Options on multiple assets

We will now consider claims which depend on more than one traded asset.

These include basket options (and hence index options), spread options and exchange options.

To consider these fully, we need to be familiar with stochastic calculus for multidimensional processes.

Multidimensional Brownian motion.

Definition: The stochastic process $W = (W^1, \dots, W^d)^T$ (here and throughout the rest of the course, we denote the transpose of a matrix by the superscript "T") is a **standard d-dimensional Brownian motion (BM)** if its components, W^i's, are independent one-dimensional Brownian motions.

Definition: The stochastic process $W = (W^1, \dots, W^d)^T$ is a d-dimensional BM with **correlation matrix Σ** (which is a symmetric positive semidefinite $d \times d$ matrix) if $W = A \cdot B$, for some matrix $A \in \mathbf{Mat}(d \times n)$, with $AA^T = \Sigma$, and a standard n-dimensional BM $B = (B^1, \dots, B^n)^T$.

A **standard** multidimensional BM is a multidimensional BM with the **identity correlation matrix** I.

Consider a d-dimensional BM W with correlation matrix Σ. Then any linear combination of its components produces a one-dimensional BM times a scalar: for any $\lambda = (\lambda_1, \dots, \lambda_d)^T \in \mathbb{R}^d$, we have

$$W_t\lambda = \sum_{i=1}^{d} \lambda_i W_t^i = \sqrt{\lambda^T \Sigma \lambda}\, \tilde{W}_t,$$

where $\tilde{W}$ is a one-dimensional BM.

If Σ is invertible, then, for each $t \geq 0$, the distribution of the random vector $W_t = (W^1, \ldots, W^d)$ has Gaussian density

$$p_t(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det \Sigma t}} \exp\left(-\frac{1}{2t} x^T \Sigma^{-1} x\right)$$

Two-dimensional case.

The case of dimension $d = 2$ is special, because it allows to illustrate the key properties of financial models with several risky assets without going into too much technicalities. In addition, the two-dimensional case is most relevant for the models of foreign-exchange (FX) markets.

The distribution of a two-dimensional BM $W = (W^1, W^2)$ is uniquely determined by the **"instantaneous correlation"** $\rho \in [-1, 1]$ between its components:

$$\frac{E(W_t^1 W_t^2)}{\sqrt{E\left(W_t^1\right)^2}\sqrt{E\left(W_t^2\right)^2}} = \frac{\rho t}{t} = \rho$$

In this case, the correlation matrix is given by

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

And the component W^2 can be represented as

$$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2} B_t$$

for a one-dimensional BM B independent of W^1.

In addition, for any numbers λ_1 and λ_2, we have, for all $t \geq 0$:

$$\lambda_1 W_t^1 + \lambda_2 W_t^2 = \sqrt{\lambda_1^2 + 2\rho\lambda_1\lambda_2 + \lambda_2^2}\, \tilde{W}_t,$$

where $\tilde{W}$ is a one-dimensional BM.

Two-dimensional Ito formula.

Consider a two-dimensional Ito process (X_1, X_2):

$$dX_{1,t} = \mu_t^1 dt + \sigma_t^1 dW_t^1$$
$$dX_{2,t} = \mu_t^2 dt + \sigma_t^2 dW_t^2$$

where W^1 and W^2 are BM's with (instantaneous) correlation ρ (this is typically written as $dW_t^1 dW_t^2 = \rho dt$), and μ^j and σ^j are stochastic processes satisfying the usual integrability conditions, for $j = 1, 2$.

Assume function $V : (0,\infty)^2 \times (0,T) \to \mathbb{R}$ is of class $C^{2,1}$. Then we have

$$\begin{aligned} dV(X_{1,t}, X_{2,t}, t) = \Big[&\frac{\partial}{\partial t} V(X_{1,t}, X_{2,t}, t) \\ &+ \frac{1}{2}\Big(\left(\sigma_t^1\right)^2 \frac{\partial^2}{\partial X_1^2} V(X_{1,t}, X_{2,t}, t) \\ &\quad + 2\rho\sigma_t^1\sigma_t^2 \frac{\partial^2}{\partial X_1 \partial X_2} V(X_{1,t}, X_{2,t}, t) \\ &\quad + \left(\sigma_t^2\right)^2 \frac{\partial^2}{\partial X_2^2} V(X_{1,t}, X_{2,t}, t)\Big)\Big] dt \\ &+ \frac{\partial}{\partial X_1} V(X_{1,t}, X_{2,t}, t) dX_{1,t} + \frac{\partial}{\partial X_2} V(X_{1,t}, X_{2,t}, t) dX_{2,t} \end{aligned}$$

Options on two assets.

Given the price processes of two assets S_1 and S_2, there are several popular choices of the payoffs of European options written on these assets.

Basket call has the payoff

$$(S_{1,T} + S_{2,T} - K)^+$$

It is a call option written on an index consisting of a linear combination of several assets (with positive weights).

Spread options are popular in commodities and energy trading. They have the payoff in the form of a function of the difference of two assets. For example,

$$(S_{1,T} - S_{2,T} - K)^+$$

Such payoff may allow the owner of a power plant to insure against losses arising from a high price of gas relative to the price of electricity.

Exchange option allows to trade one currency for another at a pre-specified rate. The payoff can be written as

$$\max(S_{1,T}, S_{2,T})$$

And it can be expressed through the spread option

$$\max(S_{1,T}, S_{2,T}) = S_{1,T} + (S_{2,T} - S_{1,T})^+$$

Two-dimensional Black-Scholes model.

The general assumptions of unlimited liquidity, linear pricing rule and NA are in place.

We also assume that the usual mathematical abstractions of continuous-time models of financial markets hold.

The market now consists of a bank account with constant interest rate r and **two** risky assets whose prices are modeled by the *two-dimensional geometric Brownian motion* (GBM) (S_1, S_2):

$$\frac{dS_{1,t}}{S_{1,t}} = (\mu_1 - q_1)dt + \sigma_1 dW_t^1$$

$$\frac{dS_{2,t}}{S_{2,t}} = (\mu_2 - q_2)dt + \sigma_2 dW_t^2$$

where W^1 and W^2 are BM's with (instantaneous) correlation ρ, and q_1 and q_2 are the (constant) dividend rates of the respective assets. The drifts μ^j and volatilities σ^j are constant.

The market filtration (available information) is generated by (S_1, S_2). Note that we here have two sources of risk and two risky assets. If we did not have two risky assets with which to hedge, the market would not be complete. In this case, some assets can be perfectly hedged (and so have a unique NA price) but others cannot.

The model is **arbitrage-free** if $\sigma_1 > 0$, $\sigma_2 > 0$ and $\rho \in (-1, 1)$. Otherwise, we need additional constraints on μ_1 and μ_2.

The equivalent martingale measure (EMM) $\mathbb{Q}$ exists and is unique under the above assumptions on σ^j and ρ. The risk-neutral dynamics are

$$\frac{dS_{1,t}}{S_{1,t}} = (r - q_1)dt + \sigma_1 dW_t^{\mathbb{Q},1}$$

$$\frac{dS_{2,t}}{S_{2,t}} = (r - q_2)dt + \sigma_2 dW_t^{\mathbb{Q},2}$$

where $W^{\mathbb{Q},1}$ and $W^{\mathbb{Q},2}$ are $\mathbb{Q}$-BM's with the same correlation ρ.

Pricing and hedging of options in the two-dimensional BS model.

- Assume the time t price of the option is given by $V(S_{1,t}, S_{2,t}, t)$.
- Search for a hedging portfolio $\pi = \left(\gamma_t, \Delta_t^1, \Delta_t^2\right)_{t\in[0,T]}$, which is *self-financing*:

$$\begin{aligned} &d(\gamma_t B_t + \Delta_t^1 S_{1,t} + \Delta_t^2 S_{2,t}) \\ &= \gamma_t dB_t + \Delta_t^1 dS_{1,t} + \Delta_t^2 dS_{2,t} + \Delta_t^1 q_1 S_{1,t} dt + \Delta_t^2 q_2 S_{2,t} dt \end{aligned}$$

and *replicates the price of the option*:

$$\gamma_t B_t + \Delta_t^1 S_{1,t} + \Delta_t^2 S_{2,t} = V(S_{1,t}, S_{2,t}, t) \qquad (1)$$

Applying Ito's rule, we obtain

$$\begin{aligned} &\gamma_t dB_t + \Delta_t^1 dS_{1,t} + \Delta_t^2 dS_{2,t} + \Delta_t^1 q_1 S_{1,t} dt + \Delta_t^2 q_2 S_{2,t} dt \\ &= dV(S_{1,t}, S_{2,t}, t) \\ &= (Ito)... \end{aligned}$$

and hence

$$\Delta_t^1 = \frac{\partial}{\partial S_1} V(S_{1,t}, S_{2,t}, t),$$
$$\Delta_t^2 = \frac{\partial}{\partial S_2} V(S_{1,t}, S_{2,t}, t).$$

Therefore

$$\frac{\partial}{\partial t}V + \frac{1}{2}\left(\sigma_1^2 S_1^2 \frac{\partial^2}{\partial S_1^2}V + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2}V + \sigma_2^2 S_2^2 \frac{\partial^2}{\partial S_2^2}V\right)$$
$$+(r-q_1)S_1\frac{\partial}{\partial S_1}V + (r-q_2)S_2\frac{\partial}{\partial S_2}V - rV = 0.$$

This is a two-dimensional BSPDE, solved with a terminal condition which is the payoff of the option.

Pricing a zero-strike spread option.

Consider a European option with payoff

$(S_{2,T} - S_{1,T})^+$ at maturity T.

Denote its time t price in the two-dimensional BS model by $V(S_{1,t}, S_{2,t}, t)$.

Let's show that the function V can obtained as a solution to the BSPDE in the form

$$V(S_1, S_2, t) = S_1 U(S_2/S_1, t)$$

for some function U which is to be determined.

Introduce the change of variables:

$$(S_1, S_2) \mapsto (S_1, x = S_2/S_1)$$

Then our assumption on V is that $V(S_1, S_2, t) = S_1 U(x, t)$, and we have

$$S_2 = xS_1, \quad \frac{\partial}{\partial S_1}V = U - \frac{S_2}{S_1}\frac{\partial}{\partial x}U = U - x\frac{\partial}{\partial x}U,$$

$$\frac{\partial^2}{(\partial S_1)^2}V = -\frac{S_2}{(S_1)^2}\frac{\partial}{\partial x}U + \frac{S_2}{(S_1)^2}\frac{\partial}{\partial x}U + \frac{(S_2)^2}{(S_1)^3}\frac{\partial^2}{\partial x^2}U = \frac{x^2}{S_1}\frac{\partial^2}{\partial x^2}U,$$

$$\frac{\partial}{\partial S_2}V = \frac{\partial}{\partial x}U, \quad \frac{\partial^2}{(\partial S_2)^2}V = \frac{1}{S_1}\frac{\partial^2}{\partial x^2}U, \quad \frac{\partial^2}{\partial S_1 \partial S_2}V = -\frac{x}{S_1}\frac{\partial^2}{\partial x^2}U.$$

Plugging the above equations in the pricing PDE of the two-dimensional Black-Scholes model, we obtain

$$\frac{\partial}{\partial t}U + \frac{1}{2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)x^2\frac{\partial^2}{\partial x^2}U + (q_1 - q_2)x\frac{\partial}{\partial x}U - rU = 0$$

The **terminal condition** for U becomes $U(x,T) = \max(x-1, 0)$.

We notice that the above equation is the **pricing PDE of the Black-Scholes model** with **volatility**

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

interest rate r and **dividend rate** $q = r - q_1 + q_2$.

Taking into account the terminal condition, we conclude that the price of a zero-strike spread option can be expressed through the price function of a call option in a standard one-dimensional BS model with appropriately chosen parameters:

$$U(X_t, t) = C^{BS}(S_t = X_t, t; K = 1, T, \sigma, r, q = r - q_1 + q_2),$$

and, therefore,

$$V(S_1, S_2, t) = S_1 C^{BS}(S_2/S_1, t; 1, T, \sigma, r, r - q_1 + q_2).$$

Notice that the dividend rate q, defined above, can become negative. Mathematically, there is no problem defining the Black-Scholes model with negative dividends. Practically, it also makes sense if, for example, it is **inconvenient** (costly) to hold the physical asset (due to storage costs etc.).

Change of numéraire.

The technique used above was to express the value of our claim in terms of units of S_1. With this trick, our equation was significantly simplified. In this case, we call S_1 the numeráire, as it is the units in which prices can be expressed.

Assume the market consists of a bank account B and d risky assets $(S_1, \ldots, S_d)$ which pay no dividends.

The price process of any tradable asset S_i can be used as a **numeráire**, provided it is positive. (More generally, any positive process can be a numeráire.) In other words, we can measure the price of all assets in the units of asset i.

Dividing by the numeráire, we obtain the discounted asset prices:

$$\tilde{S}_{j,t} = \frac{S_{j,t}}{S_{i,t}}, \quad j = 1, \ldots, d,$$

and

$$\tilde{B}_t = \frac{B_t}{S_{i,t}}$$

Claim: If the market is **arbitrage-free and complete**, then, for any admissible choice of the numeráire S_i, there exists a unique equivalent measure $\mathbb{Q}^i$, under which **the discounted (by the corresponding numeráire) prices of all tradable assets are martingales**.

$\mathbb{Q}^i$ is the **pricing measure that corresponds to the chosen numeráire** S_i.

Notice that, in this case, the bank account does not play any special role, since other assets can also be used as a numeráire. Therefore, we temporarily replace B by S_0.

So far, we always chose the bank account S_0 (or, B) as a numeráire. But this is not necessary: sometimes a different numeráire may be more convenient.

With the above result, we can price contingent claims using different numeráires.

- Assume that the model for the basic assets $(S_0, S_1, \ldots, S_d)$ is **arbitrage-free** and **complete**.

- Assume that, for some $i = 0, \ldots, d$, the price of the asset i, S_i, is **strictly positive price** at all times.

- Denote the payoff of a given contingent claim with the time of expiry T by the random variable $V_T \in \mathcal{F}_T$.

Then, the extended market $(S_0, S_1, \ldots, S_d, V)$ is arbitrage-free if and only if

$$E^{\mathbb{Q}^i}\left|\frac{V_T}{S_{i,T}}\right| < \infty$$

and, for any $t \in [0, T]$, we have

$$\frac{V_t}{S_{i,t}} = E^{\mathbb{Q}^i}\left(\left.\frac{V_T}{S_{i,T}}\right| \mathcal{F}_t\right),$$

where $\mathbb{Q}^i$ is the unique pricing measure that corresponds to the numeráire S_i.

In a **two-dimensional BS model** without dividends:

- The first risky asset S_1 is **strictly positive**, and, hence, it can be chosen as a numeráire.

- Then the discounted prices of the other risky asset and the bond are given by the stochastic processes

$$\tilde{S}_{2,t} = \frac{S_{2,t}}{S_{1,t}} \text{ and } \tilde{B}_t = \frac{B_t}{S_{1,t}},$$

  which are **geometric Brownian motions** (under any equivalent measure), and we can compute their volatilities.

Assuming the volatilities are strictly positive and the correlation coefficient is not equal to "± 1", we conclude that the model is **arbitrage-free** and **complete**.

Then the pricing measure $\mathbb{Q}^1$, corresponding to the numeráire S_1, **exists** and is **unique**.

Under $\mathbb{Q}^1$, both $\tilde{S}_2$ and $\tilde{B}$ are *geometric Brownian motions*, which have **the same volatilities as under the physical measure** (or under any other equivalent measure) and are **martingales**. Thus, *their distribution under* $\mathbb{Q}^1$ *is determined uniquely* (we can write down a system of SDEs that these processes satisfy).

In particular, it is easy to verify that the volatility of $\tilde{S}_2$ is given by

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Consider now a **zero-strike spread option**. Its payoff at expiry T is given by

$$V_T = (S_{2,T} - S_{1,T})^+$$

Using the above corollary, we can compute the arbitrage-free price of this option at any time t via

$$\tilde{V}_t = \frac{V_t}{S_{1,t}} = E^{\mathbb{Q}^1}\left(\left.\frac{V_T}{S_{1,T}}\right| \mathcal{F}_t\right) = E^{\mathbb{Q}^1}\left(\left.(\tilde{S}_{2,T} - 1)^+\right| \tilde{S}_{2,t}\right),$$

where, in the last equality, we used the fact that $\tilde{S}_2$ is a GBM and, in particular, a Markov process under $\mathbb{Q}^1$, and, therefore, in order to compute the conditional expectation of a function of its terminal value, we only need to condition on its current value rather than on all available information.

It only remains to notice that, under $\mathbb{Q}^1$, the dynamics (SDE) of the process $\tilde{S}_2$ are the same as the risk-neutral dynamics of the risky asset in a one-dimensional BS model with volatility σ (defined above), interest rate r and dividend rate $q = r$.

Thus, we obtain

$$V_t = S_{1,t}\tilde{V}_t = S_{1,t}C^{BS}\left(\tilde{S}_{2,t} = S_{2,t}/S_{1,t}, t; 1, T, \sigma, r, r\right).$$

Notice that we have rediscovered the previously derived formula for the price of a spread option, in the case when $q_1 = q_2 = 0$.

The above results can be adjusted to the case of non-zero dividends by multiplying the risky assets by corresponding exponentials.