the usual arguments, we arrive at the pricing equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S \frac{\partial V}{\partial S} + \frac{S^n}{n\, I_n^{n-1}} \frac{\partial V}{\partial I_n} - r\, V = 0. \qquad (13)$$

Consider now what happens to the term

$$\frac{S^n}{n\, I_n^{n-1}} = \frac{I_n}{n}\left(\frac{S}{I_n}\right)^n$$

as $n \to \infty$.

On the one hand, if $0 < S \le I_n$ this terms vanishes and we have

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S \frac{\partial V}{\partial S} - r\, V = 0, \quad 0 < S < M,$$

where we define $M = \lim_{n\to\infty} I_n$, corresponding to $M_t = \lim_{n\to\infty} I_n(t)$.

On the other hand, if $S > I_n$ then this term explodes and we must have

$$\lim_{n\to\infty} \frac{\partial V}{\partial I_n}(S, I_n, t) \to 0,$$

exponentially fast, for any $S > I_n$. With the same notation as above, this implies that

$$\frac{\partial V}{\partial M}(S, M, t) = 0 \quad \text{for} \quad S > M.$$

Thus we arrive at the boundary condition

$$\frac{\partial V}{\partial M}(M, M, t) = 0.$$

In total, the limiting problem is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0,$$

for $0 < S < M$ and $t < T$, the boundary condition

$$\frac{\partial V}{\partial M}(M, M, t) = 0$$

at $S = M > 0$ and $t < T$ and the terminal condition

$$V(S, M, T) = P_o(S, M).$$

Appendix B: justification of limits

Here we sketch a proof that if $S_t > 0$ is continuous for $0 \le t \le T < \infty$ then

$$\lim_{n\to\infty} I_n(t) = \lim_{n\to\infty} \left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} = \max_{0\le\tau\le t} (S_\tau) = M_t.$$

First note that since $[0, t]$ is closed and bounded and S_τ is continuous on it, S_τ achieves its maximum $0 < M_t < \infty$. Next note that

$$0 < \int_0^t S_\tau^n \, d\tau \le \int_0^t M_t^n \, d\tau = M_t^n \, t,$$

which implies that for $t > 0$

$$\left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} \le M_t \, t^{1/n}. \tag{14}$$

Note that if $t > 0$ then $\lim_{n\to\infty} t^{1/n} = 1$, so taking the limit $n \to \infty$ of (14) shows that

$$\lim_{n\to\infty} \left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} \le M_t.$$

Now suppose $0 < \epsilon \le M_t$ is given. Define $\mathcal{A}_\epsilon(t)$ by

$$\mathcal{A}_\epsilon(t) = \{\tau \in [0, t] : S_\tau \ge M_t - \epsilon\}.$$

Let $L_\epsilon(t)$ be the measure of $\mathcal{A}_\epsilon(t)$; since S_τ is continuous $\mathcal{A}_\epsilon(t)$ is measurable with nonzero measure. Clearly we have

$$\int_0^t S_\tau^n \, d\tau \ge \int_{\mathcal{A}_\epsilon(t)} S_\tau^n \, d\tau \ge \int_{\mathcal{A}_\epsilon(t)} (M_t - \epsilon)^n \, d\tau = (M_t - \epsilon)^n \, L_\epsilon(t)$$

and hence

$$\left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} \ge (M_t - \epsilon) \, L_\epsilon(t)^{1/n}. \tag{15}$$

Since $L_\epsilon(t) > 0$, taking the limit $n \to \infty$ of (15) shows

$$\lim_{n\to\infty} \left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} \geq (M_t - \epsilon)$$

Thus, for any $0 < \epsilon < M_t$ we have

$$(M_t - \epsilon) \leq \lim_{n\to\infty} \left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} \leq M_t.$$

The only way this is possible is if

$$\lim_{n\to\infty} \left(\int_0^t S_\tau^n \, d\tau \right)^{1/n} = M_t.$$

A similar line of reasoning establishes that

$$\lim_{n\to\infty} \left(\int_0^t S_\tau^{-n} \, d\tau \right)^{-1/n} = m_t = \min_{0\leq\tau\leq t} (S_\tau).$$

Appendix C: formulæ

There are formulæ for the options listed above. We present them assuming the underlying asset pays a continuous dividend yield at the rate y.* The following notation is used in all there formulæ,

$$\begin{aligned} d_\pm(K) &= \frac{\log(S/K)+(r-y\pm\frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}, \\ \tilde{d}_\pm(K) &= \frac{\log(S/K)\pm\frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma^2\,(T-t)}}, \\ d_\star(K) &= \frac{\log(S/K)-(r-y-\frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}, \\ \alpha &= 2\,(r-y)/\sigma^2. \end{aligned}$$

*In which case the coefficients of $\partial V/\partial S$ in (2) and (3) are changed from $r\,S$ to $(r-y)\,S$.

lookback strike put with payoff $M - S$ and $0 \leq S \leq M$,

$$
\begin{aligned}
V &= M\, e^{-r(T-t)}\, \mathsf{N}\left(-d_-(M)\right) - S\, e^{-y(T-t)}\, \mathsf{N}\left(-d_+(M)\right) \\
&+ \frac{1}{\alpha} S\, e^{-r(T-t)} \left(e^{(r-y)(T-t)}\, \mathsf{N}\left(d_+(M)\right) - (M/S)^{\alpha}\, \mathsf{N}\left(d_\star(M)\right) \right),
\end{aligned}
$$

unless $r = y$, in which case $\alpha = 0$ and

$$
\begin{aligned}
V &= e^{-r(T-t)} \left(M\, \mathsf{N}\left(-\tilde{d}_-(M)\right) - S\, \mathsf{N}\left(-\tilde{d}_+(M)\right) \right) \\
&+ \sqrt{\sigma^2(T-t)}\; S\, e^{-r(T-t)} \left(\frac{e^{-\frac{1}{2}\tilde{d}_+(M)^2}}{\sqrt{2\pi}} + \tilde{d}_+(M)\, \mathsf{N}\left(\tilde{d}_+(M)\right) \right).
\end{aligned}
$$

lookback strike call with payoff $S - m$ and $0 \leq m \leq S$,

$$
\begin{aligned}
V &= S\, e^{-y(T-t)}\, \mathsf{N}\left(d_+(m)\right) - m\, e^{-r(T-t)}\, \mathsf{N}\left(d_-(m)\right) \\
&- \frac{1}{\alpha} S\, e^{-r(T-t)} \left(e^{(r-y)(T-t)}\, \mathsf{N}\left(-d_+(m)\right) - (m/S)^{\alpha}\, \mathsf{N}\left(-d_\star(m)\right) \right),
\end{aligned}
$$

unless $r = y$, in which case

$$
\begin{aligned}
V &= e^{-r(T-t)} \left(S\, \mathsf{N}\left(\tilde{d}_+(m)\right) - m\, \mathsf{N}\left(\tilde{d}_-(m)\right) \right) \\
&+ \sqrt{\sigma^2(T-t)}\; S\, e^{-r(T-t)} \left(\frac{e^{-\frac{1}{2}\tilde{d}_+(m)^2}}{\sqrt{2\pi}} - \tilde{d}_+(m)\, \mathsf{N}\left(-\tilde{d}_+(m)\right) \right).
\end{aligned}
$$

The **lookback straddle** with payoff $M - m$ is the sum of the strike put and strike call.

lookback rate put with payoff max$(K - m, 0)$

For $m \geq K$

$$V = K\,e^{-r(T-t)}\,\mathsf{N}\left(-d_-(K)\right) - S\,e^{-y(T-t)}\,\mathsf{N}\left(-d_+(K)\right)$$

$$- \frac{1}{\alpha} S\,e^{-r(T-t)} \left((K/S)^\alpha\,\mathsf{N}\left(-d_\star(K)\right) - e^{(r-y)(T-t)}\,\mathsf{N}\left(-d_+(K)\right)\right),$$

while for $m < K$ it is

$$V = K\,e^{-r(T-t)} - m\,e^{-r(T-t)}\,\mathsf{N}\left(d_-(m)\right) - S\,e^{-y(T-t)}\,\mathsf{N}\left(-d_+(m)\right)$$

$$+ \frac{1}{\alpha} S\,e^{-r(T-t)} \left((m/S)^\alpha\,\mathsf{N}\left(-d_\star(m)\right) - e^{(r-y)(T-t)}\,\mathsf{N}\left(-d_+(m)\right)\right),$$

provided $r \neq y$. If $r = y$ these formulæ are singular and limits must be taken.

lookback rate call with payoff max$(M - K, 0)$

For $M \leq K$

$$V = S\,e^{-y(T-t)}\,\mathsf{N}\left(d_+(K)\right) - K\,e^{-r(T-t)}\,\mathsf{N}\left(d_-(K)\right)$$

$$+ \frac{1}{\alpha} S\,e^{-r(T-t)} \left(e^{(r-y)(T-t)}\,\mathsf{N}\left(d_+(K)\right) - (K/S)^\alpha\,\mathsf{N}\left(d_\star(K)\right)\right),$$

while for $M > K$ it is

$$V = S\,e^{-y(T-t)}\,\mathsf{N}\left(d_+(M)\right) + M\,e^{-r(T-t)}\,\mathsf{N}\left(-d_-(M)\right) - K\,e^{-r(T-t)}$$

$$+ \frac{1}{\alpha} S\,e^{-r(T-t)} \left(e^{(r-y)(T-t)}\,\mathsf{N}\left(d_+(M)\right) - (M/S)^\alpha \mathsf{N}\left(d_\star(m)\right)\right),$$

provided $r \neq y$. If $r = y$ these formulæ are again singular and limits must be taken.