Black–Scholes Theory

In Financial Derivatives, you saw how to construct a basic model of a financial market.

The fundamental modelling assumption was that of 'no arbitrage', that is that the price of an asset should evolve in such a way that it is impossible to obtain a risk-free payoff without investing any initial capital.

We only considered models with some abstractions: unlimited liquidity, linear pricing, no contract default. These restrictions avoid any modelling of transaction costs, different types of market execution (limit orders, etc) but allow us to build a tractable mathematical model.

As in many areas of applied mathematics, the key is to balance the simplicity and tractability of our model against its accuracy in modelling reality. All important phenomena should be included, but this depends on how the model is to be used.

With only these assumptions, there are still some results that one can obtain. These are often termed 'model-free' (but still have the assumptions from before).

A classic example of this is the time-T forward price for a traded asset, which is given by $F_t = S_t e^{r(T-t)}$, where r is the continuously compounded interest rate.

Another classic example is Put-Call parity

$$P_t - C_t = Ke^{-r(T-t)} - S_t.$$

You should be familiar with the proofs of both of these results.

In order to say much more than this, it is necessary to build a probabilistic model for the financial market. You have seen two such models.

The first is the binomial model, where time is discrete, but the stock can take only two values at each step, and there is a risk-free bond with constant growth $B_{t+1} = RB_t$.

In this model we can calculate the optimal hedge ratio by solving a pair of linear equations, and then use backward induction to determine the price at every point of the possibility tree.

The second is the Black–Scholes model, where the stock price is assumed to satisfy the SDE (under the real-world measure $\mathbb{P}$)

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dW_t.$$

Here μ is allowed to be any 'nice' stochastic process (e.g. bounded, piecewise continuous and adapted), while σ is a constant, W is a Brownian motion. There is also a risk-free bond B, with constant interest rate r (so $dB_t = rB_t dt$). The filtration is generated by S.

The key to pricing is then 'replication'. To hedge a (European) claim with terminal value $\xi(S_T)$, we seek to find a portfolio where a fraction π_t of our wealth is in the stock, such that we satisfy the *self-financing condition*

$$dV_t = (1 - \pi_t)V_t dB_t + \pi_t V_t dS_t$$

and at maturity matches the price of the asset $V_T = \xi(S_T)$.

Given this, the position $\xi(S_T) - V_T = 0$, so should cost nothing by no-arbitrage. This allows us to price the claim. Amazingly, for the model stated, this is indeed possible.

By holding $(-\pi_t V_t)$ stocks, we *hedge* a long position in the claim.

To calculate prices, we can then work in either of two ways.

First option: Assume that $V_t = v(t, S_t)$, for v a $C^{1,2}$ function. Then we apply Ito's formula, to obtain

$$dV_t = (\partial_t v + \frac{\sigma^2 S^2}{2}\partial_{ss} v)dt + \partial_s v dS_t$$

and matching terms with our earlier statement, we obtain $\pi = (\partial_s v)/v$ and

$$(1-\pi)vr = \partial_t v + \frac{\sigma^2 S^2}{2}\partial_{ss} v$$

which simplifies to the Black–Scholes PDE

$$\partial_t v + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss} v + rS\partial_s v - rv = 0$$

We solve this PDE with the terminal condition $v(T, s) = \xi(s)$ for $s > 0$.

Second option: We have the first and second fundamental theorems of asset pricing.

First Theorem: No arbitrage (+no approximations of arbitrage) among the portfolios with value bounded below $\Leftrightarrow$ all traded asset prices are (local) martingales under some measure $\mathbb{Q}$.

Second Theorem: All contingent claims can be replicated (the market is complete) $\Leftrightarrow$ the measure $\mathbb{Q}$ is unique.

Proofs of these in general are delicate and depend on the Hahn–Banach theorem ($\mathbb{Q}$ is not explicitly constructed).

In the Black–Scholes model, all measure changes are described by stochastic exponentials. That is, if we define

$$\Lambda_t = \mathcal{E}\Big(\int_0^{\cdot} \frac{r-\mu_u}{\sigma} dW_u\Big)_t$$

$$= \exp\Big(\int_0^t \frac{r-\mu_u}{\sigma}dW_u - \int_0^t \frac{(r-\mu_u)^2}{2\sigma^2}du\Big),$$

we have a *unique* measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = \Lambda_T$ under which

$$\frac{dS_t}{S_t} = rdt + \sigma dW^{\mathbb{Q}};$$

for $W^{\mathbb{Q}}$ a $\mathbb{Q}$-Brownian motion. Hence, under $\mathbb{Q}$, given $\mathcal{F}_t$,

$$\log(S_T/S_t) \sim N((r-\sigma^2/2)(T-t), \sigma^2(T-t)).$$

As discounted prices are martingales, we can then calculate

$$V_t = e^{-r(T-t)}E^{\mathbb{Q}}\Big[V(T,S_T)\Big|\mathcal{F}_t\Big]$$

Using either of these methods, if $\xi(s) = (s-K)^+$, we obtain the famous Black–Scholes formula for the European call

$$v(t,s) = S\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-)$$

where $d_+ = \frac{\log(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $d_- = d_+ - \sigma\sqrt{T-t}$, and $\mathcal{N}$ is the standard normal cdf. Similarly for the European put (or using put-call parity).

We can also calculate the hedge ratios (*greeks*) which determine the amount of stock which should be held to offset a (short) position in the claim ($\Delta = \partial_s v$), along with further sensitivities of the price to changes in market parameters.

In Financial Derivatives, you also saw that if the stock pays a continuous dividend at rate y, then the formula changes slightly. The Black–Scholes PDE becomes

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{ss}V + (r-y)S\partial_s V - rV = 0$$

The risk-neutral dynamics are

$$dS_t/S_t = (r-y)dt + \sigma dW_t^{\mathbb{Q}}$$

By supposing we reinvest all dividends in the stock, we replace S_t with $\tilde{S}_t := e^{yt}S_t$. Then $\tilde{S}$ has the usual zero-dividend dynamics (with a different drift). This change of variables simplifies our problem with dividends to the classical one.

For example, with continuous dividends of rate y, we can develop the pricing formulas for European options by considering the expected discounted payoffs and reducing the problem to the zero-dividend case, with S_T replaced by $S_T e^{-q(T-t)}$. For example, for the price of a European call option, we have:

$$\begin{aligned} C^{BS}(S,t;K,T,\sigma,r,y) &= e^{-r(T-t)}E^{\mathbb{Q}}(S_T-K)^+ \\ &= e^{-(r+y)(T-t)}E^{\mathbb{Q}}\left(S_T e^{y(T-t)}-Ke^{y(T-t)}\right)^+ \\ &= e^{-y(T-t)}C^{BS}(S,t;Ke^{y(T-t)},T,\sigma,r,0) \end{aligned}$$

In this course we will consider extensions of this, to non-European or to path dependent options, to options on multiple assets, and to perturbations of this model. The aim is to obtain pricing and hedging formulae for each of the key options we consider.

A first and fairly simple extension is to the case where r and σ can depend on time (but are deterministic).

The BS model is based on the assumption that the market **volatility**, **interest rate** and **dividend rate** are **known constants**.

The simplest possible extension of this model, which provides a better approximation of the reality, is to assume that the above quantities are **known** but **not necessarily constant**.

In other words, we assume that $\sigma = (\sigma(t))_{t\geq 0}$, $r = (r(t))_{t\geq 0}$ and $y = (y(t))_{t\geq 0}$ are **deterministic functions of time**, so that the equation for the underlying becomes

$$dS_t/S_t = (\mu - y(t))dt + \sigma(t)dW_t$$

It is easy to show that, in this case, the risk-neutral dynamics are given by

$$dS_t/S_t = (r(t) - y(t))dt + \sigma(t)dW_t^{\mathbb{Q}}$$

The standard arguments yield the **BSPDE with time-dependent coefficients**. The equation is the same as in the constant coefficient case, except that the coefficients are now functions of time. As a result, pricing and hedging of contingent claims in the BS model with time-dependent parameters is, essentially, the same as in the constant parameter case.

Let's solve the SDE for the risk-neutral evolution of S.

Using Ito's formula we can show that

$$\log\left(S_T/S_t\right) = \int_t^T (r(u) - y(u) - \frac{1}{2}\sigma^2(u))du + \int_t^T \sigma(u)dW_u^{\mathbb{Q}}$$

Basic fact about stochastic integrals: the last term in the right hand side of the above has the distribution

$$N\Big(0, \int_t^T \sigma^2(u)du\Big).$$

With the notation

- $\bar{r}_{t,T} := \frac{1}{T-t}\int_t^T r(u)du$
- $\bar{y}_{t,T} := \frac{1}{T-t}\int_t^T y(u)du$
- $\bar{\sigma}^2_{t,T} := \frac{1}{T-t}\int_t^T \sigma^2(u)du$

we notice that, given S_t, under the risk-neutral measure,

$$\log\left(S_T/S_t\right) \sim N\left((T-t)(\bar{r}_{t,T} - \bar{y}_{t,T} - \frac{1}{2}\bar{\sigma}^2_{t,T}), (T-t)\bar{\sigma}^2_{t,T}\right)$$

Compare this with the case of constant parameters:

$$\log\left(S_T/S_t\right) \sim N\left((T-t)(r - y - \frac{1}{2}\sigma^2), (T-t)\sigma^2\right)$$

Thus, in the BS model with known (deterministic) **time-dependent parameters**, we can apply **the same pricing formulas** for **European options** as in the case of **constant parameters**, using the averages $\bar{\sigma}^2_{t,T}$, $\bar{r}_{t,T}$ and $\bar{y}_{t,T}$ in place of σ^2, r and y.