FINAL EXAM

INSTRUCTIONS: QUESTION **ONE** IS COMPULSORY AND THEN ATTEMPT ANY OTHER **TWO** QUESTIONS

QUESTION ONE: 40 MARKS

In this question all the standard Black-Scholes assumptions hold. Under these assumptions the price function, $C_{\rm d}$, of a European digital call option satisfies the problem

$$\frac{\partial C_{\rm d}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\rm d}}{\partial S^2} + (r-y)\, S\, \frac{\partial C_{\rm d}}{\partial S} - r\, C_{\rm d} = 0, \quad C_{\rm d}(S,T) = 1_{(S>K)},$$

where r, y, $K > 0$, $\sigma > 0$ and $T > 0$ have their usual meanings and are all constants, for all $S > 0$ and $t < T$ and $1_{(\cdot)}$ is the usual indicator function. You may also assume that $C_{\rm d}(S,t;K,T,r,y,\sigma) = e^{-r(T-t)}\,\mathrm{N}(d_-)$, where

$$d_- = \frac{\log(S/K) + (r - y - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}, \quad \mathrm{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-q^2}\, dq.$$

(a) Write down Black-Scholes problem for a down-and-out barrier digital call option's pricing function, $C_{\rm B}(S,t)$, where the barrier is set at a constant level, $B > 0$.

(b) Show that if $B > 0$ is a constant and we define

$$\xi = B^2/S, \quad \alpha = 1 - 2(r-y)/\sigma^2, \quad V(S,t) = (S/B)^{\alpha}\, C_{\rm d}(\xi, t)$$

then $V(S,t)$ satisfies the same Black-Scholes equation as $C_{\rm d}(S,t)$.

(c) Using the previous result, or otherwise, find the Black-Scholes pricing function $C_{\rm B}(S,t)$ for the down-and-out barrier digital call when the constant barrier B is set below the strike K, $0 < B < K$. Carefully justify the key steps in your derivation.

(d) Consider a down-and-out digital barrier option in which the barrier level varies with time as $B_t = B\, e^{\beta(T-t)}$ with β constant and $B < K$. By considering a change of variables $\hat{S} = f(t)\, S$, $C_{\rm Bt}(S,t) = W(\hat{S}, t)$, for a suitable function $f(t)$, express the Black-Scholes pricing function, $C_{\rm Bt}(S,t)$, for this option in terms of $C_{\rm d}(S,t;K,T,r,y,\sigma)$.

QUESTION TWO: 30 MARKS

Consider an American digital call option, with payoff

$$P(S) = \begin{cases} 0 \text{ if } S < K, \\ 1 \quad S \geq K, \end{cases}$$

where $K > 0$ is the strike and the option can be exercised at any time up to and including expiry, T.

(a) Using a no-arbitrage argument, derive the optimal exercise rule for this option. Is this strategy model-dependent?

(b) Under a standard Black–Scholes model, explain carefully why, and in what domain, the value function of this option should satisfy the Black–Scholes PDE.

(c) By considering the following function,

$$C(S,t) = \left(\frac{S}{K}\right)\mathcal{N}(d_+^*) + \left(\frac{K}{S}\right)^{2r/\sigma^2}\mathcal{N}(d_-^*),$$

where

$$d_\pm^* = \frac{\log\frac{S}{K} \pm (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.s$$

give a solution to the pricing equation, and verify that the linear complementarity conditions are satisfied.

QUESTION THREE: 30 MARKS

Consider a Bermudan put option, which allows the holder to obtain the payoff of a European call at either time $t = 1$ or time $t = 2$ (after which it expires). The underlying is assumed to follow a standard Black–Scholes model without dividends.

(a) Write down the value of the option at time $t = 1$, and hence derive the optimal exercise boundary at time $t = 1$.

(b) Derive an appropriate PDE (with boundary condition) which is satisfied by value of this option on the remaining time domain.

(c) Explain explicitly how a Monte–Carlo system could be used to approximate the value of this option.

(d) Prove that the value of this option must lie between the values of the European and American options.

QUESTION FOUR: 30 MARKS

Consider a standard Black–Scholes model with continuous dividends, that is where,

$$dB_t = rB_t dt, \qquad dS_t/S_t = \mu dt + \sigma dW_t$$

with dividends paid on stocks at rate $qS_t dt$, for W a Brownian motion, μ, r, q, σ constants.

(a) What are the risk-neutral dynamics of the discounted stock and the bond when using the stock price as a numéraire?

(b) Calculate the time-t value of the payoff $S_T I_{S_T>K}$ (at time T) by using the numéraire given by the stock price and the corresponding law of $\tilde{B}_T$.

(c) Using part (b) and the decomposition $(S_T - K)^+ = S_T I_{S_T>K} - K I_{S_T>K}$, calculate the value of the option using change of numéraire techniques (you may use different numeraires for different parts of the equation).

QUESTION FIVE: 30 MARKS

Consider a foreign exchange model, where X denotes the price of 1 USD denominated in GBP. Assume $X > 0$. An GBP denominated account pays interest r_d, a USD denominated account pays interest r_f.

(a) Explain why X can be considered a traded asset. What does this tell you (in general) about the dynamics of X under the risk neutral measure (with numéraire GBP).

(b) If $\mathbb{Q}^d$ denotes the risk neutral measure with numéraire GBP, show that $\{e^{t(r_f - r_d)}/X_t\}_{t\geq 0}$ (the dynamics of the price of 1 GBP denoted in USD) is not a martingale unless $e^{t(r_d - r_f)}X$ is constant.

(c) Show by no-arbitrage that $1/X$ must be the price of 1 GBP denominated in USD. Hence or otherwise, show that $\mathbb{Q}^d$ is *not* the risk neutral measure under numéraire USD.

(d) Assuming that X is continuous and has constant volatility, write down an SDE satisfied by X under each of the risk-neutral measures *with numéraires USD and GBP.*