Barrier options

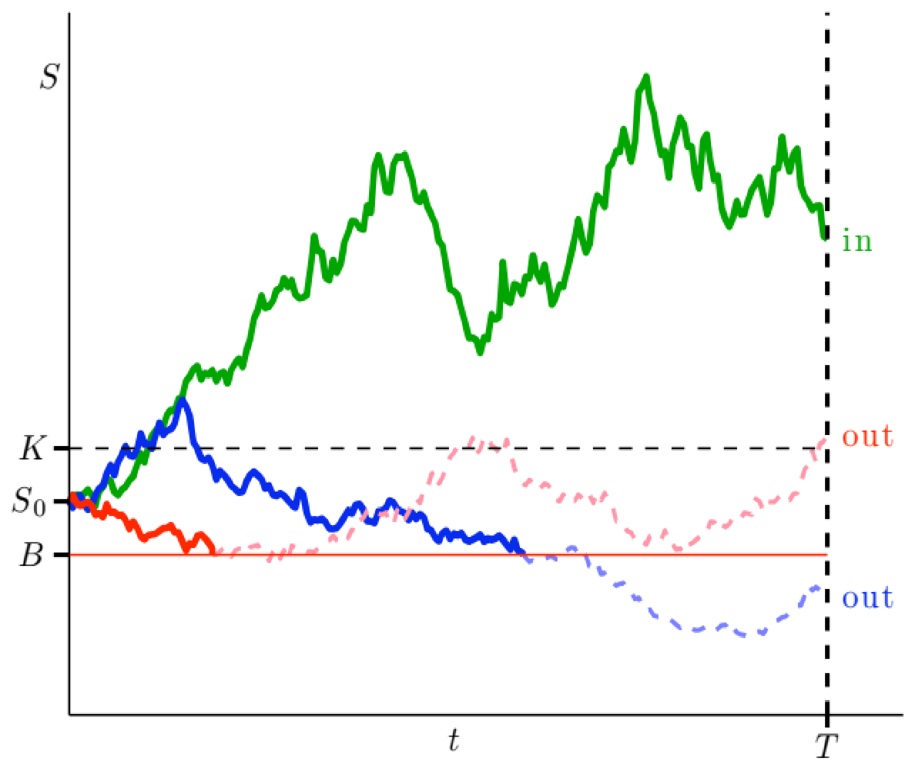
A typical barrier option contract changes if the asset hits a specified level, the *barrier*. Barrier options are therefore path-dependent.

Out options expire worthless if S_t reaches the barrier value B before expiry, T. For a *down-and-out call option*, if $S_t \leq B$ for some $0 \leq t \leq T$ then the option dies, if $S_t > B$ for $0 \leq t \leq T$ it survives and has the usual payoff $\max(S_T - K, 0)$.*

In options only come into being if S_t reaches B for some $0 \leq t \leq T$, at which point they become an ordinary option.

*This is equivalent to a payoff $\max(S_T - K, 0)\, \mathbf{1}_{\{m_T > B\}}$, where $m_T = \min_{0 \leq t \leq T} \{S_t\}$.

Price paths with a down-and-out barrier



The down-and-out call

If S_t falls to B, the option dies and becomes worthless. If S_t stays above B, it has the usual payoff $\max(S_T - K, 0)$. For the moment we'll assume that $B < K$.

While the option remains in its price function, $C_{do}(S, t)$, satisfies the Black-Scholes equation,

$$\frac{\partial C_{do}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{do}}{\partial S^2} + (r - y)S \frac{\partial C_{do}}{\partial S} - r\, C_{do} = 0.$$

This only holds if the option hasn't knocked out, i.e., only if $S_u > B$ for all $0 \le u \le t$.

If $S_t = B$ at any time $0 \le t \le T$ the option knocks out and becomes worthless, so

$$C_{do}(B, t) = 0.$$

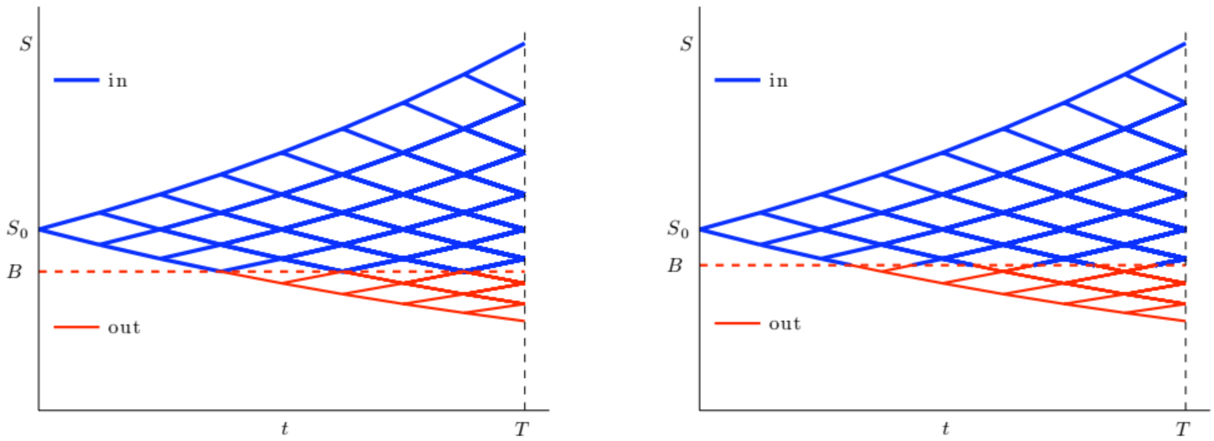
If the option is still alive at expiry then

$$C_{do}(S, T) = \max(S - K, 0).$$

Note that the barrier feature manifests itself as a *boundary* condition.

If the option has already knocked out then $C_{do}(S_t, t) = 0$ no matter what the value of S_t.

A binomial method can incorporate this path dependence:



The barrier may hit grid points, which is good. Usually, however, it misses them all which is less good.

A 'reflection' result for Black–Scholes

Suppose $V(S,t)$ is a solution of the Black–Scholes equation,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)S\frac{\partial V}{\partial S} - rV = 0, \qquad (1)$$

for all $S > 0$ and $t < T$.

If $B > 0$ is a constant and

$$\alpha = \tfrac{1}{2} - (r-y)/\sigma^2$$

then

$$W(S,t) = (S/B)^{2\alpha}\, V\Big(B^2/S, t\Big)$$

also satisfies the Black–Scholes equation (1) for $S > 0$ and $t < T$.

There is a derivation of this result in the introductory notes on the Black–Scholes equation, reproduced along with two alternative proofs in the Appendix of these notes.

In general, B may be any constant, but in what follows we will take it to be the level of our barrier.

If $V(S,t)$ satisfies the Black–Scholes equation for all $S > 0$ then so too does $W(S,t)$ and so, by linearity,

$$V(S,t) + A\,W(S,t)$$

is also a solution of the Black–Scholes equation for all $S > 0$ for any constant A.

An explicit solution for a down-and-out call

The down-and-out call's price function, $C_{\text{do}}(S,t)$, satisfies the Black–Scholes equation

$$\frac{\partial C_{\text{do}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\text{do}}}{\partial S^2} + (r-y)\,S\,\frac{\partial C_{\text{do}}}{\partial S} - r\,C_{\text{do}} = 0, \qquad (2)$$

for $t < T$ and $S > B$, has payoff

$$C_{\text{do}}(S,T) = \max(S-K,0), \quad S \geq B \qquad (3)$$

and vanishes on the barrier B,

$$C_{\text{do}}(B,t) = 0, \quad t \leq T. \qquad (4)$$

Recall that we only need to find $C_{\text{do}}(S,t)$ for $S > B$.

Let us assume that $B < K$.

Let C_{bs} be the price of a vanilla version of our down-and-out call, i.e., a vanilla call on the same asset with the same strike and expiry.

Its price function, $C_{\text{bs}}(S,t)$, is necessarily a solution of the Black–Scholes equation (2), for all $S > 0$ and $t < T$.

We look for a solution of the form

$$C_{\text{do}}(S,t) = C_{\text{bs}}(S,t) - W(S,t).$$

As the Black–Scholes equation is linear, $W(S,t)$ must also be a solution of (2).

The barrier condition (4) is satisfied if

$$W(B,t) = C_{\text{bs}}(B,t).$$

The obvious candidate is

$$W(S,t) = (S/B)^{2\alpha}\, C_{\text{bs}}\big(B^2/S, t\big).$$

The reflection principle assures us that this satisfies (2) if $C_{\text{bs}}(S,t)$ does, and setting $S = B$ shows

$$W(B,t) = C_{\text{bs}}(B,t).$$

The only remaining condition is (3), the payoff condition

$$C_{\text{do}}(S,T) = C_{\text{bs}}(S,T) - W(S,t) = \max(S-K, 0),$$

and this requires that

$$W(S,T) = 0, \quad S > B.$$

To see that this is indeed the case we argue as follows.

If $0 < B < S$ then $0 < 1/S < 1/B$. Multiplying by $B^2 > 0$ shows

$$B^2/S < B.$$

Our assumption that $B < K$ shows that

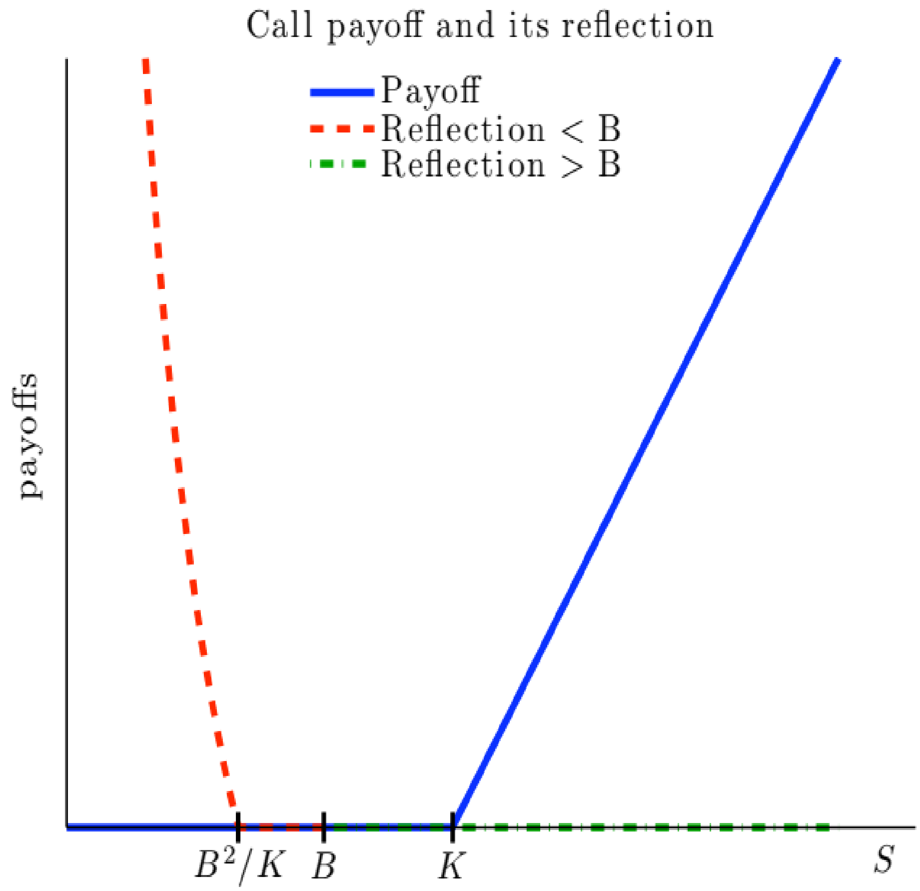
$$S > B \quad \Longrightarrow \quad B^2/S < B < K.$$

Thus if $S > B$ then

$$(S/B)^{2\alpha}\, C_{\text{bs}}(B^2/S, T) = (S/B)^{2\alpha} \max\big(B^2/S - K, 0\big) = 0,$$

since $B^2/S - K < 0$ and so $\max(B^2/S - K, 0) = 0$.

This establishes that the payoff condition (3) is indeed satisfied.



Thus, if the option has not yet knocked out its value for $S > B$ is given by

$$C_{\mathrm{do}}(S,t) = C_{\mathrm{bs}}(S,t) - (S/B)^{2\alpha}\, C_{\mathrm{bs}}(B^2/S,t).$$

For $S \leq B$ the option has knocked out and $C_{\mathrm{do}}(S,t) = 0$.

Provided the option is alive and $S > B$, its delta is found using

$$\begin{aligned}\Delta_{\mathrm{do}}(S,t) &= \Delta_{\mathrm{bs}}(S,t) + (S/B)^{2\alpha-2}\,\Delta_{\mathrm{bs}}(B^2/S,t) \\ &- 2\alpha\,(S^{2\alpha-1}/B^{2\alpha})\,C_{\mathrm{bs}}(B^2/S,t),\end{aligned}$$

where

$$\Delta_{\mathrm{bs}} = \frac{\partial C_{\mathrm{bs}}}{\partial S}.$$

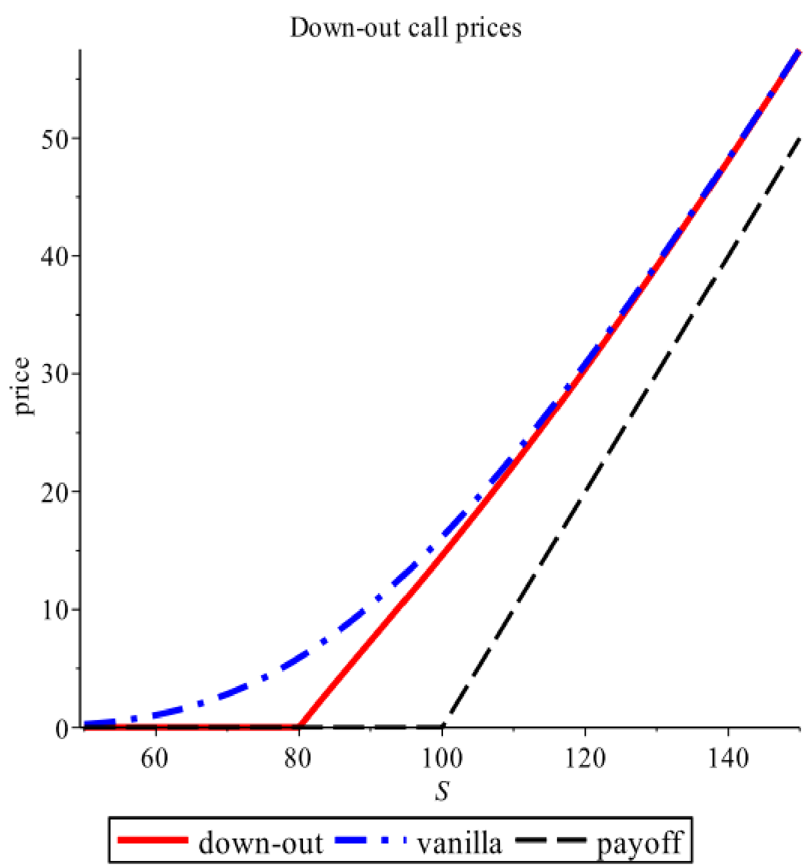
While the option is alive and $S > B$, its gamma is given by

$$\begin{aligned}\Gamma_{\mathrm{do}}(S,t) &= \Gamma_{\mathrm{bs}}(S,t) - (S/B)^{2\alpha-4}\,\Gamma_{\mathrm{bs}}(B^2/S,t) \\ &+ 2\,(2\alpha-1)\,(S^{2\alpha-3}/B^{2\alpha-2})\,\Delta_{\mathrm{bs}}(B^2/S,t) \\ &- 2\alpha\,(2\alpha-1)\,(S^{2\alpha-2}/B^{2\alpha})\,C_{\mathrm{bs}}(B^2/S,t),\end{aligned}$$

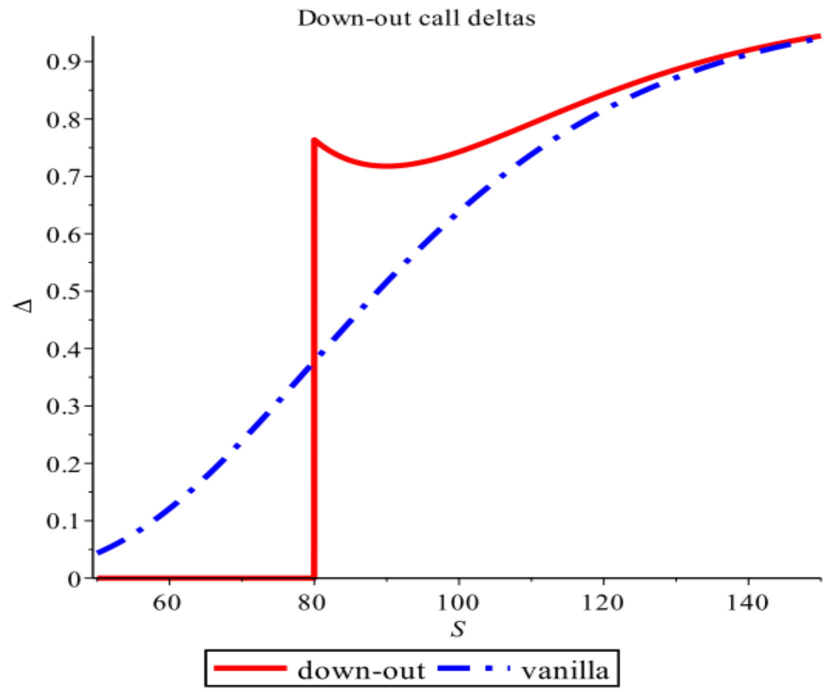
where

$$\Gamma_{\mathrm{bs}} = \frac{\partial^2 C_{\mathrm{bs}}}{\partial S^2}.$$

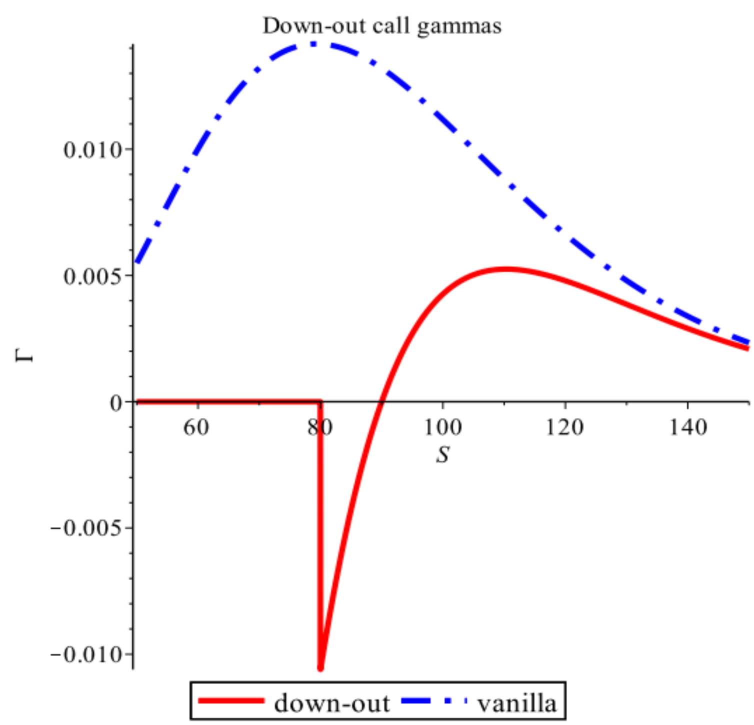
Similar results apply to the other greeks.



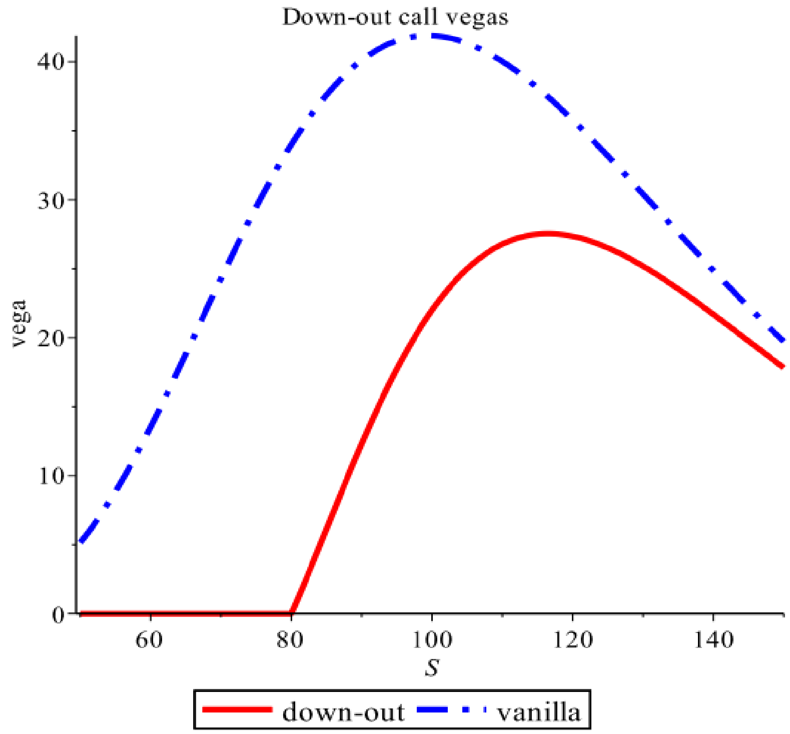
$C_{\mathrm{d/o}}$ against S with $B = 80$, $K = 100$, $T - t = 1.25$, $r = 0.05$, $q = 0$ and $\sigma = 0.3$.



$\Delta_{d/o}$ against S with $B = 80$, $K = 100$, $T - t = 1.25$, $r = 0.05$, $q = 0$ and $\sigma = 0.3$.

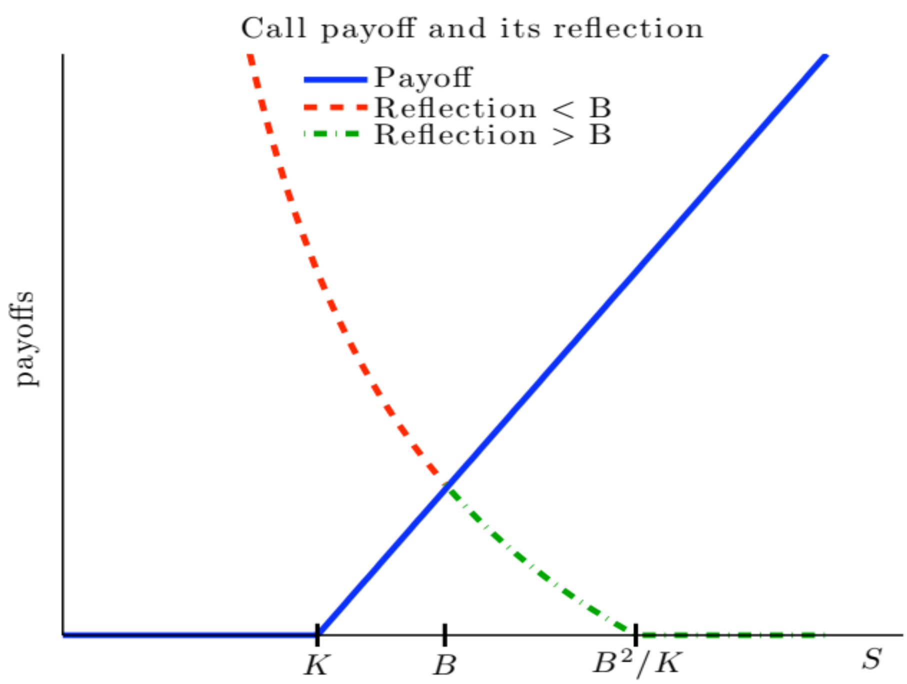


$\Gamma_{d/o}$ against S with $B = 80$, $K = 100$, $T - t = 1.25$, $r = 0.05$, $q = 0$ and $\sigma = 0.3$.



$\varphi_{d/o}$ against S with $B = 80$, $K = 100$, $T - t = 1.25$, $r = 0.05$, $q = 0$ and $\sigma = 0.3$.

Down-and-out calls: barrier above the strike



If the barrier lies above the strike, $B > K$ our trick above fails, because there are some values of $S > B$ where

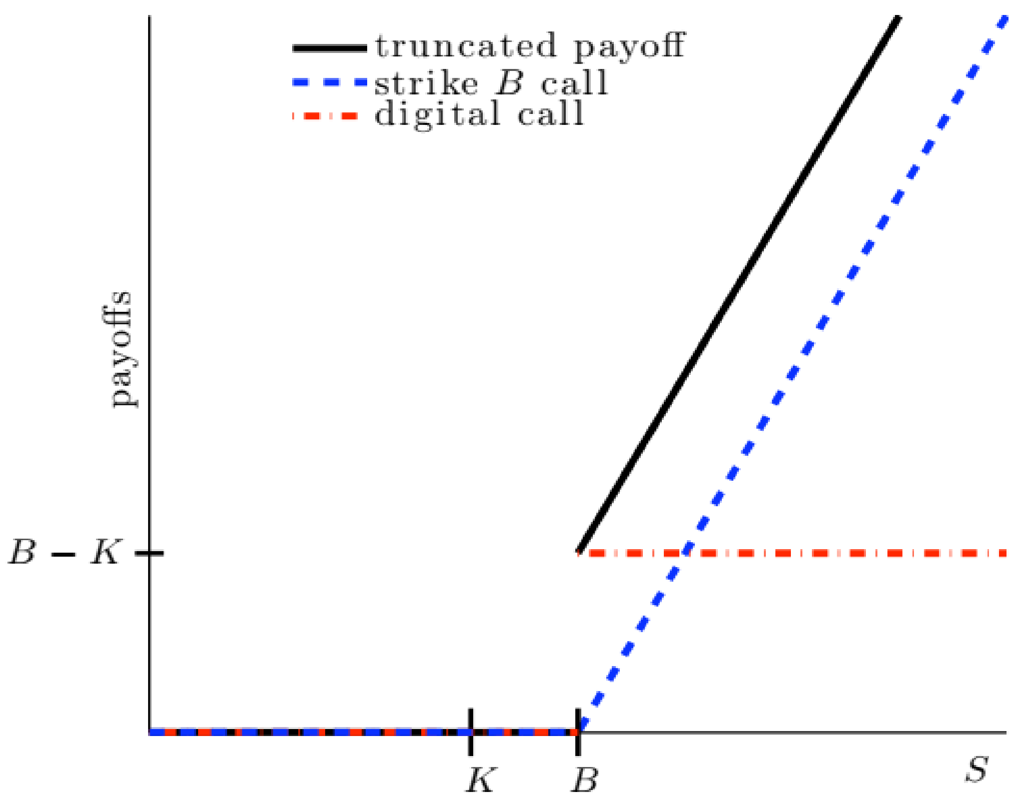
$$C_{\text{bs}}(B^2/S, T) \neq 0.$$

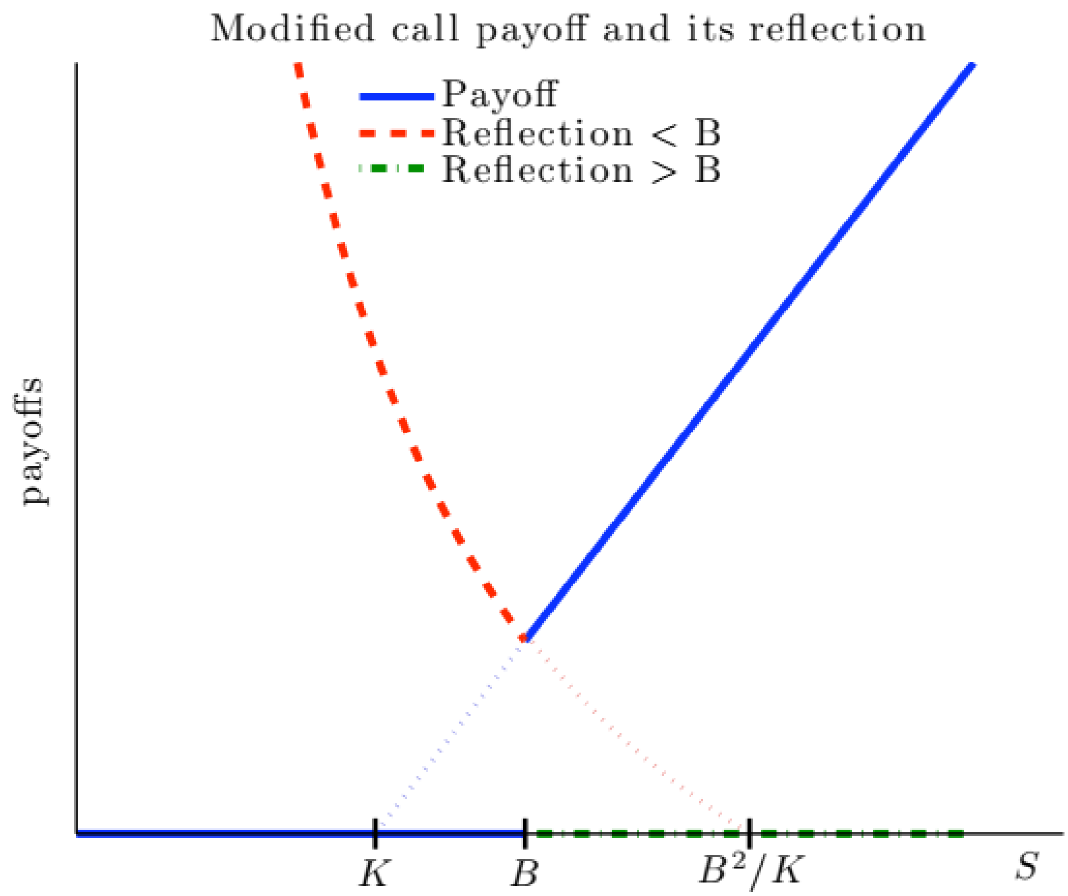
We want the reflected solution to vanish at expiry for $S > B$.*

So before reflecting, value an option with a 'truncated' call payoff which is zero for $S < B$ — we don't care about the payoff for $S < B$ because we aren't pricing the option there.

This is the payoff for a vanilla call struck at B plus $(B - K)$ standard digital calls, each paying either 0 or 1, and also struck at B.

*More generally if we find the reflected vanilla option price with payoff $P_o(S)$ does not result in the correct payoff for the barrier option, the trick is to work with the price of a new vanilla option with the truncated payoff $P_o(S)\,\mathbf{1}_{\{S>B\}}$.





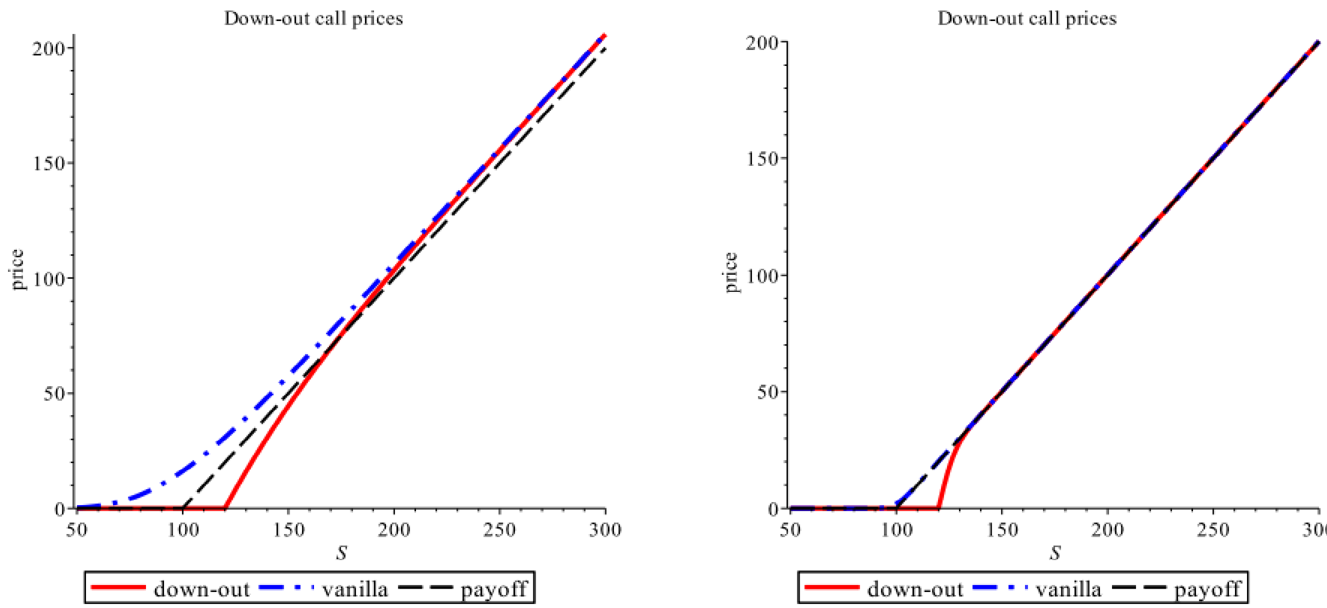
The Black–Scholes value of a down-and-out call which hasn't already knocked out is therefore

$$\begin{aligned} C_{\text{do}}(S,t) &= C_{\text{bs}}(S,t;B) + (B-K)C_{\text{d}}(S,t;B) \\ &- \left(\frac{S}{B}\right)^{2\alpha}\left(C_{\text{bs}}(B^2/S,t;B) + (B-K)C_{\text{d}}(B^2/S,t;B)\right) \end{aligned}$$

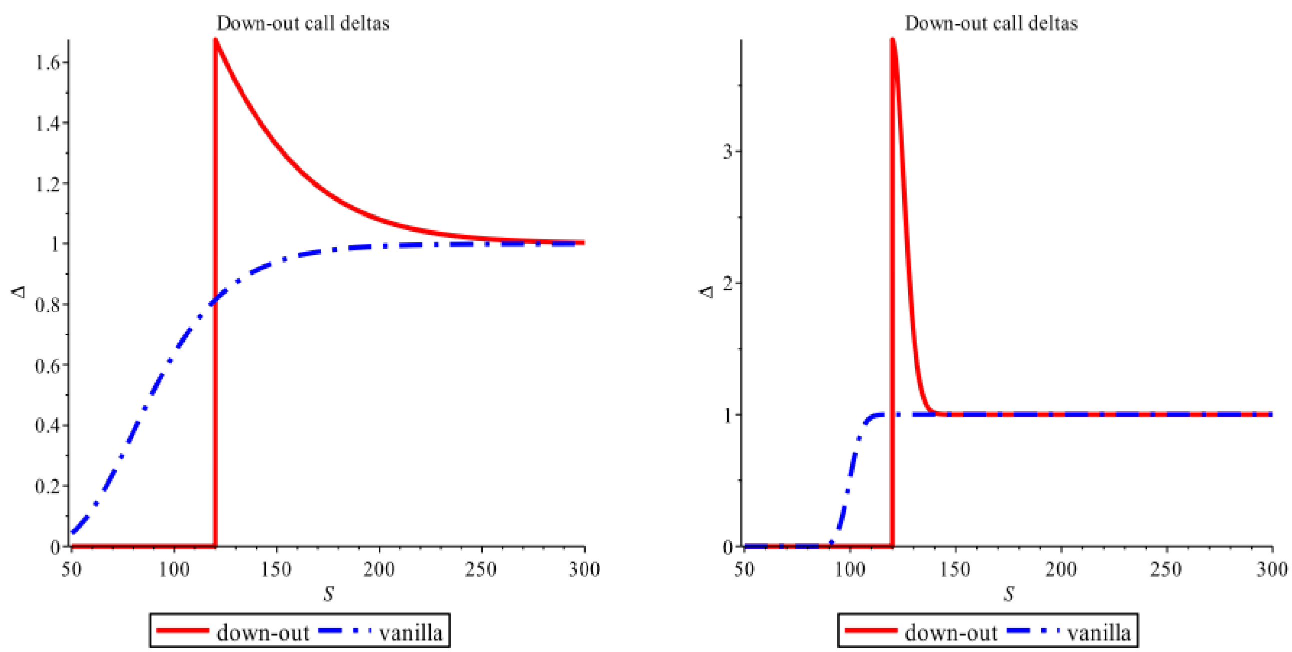
for $S > B$.

The first two terms are the unreflected solution, which takes care of the payoff. The second two are the reflection and take care of the barrier condition.

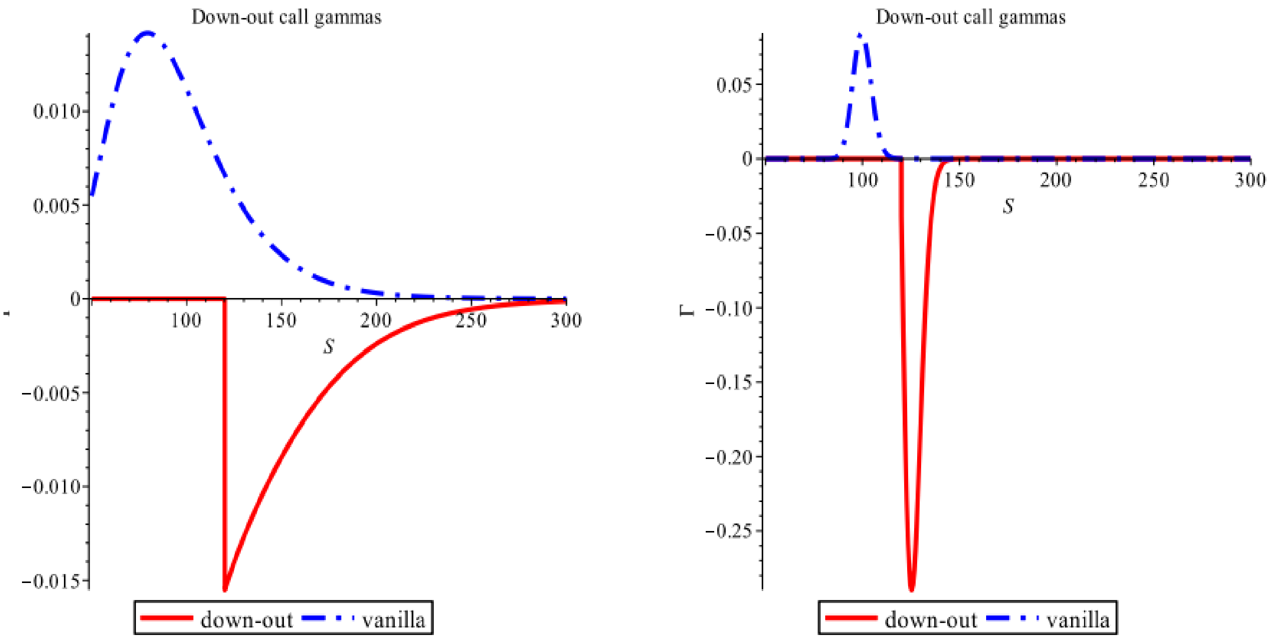
From this we may compute formulae for the other Greeks; they are lengthy.



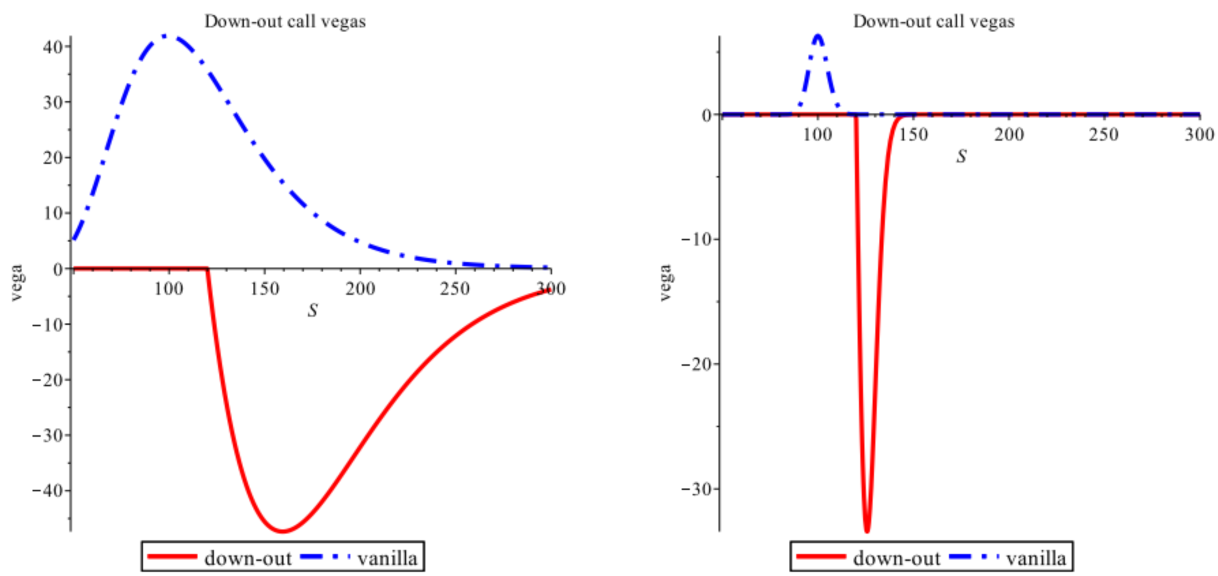
$C_{d/o}$ as a function of S with $K = 100$, $B = 120$, $r = 0.05$, $q = 0$, $\sigma = 0.3$ and $T - t = 1.25$ (left) and $T - t = 0.025$ (right).



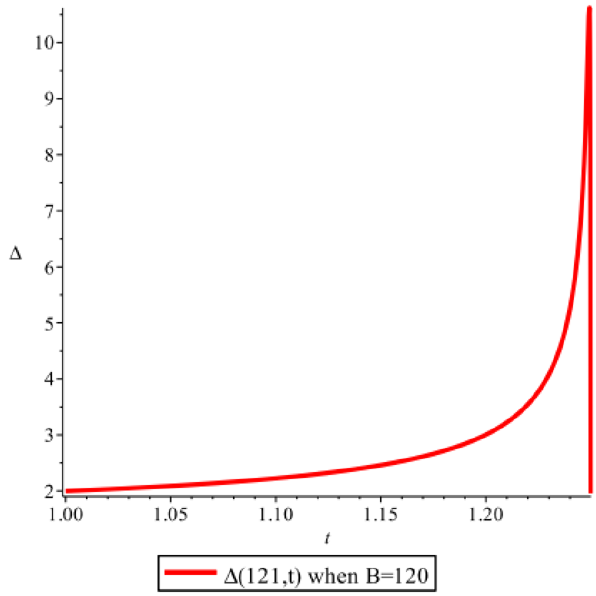
$\Delta_{d/o}$ as a function of S with $K = 100$, $B = 120$, $r = 0.05$, $q = 0$, $\sigma = 0.3$ and $T - t = 1.25$ (left) and $T - t = 0.025$ (right).



$\Gamma_{d/o}$ as a function of S with $K = 100$, $B = 120$, $r = 0.05$, $q = 0$, $\sigma = 0.3$ and $T - t = 1.25$ (left) and $T - t = 0.025$ (right).



$\text{vega}_{d/o}$ as a function of S with $K = 100$, $B = 120$, $r = 0.05$, $q = 0$, $\sigma = 0.3$ and $T - t = 1.25$ (left) and $T - t = 0.025$ (right).



When $B > K$ we find that Δ_{do} becomes singular as $t \to T$ and $S \to B$. This makes delta-hedging the option impractical in this limit.