INSTRUCTIONS TO STUDENTS

- ATTEMPT **ALL** QUESTIONS IN **SECTION A** AND ANY OTHER **TWO** IN **SECTION B.**
- A SCIENTIFIC CALCULATOR WILL BE REQUIRED FOR THE EXAMINATION.

TIME

THREE HOURS

SECTION A

1. Consider the advection diffusion with positive drift $\mu > 0$,

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial u}{\partial x}, \quad x \in \mathbb{R},\, t > 0.$$

   (a) Assume in the following $\mu \geq 0$ and, for a finite difference scheme with step size Δt and grid size Δx, $\mu\Delta t \leq \Delta x$. Give a geometric interpretation to these conditions.

   (b) Show that the implicit Euler scheme with upwinding difference for (5.66) is monotone with respect to the initial data, $u^m \geq 0$ for all $m \geq 0$ if $u^0 \geq 0$. Deduce that the scheme is stable in the maximum norm.

   (c) Define the implicit Euler discretisation in Lagrangian coordinates for (5.66), with a general interpolating function. Show that for piecewise linear interpolation this scheme is identical to the upwinding scheme, but with the modification that only the second order difference is implicit, whereas the (first order) upwinding difference is explicit (i.e. evaluated at the previous timestep). Discuss stability of the scheme.

2. Consider the θ-η-scheme (5.4) for the PDE (5.66). Using von Neumann analysis, assess the stability of the scheme. Verify that the symbol for the special case of the central difference scheme is given by (5.10). [*Hint: You may want to prove and use the identity* $(1-\eta)\exp(ik) + (2\eta - 1) - \eta\exp(-ik) = 2(2\eta-1)\sin^2(k/2) + i\sin(k)$.]

3. Assume that a matrix $K \in \mathbb{R}^{n\times n}$ satisfies the following conditions:

i. $K_{ij} \leq 0, \quad 1 \leq i, j \leq n, \quad i \neq j;$

ii. $K_{ii} > \sum_{j \neq i} |K_{ij}|, \quad 1 \leq j \leq n.$

Show that:

(a) K is invertible;

(b) $K^{-1} \geq 0$ elementwise, which is equivalent to saying that $K^{-1}u \geq 0$ for all $u \in \mathbb{R}^n$ with $u \geq 0$, where again non-negativity is elementwise;

(c) if the stronger inequality

$$K_{ii} \geq 1 + \sum_{j \neq i} |K_{ij}|$$

holds for an index $1 \leq j \leq n$, then for the solution $u^1 \in \mathbb{R}^n$ of

$$Ku^1 = u^0,$$

for this index j

$$u_j^1 \leq \max\left(u_j^0, \max_{k \neq j} u_k^1\right).$$

(d) Discuss what the above properties mean for implicit finite difference schemes defined by the matrix K.

4. (a) Show that (5.40) and (5.41) are the eigenvalues and eigenvectors of (5.39).

(b) Use Gershgorin's theorem (Lemma 5.2.13) to estimate the eigenvalues of the constant coefficient matrix (5.39) and compare with the exact values.

5. Consider an infinite version of the matrix K in (5.39), i.e. an operator $K : l_2 \to l_2$, $(Ku)_n = au_{n-1} + bu_n + cu_{n+1}$.

(a) Find the adjoint operator K^* such that $(Ku, v) = (u, K^*v)$ where $(\cdot, \cdot)$ is the standard l_2 inner product. When is K self-adjoint?

(b) Show that K is a *normal* operator, i.e. $KK^* = K^*K$.

(c) Find the eigenvalues of K corresponding to "eigenvectors" e^{ikn}. Are these "eigenvectors" in l_2?

(d) Show that the operator K for non-constant a_n, b_n, c_n is not normal.

6. Given is a tridiagonal matrix $A = \text{tridiag}(a_i, b_i, c_i, n)$, i.e. of the shape of the matrix K_1 from (5.27), with $0 < a_{i+1}/c_i$ for $i = 1, \ldots, n-1$.

   (a) Show that for a diagonal matrix $D = \text{diag}(d_i, n)$ such that

   $$\frac{d_i^2}{d_{i+1}^2} = \frac{a_{i+1}}{c_i},$$

   the *similarity transformation*

   $$\widetilde{A} = DAD^{-1}$$

   gives a symmetric, tridiagonal matrix $\widetilde{A} = \text{tridiag}(\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, n)$ with $\tilde{b}_i = b_i, \tilde{a}_i = \tilde{c}_i = \sqrt{a_{i+1}c_i}$.

   (b) Show that A and $\widetilde{A}$ have the same eigenvalues, $\mathcal{S}(A) = \mathcal{S}(\widetilde{A})$, and eigenvectors.

   (c) Conclude that

   $$\|A^m\|_2 \leq \frac{\max_i |d_i|}{\min_i |d_i|}\rho(\widetilde{A})^m. \tag{5.67}$$

7. Consider the IBVP

   $$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1), t > 0, \tag{5.68}$$

   where σ and μ are differentiable functions of x and t with

   $$0 < \underline{\mu} \leq \mu(x,t) \leq \overline{\mu}, \quad 0 < \underline{\sigma} \leq \sigma(x,t) \leq \overline{\sigma}, \quad x \in (0,1), t > 0. \tag{5.69}$$

   (a) Define the fully implicit central difference scheme for (5.68).

   (b) Use Exercise 6 to calculate a symmetric matrix $\widetilde{K}$ which is similar to the discretisation matrix K, by transformation with a diagonal matrix D.

   (c) Use the Gerschgorin's theorem (Lemma 5.2.13) to find a lower bound for the eigenvalues of $\widetilde{K}$.

   (d) Find upper and lower bounds for the diagonal elements of D.

   (e) Using (5.67) and the previous two items, deduce that the implicit Euler scheme is stable in the l_2-norm.

   (f) Explain briefly what you would expect to happen for the θ-scheme?

   (g) Retrace the steps to find out if (5.69) is crucial here, with a view of extending the analysis to the case where the coefficients approach zero at the boundaries.

SECTION B

1. For the Black-Scholes PDE, calculate analytically the truncation error of the implicit Euler central difference scheme. What behaviour of the discretisation error do you predict as

   (a) $\Delta t \to 0$, ΔS fixed;

   (b) $\Delta S \to 0$, Δt fixed;

   (c) $\Delta t, \Delta S \to 0$, $\Delta t/\Delta S$ fixed;

   (d) $\Delta t, \Delta S \to 0$, $\Delta t/\Delta S^2$ fixed.

2. The Black-Scholes equation in log-price is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial V}{\partial X} - rV = 0, \quad X \in \mathbb{R}, t \in (0,T). \tag{6.95}$$

   (a) Write down the explicit finite difference scheme for this equation with M timesteps and on an infinite grid with spacing ΔX. Denote V_n^m the numerical approximation at $t = m\Delta t$, $X = n\Delta x$, $n \in \mathbb{Z}$, $m = 0, \ldots, M$.

   (b) Show that for a terminal condition of the form

$$V_n^M = \mathrm{e}^{ikn},$$

   the solution is of the form

$$V_n^m = R_0(\Delta x, \Delta t; k)^{M-m} V_n^M$$

   and find R_0. *[Hint: Prove and use that* $\mathrm{e}^{ik(n+1)} - \mathrm{e}^{ik(n-1)} = 2i\mathrm{e}^{ikn}\sin k$.*]*

   (c) Give sufficient conditions for stability of the scheme. Are these necessary?

   (d) Discuss, without proof, what this result indicates for l_2 stability of explicit finite differences for the Black-Scholes PDE in original coordinates.

3. Consider a European put option with parameters $\sigma = 0.4$, $r = 0.05$, $K = 0.25$, $T = 1$. Set $S_{max} = 1$. Implement the θ-method for a PDE of the form (6.8), and specify the coefficients μ, σ and r to compute numerical approximations to the Black-Scholes price of the put, solving

(a) the Black-Scholes PDE in S and t;

(b) the Black-Scholes PDE in log-price X and t as per (6.95);

(c) the heat equation in x and τ as in (6.18).

The terminal condition in cases 3b. and 3c. is the payoff in transformed coordinates. Use suitable upper and lower bounds for the computational domain (e.g. $X_{min} = \log(K^2/S_{max})$ etc) and asymptotic boundary conditions as appropriate.

Compare the l_∞ error (measured against the known analytical solution) of the numerical solutions, for an increasing number of grid points, say $N = 100$, 200, 400, 800, identical for all three approaches. Which method is most accurate?

4. Consider a European put with identical parameters to Exercise 3. For Crank-Nicolson with and without Rannacher start-up, compute a finite difference solution with $N = 512$ grid intervals and $M = 64$ timesteps. Plot V_n^M and V_n^0 versus S_n for $n = 0, \ldots, N$. From these finite difference solutions, compute sensitivities

$$\Delta_n^m = \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta S}, \quad \Gamma_n^m = \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{\Delta S^2}$$

for $m = 0$ and $m = M$, and plot them over S_n, $n = 0, \ldots, N$ (using appropriate modifications for $n = 0, N$, e. g. $\Delta_0^m = -1$ etc or appropriate one-sided differences).

5. Write the Crank-Nicolson scheme with Rannacher start-up as

$$V^m = K_{1/2,\Delta t}^{M-m-r} K_{1,\Delta t/2}^{2r} V^M$$

where you have to define $K_{\theta,\Delta t}$.

(a) Show that the eigenvectors of $K_{\theta,\Delta t}$ are identical for all θ, Δt, and find the eigenvalues.

(b) If we introduce $W = (W_0, \ldots, W_N)$, the matrix of eigenvectors of K, $\widehat{V}^m = W^{-1}V^m$ contains the coordinates of $V^m = (V_0^m, \ldots, V_N^m)'$ in the eigenvector basis. Show that for $\Lambda_{\theta,\Delta t}$ the diagonal matrix of eigenvalues of $K_{\theta,\Delta t}$,

$$\widehat{V}^0 = \Lambda_{1/2,\Delta t}^{M-r} \Lambda_{1,\Delta t/2}^{2r} \widehat{V}^M.$$

(c) Plot $\widehat{V}^M$, and $\widehat{V}^0$ for $r = 0, 1, 2$.

(d) Similarly, define $\Delta^m = (\Delta_0^m, \ldots, \Delta_N^m)'$, $\Gamma^m = (\Gamma_0^m, \ldots, \Gamma_N^m)'$, and $\widehat{\Delta}^m = W^{-1}\Delta^m$, $\widehat{\Gamma}^m = W^{-1}\Gamma^m$. Compute $\widehat{\Delta}^M$ and $\widehat{\Delta}^0$, $\widehat{\Gamma}^M$, and $\widehat{\Gamma}^0$, the finite difference delta and gamma represented in the eigenvector basis, and plot them, again for $r = 0, 1, 2$. Interpret the result.