Other barrier options

Up-and-out call: this is superficially similar to a down-and-out call, except that it knocks out when the asset *rises* to B.

Its risk characteristics, however, are completely different as its value increases with S but then falls sharply as the barrier is approached.

Down-and-in call : this expires worthless *unless* S_t falls to B, in which case it turns into a vanilla call with strike K. The latter is worth $C_{\text{bs}}(B, t; K)$ at exchange.

The corresponding **put** options have obvious definitions.

Up-and-out call

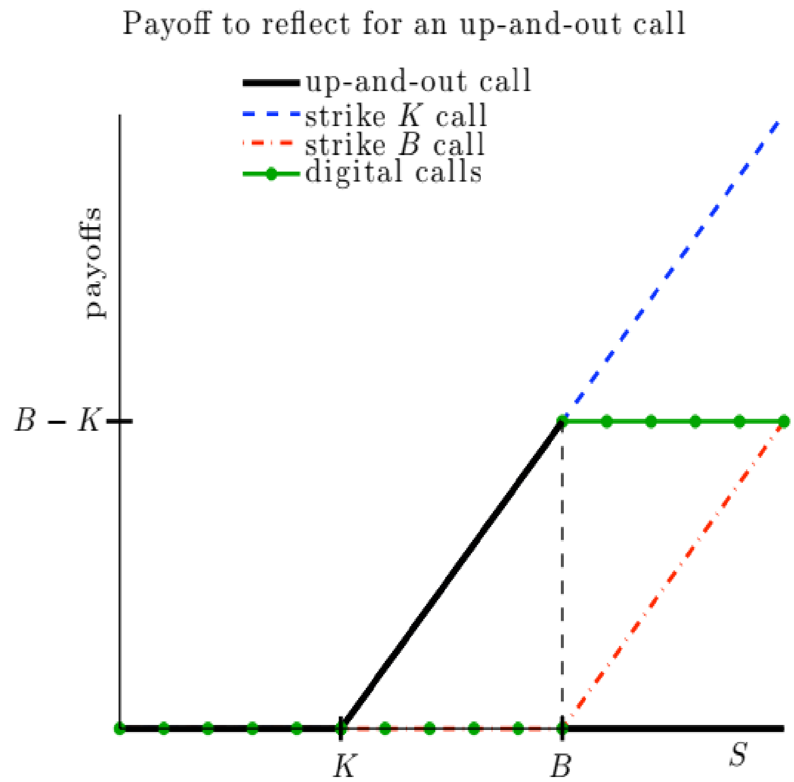
To deal with this option, note that we must have $B > K$ or the option is worthless. The up-and-out option has the same payoff as the vanilla option for $S < B$, but we have to truncate this payoff for $S > B$ if we wish to use a reflection argument.

Therefore, let $V_{\text{bs}}(S, t)$ be the price of an option with the payoff

$$V_{\text{bs}}(S, T) = \begin{cases} 0 & \text{if } S \leq K, \\ S - K & \text{if } K < S \leq B, \\ 0 & \text{if } S > B. \end{cases}$$

This payoff may be written as

$$\max(S - K, 0) - \max(S - B, 0) - (B - K)\, \mathbb{1}_{\{B > S\}}.$$



We can write this as

$$V_{\mathsf{bs}}(S,T) = C_{\mathsf{bs}}(S,T;K) - C_{\mathsf{bs}}(S,T;B) - (B-K)\,C_{\mathsf{d}}(S,T;B),$$

so we have

$$V_{\mathsf{bs}}(S,t) = C_{\mathsf{bs}}(S,t;K) - C_{\mathsf{bs}}(S,t;B) - (B-K)\,C_{\mathsf{d}}(S,t;B).$$

If $S < B$ then $B^2/S > B$, hence

$$V_{\mathsf{bs}}(B^2/S,T) = 0,$$

and, as usual, at $S = B$

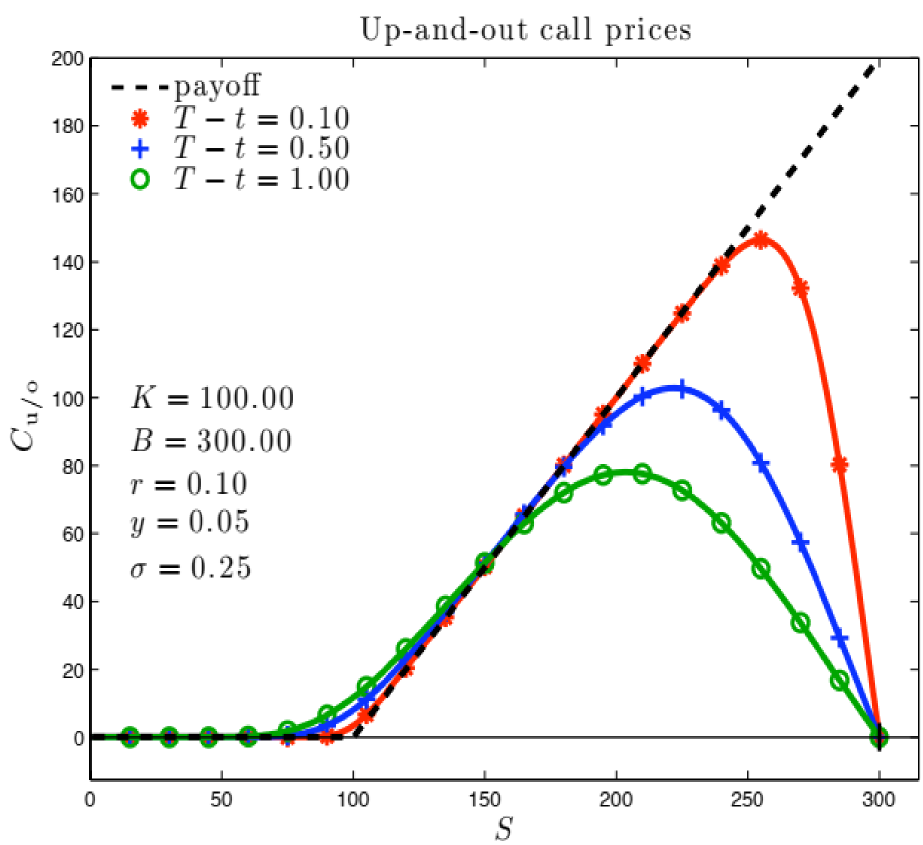
$$\left(\frac{S}{B}\right)^{2\alpha} V_{\mathsf{bs}}(B^2/S,t) = V_{\mathsf{bs}}(B,t).$$

The value of an up-and-out call which has not yet knocked out is

$$\begin{aligned} C_{\text{uo}}(S,t) &= V_{\text{bs}}(S,t) - \left(\tfrac{S}{B}\right)^{2\alpha} V_{\text{bs}}(B^2/S,t) \\ &= C_{\text{bs}}(S,t;K) - \left(\tfrac{S}{B}\right)^{2\alpha} C_{\text{bs}}\left(\tfrac{B^2}{S},t;K\right) \\ &- \left(C_{\text{bs}}(S,t;B) - \left(\tfrac{S}{B}\right)^{2\alpha} C_{\text{bs}}\left(\tfrac{B^2}{S},t;B\right)\right) \\ &- (B-K)\left(C_{\text{d}}(S,t;B) - \left(\tfrac{S}{B}\right)^{2\alpha} C_{\text{d}}\left(\tfrac{B^2}{S},t;B\right),\right) \end{aligned}$$

for $S < B$.

The singularity in Δ near $S = B$ as $t \to T$ means it is impractical to delta-hedge the option in this limit.



Out-in parity

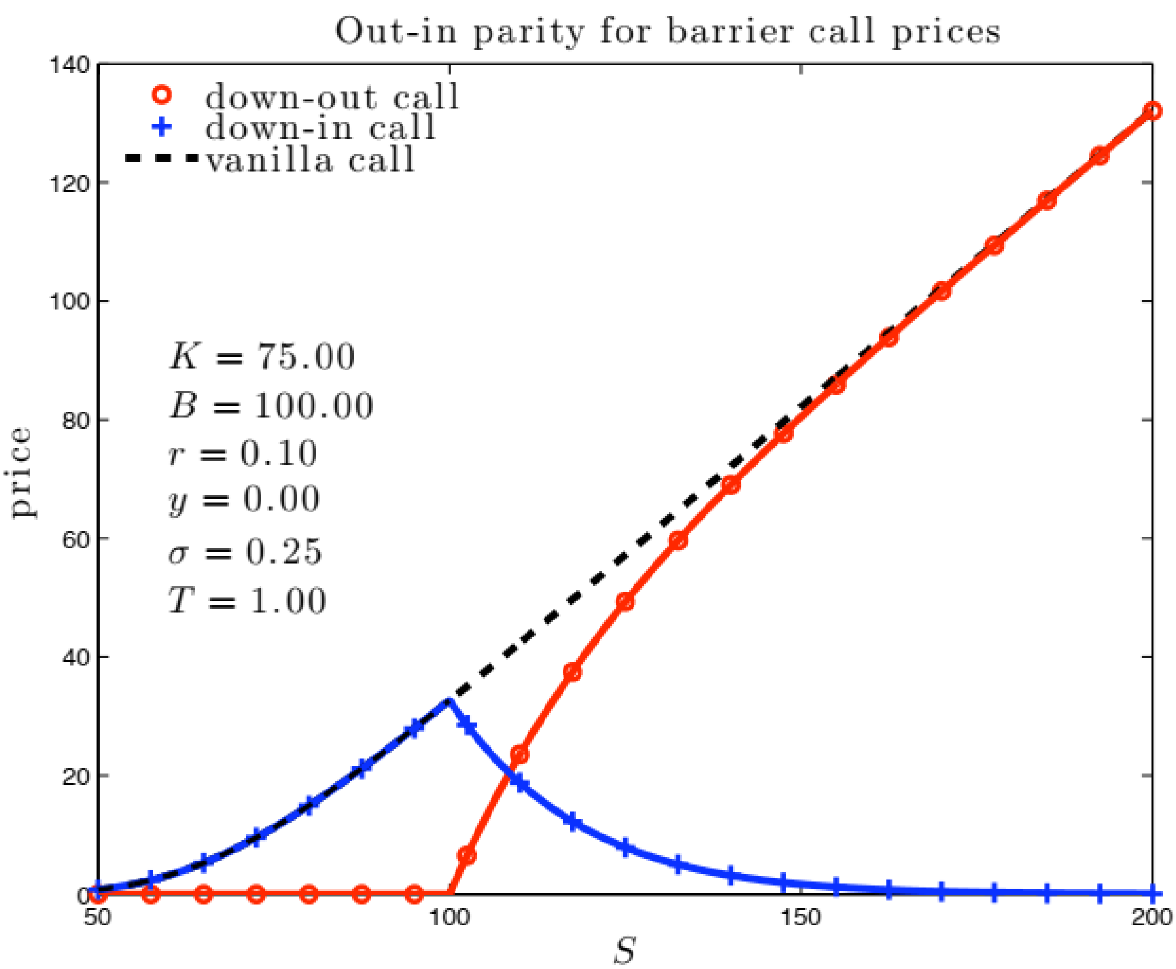
Without rebates, for any barrier options with just one barrier,

$$\text{(down-and-out)} + \text{(down-and-in)} = \text{vanilla},$$

because whether the asset hits the barrier or not, one and only one of the barrier options is exercised, the other expiring worthless. So

$$V_{\text{di}}(S,t) + V_{\text{do}}(S,t) = V_{\text{bs}}(S,t).$$

This applies to up options, with the obvious modification, as well.



Forward-start barrier options

Many barrier options have a forward-start condition. At time $t = 0$, it is agreed that at an intermediate time T_1 the holder will receive a barrier contract with later expiry $T > T_1$. The strike and barrier are set by reference to the asset value at T_1.

For example, a six-month down-and-out call, starting in three months, with the strike to be set at-the-money (i.e., at the spot price three months from initiation) and the barrier to be set at 90% of the strike.

Here T_1 is three months and T is nine months. There is no barrier for the first three months; for the following six months it is a regular down-and-out call with $B = 0.9\,K$.

These options are very like forward-start vanilla options. To value them, first work back to the intermediate date T_1, then use the values at that time as an intermediate payoff.

We'll assume that the barrier is continuously monitored. So, with the example above, the barrier option is at-the-money at time T_1,

$$K = S(T_1),$$

and the barrier is set to

$$B = c\,S(T_1),$$

where $0 < c < 1$ — in our case $c = 0.9$.

At time T_1, set $K = S$ and $B = c\,S$ in the earlier formula for a down-and-out call,

$$\begin{aligned} C^{\mathrm{fs}}_{\mathrm{do}}(S,T_1) &= C_{\mathrm{do}}(S,T_1;\ K=S,\ B=c\,S,\ T) \\ &= C_{\mathrm{bs}}(S,T_1;S,T) - (S/c\,S)^{2\alpha}\,C_{\mathrm{bs}}((c\,S)^2/S,T_1;S,T) \\ &= A_{\mathrm{do}}(T_1,T)\,S \end{aligned}$$

where $A_{\mathrm{do}}(T_1,T)$ is easily worked out and does not depend on S or t. For earlier times, $t < T_1$,

$$C^{\mathrm{fs}}_{\mathrm{do}}(S,t) = A_{\mathrm{do}}(T_1,T)\,S\,e^{-y(T_1-t)}.$$

Note that this procedure also works if different constant volatilities are used for each leg of the contract, as is sometimes done for forward-start vanilla options.

Appendix: three proofs of the reflection principle

We present three proofs of the reflection principle, which asserts that if $V(S,t)$ is a solution of the Black–Scholes equation then so too is

$$W(S,t) = (S/B)^{2\alpha}\,V(B^2/S,t),$$

where $\alpha = \frac{1}{2} - (r-y)/\sigma^2$.

We also note that this change of variables is its own inverse,

$$V(S,t) = (S/B)^{2\alpha}\,W(B^2/S,t),$$

which is one reason for the term ‘reflection principle’. The proof of this fact is left as an exercise.

We now proceed with the proofs of the principle.

Proof by reduction to a diffusion equation

We start with the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)\, S\, \frac{\partial V}{\partial S} - r\, V = 0$$

and introduce the change of variables

$$x = \log(S/B), \quad \tau = \sigma^2(T-t), \quad V(S,t) = e^{\alpha x - \beta\tau} u(x,\tau),$$

where*

$$\alpha = \tfrac{1}{2} - (r-y)/\sigma^2 \quad \text{and} \quad \beta = \tfrac{1}{2}\alpha^2 - r/\sigma^2.$$

*Note that $\hat{\mathcal{L}}_\tau = e^{\alpha W_\tau - \beta\tau} = e^{-rt}\, e^{\alpha W_\tau - \alpha^2\tau/2} = e^{-rt}\,\mathcal{L}_\tau$ is essentially a *discounted* change of measure.

We find that $u(x,\tau)$ satisfies the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},$$

which has the property* that if $u(x,\tau)$ is a solution then so too is $u(-x,\tau)$. The map $x \to -x$ is a reflection in the origin, which is another reason for the term ‘reflection principle’.

Therefore, we get two solutions of the Black–Scholes equation for the price of one! These are

$$V(S,t) = e^{\alpha x - \beta\tau}\, u(x,\tau),$$
$$W(S,t) = e^{\alpha x - \beta\tau} u(-x,\tau).$$

*This is easily verified by setting $z = -x$ and using the chain rule.

Now observe that

$$W(S,t) = e^{2\alpha x}\left(e^{-\alpha x - \beta\tau}\, u(-x,\tau)\right),$$

and that

$$e^{-\alpha x - \beta\tau}\, u(-x,\tau)$$

is simply the expression for $V(S,t)$ with *all* of the x's replaced by $-x$. Since $x = \log(S/B)$,

$$-x = \log\,(B/S) = \log\left(\frac{B^2}{S}\Big/B\right)$$

and so, to get from x to $-x$, all we need do is replace S by B^2/S in the expression $x = \log(S/B)$.

Thus

$$e^{-\alpha x - \beta\tau}\, u(-x,\tau) = V(B^2/S,t),$$

and hence we have the reflection principle,

$$W(S,t) = e^{2\alpha x}\, V(B^2/S,t) = \left(\frac{S}{B}\right)^{2\alpha} V(B^2/S,t).$$

We leave it as an exercise to interpret this in terms of the risk-neutral processes

$$\frac{dS_t}{S_t} = (r-y)dt + \sigma\, dW_t \quad \text{and} \quad dx_t = dW_t.$$

[Hint: it comes down to showing the two processes x_t^+ and x_t^- with $dx_t^+ = dW_t$, $dx_t^- = -dW_t$ and $x_0^+ = x_0^- = x_0$ have the same terminal distributions.]

Proof by brute force

Suppose that

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2\,\frac{\partial^2 V}{\partial S^2} + (r-y)\, S\,\frac{\partial V}{\partial S} - r\,V = 0.$$

Set

$$W(S,t) = S^{\beta}\, V(\xi,t), \quad \xi = \frac{B^2}{S},$$

where β is an as-yet-undetermined constant.

The chain rule shows that

$$\frac{\partial}{\partial S} = -\frac{\xi}{S}\,\frac{\partial}{\partial \xi}.$$

Using the chain and product rules, we find that

$$\begin{aligned}
\frac{\partial W}{\partial t} &= S^{\beta}\,\frac{\partial V}{\partial t}, \\
\frac{\partial W}{\partial S} &= S^{\beta-1}\left(\beta\, V - \xi\,\frac{\partial V}{\partial \xi}\right), \\
\frac{\partial^2 W}{\partial S^2} &= S^{\beta-2}\left(\beta(\beta-1)\, V - 2(\beta-1)\,\xi\,\frac{\partial V}{\partial \xi} + \xi^2\,\frac{\partial^2 V}{\partial \xi^2}\right).
\end{aligned}$$

We now try to choose β so that W satisfies the same Black–Scholes equation as V,

$$\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2\,\frac{\partial^2 W}{\partial S^2} + (r-y)\, S\,\frac{\partial^2 W}{\partial S^2} - r\, W = 0.$$

This implies that we must have

$$\begin{aligned}
\frac{\partial V}{\partial t} \;+\; \tfrac{1}{2}\sigma^2\,\xi^2\,\frac{\partial^2 V}{\partial \xi^2} \;-\; \Big((\beta-1)\sigma^2 + r - y\Big)\xi\,\frac{\partial V}{\partial \xi} \\
+ \;\Big(\tfrac{1}{2}\,\beta(\beta-1)\sigma^2 + \beta(r-y) - r\Big)V = 0,
\end{aligned}$$

which will be the case if we can find β so that

$$\begin{aligned}
-(\beta-1)\sigma^2 - r - y &= r - y, \\
\tfrac{1}{2}\,\beta(\beta-1)\sigma^2 + \beta(r-y) - r &= -r,
\end{aligned}$$

as this reduces the expression above to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\,\xi^2\,\frac{\partial^2 V}{\partial \xi^2} + (r-y)\,\xi\,\frac{\partial V}{\partial \xi} - r\, V = 0$$

which is certainly true.

Fortunately, these two equations for β are not independent. They both reduce to

$$(\beta-1)\sigma^2 = -2(r-y),$$

which implies that

$$\beta = 1 - \frac{2(r-y)}{\sigma^2}.$$

Setting $\beta = 2\alpha$ gives the reflection formula that we have used in the main text of these lectures, namely,

$$W(S,t) = S^{2\alpha} V(B^2/S, t)$$

with

$$\alpha = \tfrac{1}{2} - (r-y)/\sigma^2.$$

Proof by probability

The Gaussian shift theorem says that if Z is drawn from an $N(0,1)$ normal distribution, F is a function and c is a constant, then

$$\begin{aligned}
\mathbb{E}[\, e^{cZ} F(Z)\,] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(Z)\, e^{-Z^2/2+cZ}\, dZ \\
&= \frac{e^{c^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(Z)\, e^{-(Z-c)^2/2}\, dZ \\
&= \frac{e^{c^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(Y+c)\, e^{-Y^2/2}\, dY \\
&= e^{c^2/2}\, \mathbb{E}[\, F(Z+c)\,],
\end{aligned} \qquad (5)$$

provided, of course, that the expectations involved exist.

Under the usual Black–Scholes assumptions we have

$$\begin{aligned} V(S,t) &= e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\Big[\,P_o(S_T)\,\Big|\,S_t = S\,\Big], \\ S_T &= S_t \exp\big(\xi + \eta\, Z\big), \end{aligned} \tag{6}$$

where

$$\tau = T - t, \quad \xi = (r - y - \tfrac{1}{2}\sigma^2)\,\tau, \quad \eta = \sigma\sqrt{\tau}, \quad Z \sim N(0,1).$$

Note that

$$\beta = 2\alpha = 1 - 2(r-y)/\sigma^2 = -2\xi/\eta^2. \tag{7}$$

We may write (6) in the form

$$V(S,t) = e^{-r\tau}\mathbb{E}\Big[\,P_o\Big(S\,e^{\xi+\eta\,Z}\Big)\,\Big].$$

Therefore, we can write

$$\begin{aligned} W(S,t) &= \left(\frac{S}{B}\right)^{\beta} V\left(\frac{B^2}{S}, t\right) \\ &= e^{-r\tau}\mathbb{E}\left[\left(\frac{S}{B}\right)^{\beta} P_o\left(\frac{B^2}{S}\,e^{\xi+\eta Z}\right)\right] \\ &= e^{-r\tau}\,e^{\beta\xi}\,\mathbb{E}\left[e^{\beta\eta Z}\left(\frac{Se^{-\xi-\eta Z}}{B}\right)^{\beta} P_o\left(\frac{B^2}{S}\,e^{\xi+\eta Z}\right)\right] \\ &= e^{-r\tau+\beta(\xi+\beta\eta^2/2)}\,\mathbb{E}\left[\left(\frac{Se^{-\xi-\eta(Z+\beta\eta)}}{B}\right)^{\beta} P_o\left(\frac{B^2}{S}\,e^{\xi+\eta\,(Z+\beta\eta)}\right)\right] \\ &= e^{-r\tau}\,\mathbb{E}\left[\left(\frac{Se^{-\xi-\beta\eta^2-\eta Z)}}{B}\right)^{\beta} P_o\left(\frac{B^2}{S}\,e^{\xi+\beta\eta^2+\eta\,Z}\right)\right], \end{aligned}$$

since $\beta = -2\xi/\eta^2$ implies that $\beta\,(\xi + \beta\,\eta^2/2) = 0$.

Furthermore, $\beta = -2\xi/\eta^2$ also implies that

$$-\xi - \beta\eta^2 = \xi, \quad \xi + \beta\eta^2 = -\xi$$

and, since $Z \sim N(0,1)$ if and only if $-Z \sim N(0,1)$, we have

$$\begin{aligned} W(S,t) &= e^{-r\tau}\,\mathbb{E}\Big[\Big(\frac{S\,e^{\xi+\eta Z}}{B}\Big)^{\beta} P_o\Big(\frac{B^2}{S}\,e^{-\xi-\eta Z}\Big)\Big] \\ &= e^{-r\tau}\,\mathbb{E}\Big[\Big(\frac{S\,e^{\xi+\eta Z}}{B}\Big)^{\beta} P_o\Big(\frac{B^2}{S\,e^{\xi+\eta Z}}\Big)\Big], \end{aligned}$$

which is equivalent to

$$W(S,t) = e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\Big[\Big(\frac{S_T}{B}\Big)^{\beta} P_o\Big(\frac{B^2}{S_T}\Big)\,\Big|\,S_t = S\Big].$$

Therefore we have

$$W(S,t) = e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\big[F(S_T)\,\big|\,S_t = S\big]$$

which satisfies the Black–Scholes equation and where the the payoff function is

$$F(S_T) = (S_T/B)^{\beta} P_o\big(B^2/S_T\big).$$