INSTRUCTIONS TO STUDENTS

ATTEMPT ALL QUESTIONS. A SCIENTIFIC CALCULATOR WILL BE ALLOWED FOR THE ASSESSMENT TEST. ANY OTHER REQUIRED MATERIAL WILL BE SUPPLIED BY THE INVILATOR PRESENT.

TIME: TWO AND A HALF HOURS

1. Consider a solution $u : [0, T] \to \mathbb{R}$ to the scalar ODE

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= ru \\ u(0) &= 1 \end{aligned}$$

You can think of u as the money in a bank account with interest rate $r \geq 0$.

(a) Compute an approximate solution u^M at time T, for the explicit Euler method with M timesteps of length $\Delta t = T/M$, where

$$u^m = (1 + r\Delta t)u^{m-1}, \quad m > 0$$

and $u^0 = 1$. Hence show that $u^M \to \mathrm{e}^{rT}$ for $M \to \infty$ such that $T = M\Delta t$ fixed.

(b) Find bounds $c = c(T)$ and $C = C(T)$ such that

$$|(1 + rT/M)^m| \leq C \quad \forall m \leq M$$

and

$$\frac{1}{\Delta t}|(1 + r\Delta t)\mathrm{e}^{r(t-\Delta t)} - \mathrm{e}^{rt}| \leq c\,\Delta t \quad \forall t \leq T.$$

Using the Lax equivalence theorem, deduce convergence and show that a simple bound for the approximation error is given by

$$|u^M - u(T)| \leq \tfrac{1}{2}r^2\mathrm{e}^{2rT}\,\Delta t.$$

(c) Retracing the proof of the Lax equivalence theorem, by estimating carefully the propagation of truncation errors introduced at individual timesteps, show how you can improve the error bound for this specific example.

2. Show that the truncation error of the θ-scheme for the heat equation is of the form

$$(1-2\theta)O(\Delta t)+O(\Delta t^2)+O(\Delta x^2)$$

and deduce that the Crank-Nicolson scheme is of second order accurate.

Hint: Write the truncation error as

$$\delta_t^+ u(x,t)-\theta\tfrac{1}{2}\delta_x^2 u(x,t+\Delta t)-(1-\theta)\tfrac{1}{2}\delta_x^2 u(x,t)=\delta_t u(x,t)-\tfrac{1}{2}\delta_x^2\left\{\theta u(x,t+\Delta t)+(1-\theta)u(x,t)\right\}$$

where $\delta_t u(x,t)=(u(x,t+\Delta t)-u(x,t)/\Delta t$ *and* $\delta_x^2 u(x,t)=(u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t))/\Delta x^2$ *as usual. Expand* $u(x,t)$, $u(x,t+\Delta t)$, $u(x+\Delta x,t)$ *etc around* $(x,t+\theta\Delta t)$ *and use the heat equation at this point to find the truncation error.*

3. Show that if $\theta=\frac{1}{2}$ is replaced by $\theta=\frac{1}{2}-\frac{1}{12}\frac{\Delta x^2}{\Delta t}$ in the definition of the θ-scheme, the accuracy improves to $\mathcal{O}(\Delta t^2)+\mathcal{O}(\Delta x^4)$. Is the scheme stable, and in what sense? How should you choose Δt in relation to Δx to maximise efficiency?

4. (a) Show that the symbol of the θ-method for the heat equation is

$$R_\theta(\Delta t,\Delta x;k)=\frac{1-2(1-\theta)\lambda\sin^2(k/2)}{1+2\theta\lambda\sin^2(k/2)}$$

   where $\lambda=\Delta t/\Delta x^2$.

   (b) By expanding for small Δt, deduce the order of consistency of the explicit, implicit, and Crank-Nicolson schemes.

5. Given the symbol of the fully implicit method for the heat equation,

$$R_1(\Delta t,\Delta x;k)=\frac{1}{1+2\theta\lambda\sin^2(k/2)}$$

with $\lambda=\Delta t/\Delta x^2$, the finite difference solution can be written as

$$u_n^m=\int_{-\pi}^{\pi}R_1(\Delta x,\Delta t;k)^m\widehat{u}^0(k)\mathrm{e}^{ink}\,\mathrm{d}k, \qquad (4.17)$$

where $\widehat{u}^0(k)$ is the Fourier transform of the discrete initial condition.

   (a) Find $\widehat{u}^0(k)$ for Dirac initial data.

   (b) Using calculus of residues, or otherwise, evaluate the integral in (4.17) to find u_n^m for all $m\geq 0$, $n\in\mathbb{Z}$.

   (c) Compare your result to the one found in Section 2.5, Exercise 2., for $n=1$ and $n=2$.

6. Consider the semi-discrete implicit Euler scheme for the heat equation,

$$\frac{u^m-u^{m-1}}{\Delta t}=\frac{1}{2}\frac{\partial^2 u^m}{\partial x^2},\qquad x\in\mathbb{R},m\geq 1,$$

where time is discretised and the space coordinate left continuous (*horizontal method of lines*).

(a) Show that the first step with $u^0 = \delta$ has a solution of the form

$$u^1 = b \cdot \mathrm{e}^{-c|x|},$$

where you have to find b and c (compare Exercise 2, Section 2.5).

(b) By Fourier transform and calculus of residues, or otherwise, find the general solution of the semi-discrete implicit Euler scheme (compare Exercise 5 in this section). How smooth is the solution after m steps?

(c) Why would a semi-discrete explicit Euler scheme be problematic?

7. The Leapfrog scheme for the heat equation is defined as

$$\frac{u_n^{m+1} - u_n^{m-1}}{2\Delta t} = \frac{1}{2} \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\Delta x^2},$$

where two initial data u_n^0, u_n^1 need to be given.

(a) Compute the truncation error at a point (x_n, t_m). Show that the scheme is consistent with the heat equation and determine the consistency order.

(b) Find all solutions of the form

$$u_n^m = R(\Delta x, \Delta t; k)^{M-m} \mathrm{e}^{ikn}$$

and give an expression for the general solution (i.e. for general u_n^0, u_n^1). Show that for any choice of u_n^1 which is consistent with the heat equation, the scheme is unstable in l_2.

8. The Du Fort and Frankel scheme for the heat equation is defined as

$$\frac{u_n^{m+1} - u_n^{m-1}}{2\Delta t} = \frac{1}{2} \frac{u_{n-1}^m - (u_n^{m-1} + u_n^{m+1}) + u_{n+1}^m}{\Delta x^2},$$

where two initial data u_n^0, u_n^1 need to be given.

(a) Find all solutions of the form

$$u_n^m = R(\Delta x, \Delta t; k)^{M-m} \mathrm{e}^{ikn}$$

and give an expression for the general solution (i.e. for general u_n^0, u_n^1). Deduce that the scheme is unconditionally stable in l_2. [2]

(b) Compute the truncation error at a point (x_n, t_m). Show that the scheme is consistent with the heat equation only if $\lim_{\Delta t, \Delta x \to 0} \Delta t/\Delta x = 0$. Which PDE is the scheme consistent with in the limit $\Delta t, \Delta x \to 0$ with $\Delta t/\Delta x = \lambda$ fixed?

(c) Explain briefly how you would choose Δt and Δx optimally. Compare the performance of the scheme to that of the explicit Euler method.

9. This exercise offers a more abstract take on the question of convergence. Consider a "generic" equation

$$Au = b$$

where $u \in X$, $b \in Y$, X, Y linear spaces, $A \in L(X, Y)$ a linear operator from X to Y.

An approximation to this equation is written as

$$A_h u_h = b_h$$

where $u_h \in X_h$, $b_h \in Y_h$, X_h, Y_h normed linear spaces with norms $|\cdot|_{X_h}$ and $|\cdot|_{Y_h}$, and $A_h \in L(X_h, Y_h)$ a linear operator from X_h to Y_h.Assume A_h is invertible for all h, $A_h^{-1} \in L(Y_h, X_h)$ with standard operator norm $\|\cdot\|_h$.Define further operators $R_h : X \to X_h$ and $Q_h : Y \to Y_h$ and assume $b_h = Q_h b$.

Show that if

$$|Q_h Au - A_h R_h u|_{Y_h} \to 0 \quad \text{for } h \to 0$$

and if there exists a constant C with

$$\|A_h^{-1}\|_h \leq C,$$

then

$$|u_h - R_h u|_{X_h} \to 0 \quad \text{for } h \to 0.$$

Explain briefly how this relates to the convergence theory of timestepping schemes.