Implementing different payoffs

Different payoff functions can – conceptually – be incorporated very easily in the finite difference framework. What changes is the terminal condition (6.4) to the pricing PDE, and usually this means that asymptotic boundary conditions in the stock direction have to be adapted as well. So, mathematically speaking, this is a section about the accurate and stable approximation of boundary conditions.

Take our running example the Black-Scholes model. The three boundaries, $S = 0$, $S \to \infty$, and $t = T$, are very different in their nature.

1. At $S = 0$, the boundary values to the PDE solution come out naturally, as the PDE (6.1) reduces to the ODE (6.9), (6.10). All that changes is the terminal value $V(0, T) = G(0)$, the payoff for $S = 0$.
2. The boundary $S \to \infty$ has to be approximated at a large but finite value S_{max}. More often than not, we need to set approximate values for the solution, e.g. obtained from an asymptotic analysis of the PDE for large S. Ideally, one would like a "generic" boundary condition which works for all situations, or at least a large class. This has the practical advantage not to have to derive, implement, and possibly analyse different boundary conditions for each new problem. This leads to a discussion on derivative boundary conditions. We discuss this in 6.4.1.
3. At $t = T$, a terminal condition is defined by the payoff of the option. In the previous sections, the numerical solution at T is picked pointwise from the payoff, and this works fine for continuous payoff functions. In the presence of discontinuous, say digital option payoffs, or even Dirac data, say the gamma of a vanilla call at expiry, the accurate approximation on a grid is less trivial, and we analyse this in 6.4.2.

We are not going to say much about the implementation of boundary conditions as in item 1. If implemented diligently, the finite difference scheme for the PDE reduces to a timestepping scheme for an ODE at the lowest node $S_0 = 0$. To exemplify this, setting $n = 0$ in (6.20) to (6.23) gives (6.24), and this serves as boundary condition for (6.20) for $n > 0$. It is worth pointing out, however, that the question of setting boundary conditions is related to the boundedness of the underlying stochastic process. Recall that the PDE (6.8) is the backward equation for a process of the form

$$\mathrm{d}S_t = \mu(S_t, t)\,\mathrm{d}t + \sigma(S_t, t)\,\mathrm{d}W_t. \tag{6.74}$$

Assume $\sigma(S,t) > 0$ for $S > 0$. For S_t to stay non-negative, it is intuitively necessary that

$$\begin{aligned} \sigma(0,t) &= 0, && (6.75) \\ \mu(0,t) &\geq 0, && (6.76) \end{aligned}$$

where we assume that σ and μ are smooth. If the variance was non-zero, the process could surely diffuse beyond zero with positive probability, independent of the drift. Given that, the drift has to be non-negative or the process would drift into the negative range.

More precisely, and adapted to the question of boundary conditions, it is shown in [Sun et al., 2003] that the solution to the PDE is unique without boundary condition at $S = 0$, if

$$-\frac{1}{2}\frac{\partial \sigma^2}{\partial S}(0+,t) + \mu(0+,t) \geq 0,$$

where $0+$ indicates the limit coming from the positive side. This is clearly true for the Black-Scholes model. In other cases, this depends on the order by which the diffusion coefficient goes to zero. The first term is zero for say $\sigma = S^\alpha$ with $\alpha > 1/2$ and ∞ if $\alpha < 1/2$. In the limiting case of a square-root diffusion, as in the CIR model, the size of the drift becomes relevant. We refer to [Sun et al., 2003] for an analysis.

We focus in the following on the other boundary conditions 2. and 3. These depend largely on the payoff, and we have in mind as principal examples standard and digital calls and puts. Most other common payoffs are combinations of these. The payoff of a European call,

$$V(S,T) = \max(S-K,0), \tag{6.77}$$

is continuous, with a discontinuity in the first derivative at the strike. It is asympotically linear for large S. Its value is in the Black-Scholes model given by the Black-Scholes formula

$$C(S,t) = SN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2), \tag{6.78}$$

which is related to the put value (6.5) by *put-call parity*,

$$C(S,t) - P(S,t) = S - K\mathrm{e}^{-r(T-t)}. \tag{6.79}$$

A binary *cash-or-nothing* call pays 1 if the stock is above the strike at expiry, and 0 otherwise. It has discontinuous payoff

$$V(S,T) = \begin{cases} 1 & \text{if } S > K, \\ 0 & \text{else.} \end{cases} \tag{6.80}$$

The Black-Scholes price of this option is

$$V(S,0) = \mathrm{e}^{-rT}N(d_2), \tag{6.81}$$

with d_2 from (6.7) the same as in the Black-Scholes formula. Payoff and solution are asymptotically constant for large S. At expiry, the solution has a discontinuity at the strike, identical to the delta of the standard European call. The delta of the digital call at expiry is a Dirac distribution, and its gamma therefore the derivative of a Dirac distribution.

6.4.1 Asymptotic boundary condations

In this section, we revisit the question of truncating the compuational range at a large asset value, and the suitability of numerical boundary conditions. This aspect was first taken up in 5.2 with a focus on their impact on stability of numerical schemes. We extend this discussion here to a range of possible payoff functions, and discuss the accuracy of asymptotic values themselves.

Dirichlet conditions

Suitable approximations of the value of the option at the upper boundary of S depend on the specific contract. Put options are straightforward, because they are increasingly likely to expire worthless if the underlying spot price is high, and the value function goes to zero,

$$V(S,t) \to 0 \quad \text{for} \quad S \to \infty.$$

Numerically, we can approximate this by the boundary condition $V(S_{max}, t) \approx 0$ for a large value S_{max}, take this point as largest grid point, S_N, and add $V_N^m = 0$ to finite difference equations for $n = 0, 1, \ldots, N-1$. From 5.2 we know that his approximation is stable, but how accurate is it? It is convenient to take a probabilistic interpretation of V as

$$V(S,t) = \mathbb{E}(\mathrm{e}^{-r(T-t)}(K - S_T)^+ | S_t = S) \leq K\,\mathbb{P}(S_T \leq K | S_t = S). \tag{6.82}$$

For the Black-Scholes model, S_T is log-normal, $\log(S_T/S_t) \sim N((\sigma^2/2 - r)(T-t), \sigma^2(T-t))$, and $\mathbb{P}(S_T \leq K | S_t = S)$ can be estimated by the cumulative standard normal, which shows that the put value $V(S,t)$ goes down very rapidly as $S \to 0$.

The analysis so far uses the specific shape of the payoffs, and the stock price distribution of the Black-Scholes model, but a similar argument can be applied to a wider situation by noting that the majority of option payoffs are asymptotically linear (affine, more precisely),

$$G(S) = aS - b \quad \text{for} \quad S > S_{min},$$

for some constants a and b. By conditioning on hitting the range $S < S_{min}$ at expiry, we can separate out the contribution of a potentially more complex payoff there, via

$$\begin{aligned} V(S,t) &= \mathbb{E}(B(t,T)(G(S_T) - (aS_T - b)) | S_t = S, S_T \leq S_{min})\mathbb{P}(S_T \leq S_{min} | S_t = S) \quad (6.83) \\ &\quad + \underbrace{\mathbb{E}(B(t,T)(aS_T - b) | S_t = S)}_{F(S,t)}, \quad (6.84) \end{aligned}$$

where $B(t,T)$ is the time t value of a bond expiring at T, say $B(t,T) = \exp(r(t-T))$ for constant interest rate r. The point is that F is independent of the model for the stock and explicitly computable, because the expectation is taken under the *risk-neutral* measure under which the discounted stock price is a martingale, $\mathbb{E}(B(t,T)S_T | S_t = S) = S$, and the remainder term is the bond price, so e.g. for constant interest rates

$$F(S,t) = aS - b\mathrm{e}^{-r(T-t)}.$$

The first term in (6.83) will be small for large S_{max} and gives an estimate

$$|V(S,t) - F(S,t)| \leq \max_{s \leq S_{min}} |G(s) - (as - b)|\,\mathbb{P}(S_T \leq S_{min} | S_t = S), \tag{6.85}$$

which justifies the numerical boundary condition

$$V_N^m = aS_N - b\mathrm{e}^{-r(T-t_m)}.$$

Put options, $a = b = 0$, vanilla calls, $a = 1$, $b = K$, and digital calls, $a = 0$, $b = 1$, are special cases.

Generic asymptotic conditions

Taking the argument around asymptotic linearity a step further, a slightly more generic boundary condition is based on the observation that often

$$\lim_{S\to\infty} \frac{\partial^2 V}{\partial S^2} = 0,$$

which can be approximated by

$$\frac{\partial^2 V}{\partial S^2}(S_{max}, t) = 0.$$

Approximated by a finite difference at $S_N = S_{max}$,

$$\frac{V_{N+1}^m - 2V_N^m + V_{N-1}^m}{\Delta S^2} = 0. \tag{6.86}$$

Given S_N is assumed the right-most grid point, we have introduced a "ghost point" S_{N+1} outside the computational range. One can obtain a second equation for the unknown value V_{N+1}^m from a finite difference discretisation of the PDE at S_N, e.g. from a one-step finite difference scheme in the form (6.27),

$$a_N^m V_{N-1}^{m-1} + b_N^m V_N^{m-1} + c_N^m V_{N+1}^{m-1} = A_N^m V_{N-1}^m + B_N^m V_N^m + C_N^m V_{N+1}^m,$$

and can then use (6.86) to eliminate V_{N+1}^m and get

$$\bar{a}_N^m V_{N-1}^{m-1} + \bar{b}_N^m V_N^{m-1} = \bar{A}_N^m V_{N-1}^m + \bar{B}_N^m V_N^m$$

with

$$\bar{a}_N^m = a_N^m - c_N^m, \quad \bar{b}_N^m = b_N^m + 2c_N^m, \quad \bar{A}_N = A_N^m - C_N^m, \quad \bar{B}_N^m = B_N^m + 2C_N^m.$$

The question of stability of these boundary conditions arises, and falls outside the analysis of 5.2. To make this a bit more concrete, consider the explicit Euler scheme for the Black-Scholes PDE from Example 6.1.2, evaluated at S_N and incorporating the boundary condition (6.86),

$$\partial_t^- V_N^m = -\left(\frac{1}{2}S_N^2\delta_S^2 + rS_N\delta_S - r\right)V_N^m = -(rS_N\delta_S^- - r)V_N^m,$$

which can be written as

$$V_N^{m-1} = \bar{A}_N V_{N-1}^m + \bar{B}_N V_N^m, \quad \bar{A}_N = -rS_N\Delta t/\Delta S < 0, \ \bar{A}_N = r(1 + S_N\Delta t/\Delta S) > 0. \tag{6.87}$$

The second derivative term disappears, by design, but the application of the boundary condition to the first derivative term implies that we end up using a one-sided difference in the "wrong" direction, down-winding instead of up-winding, resulting in a negative off-diagonal entry, which ultimately destroys monotonicity of the scheme. This does not rule out that the scheme might be stable nonetheless, and it does appear so in practice. An (informative but not entirely conclusive) analysis of this boundary condition for the Black-Scholes PDE using Fourier techniques is found on [Wincliff et al., 2004].

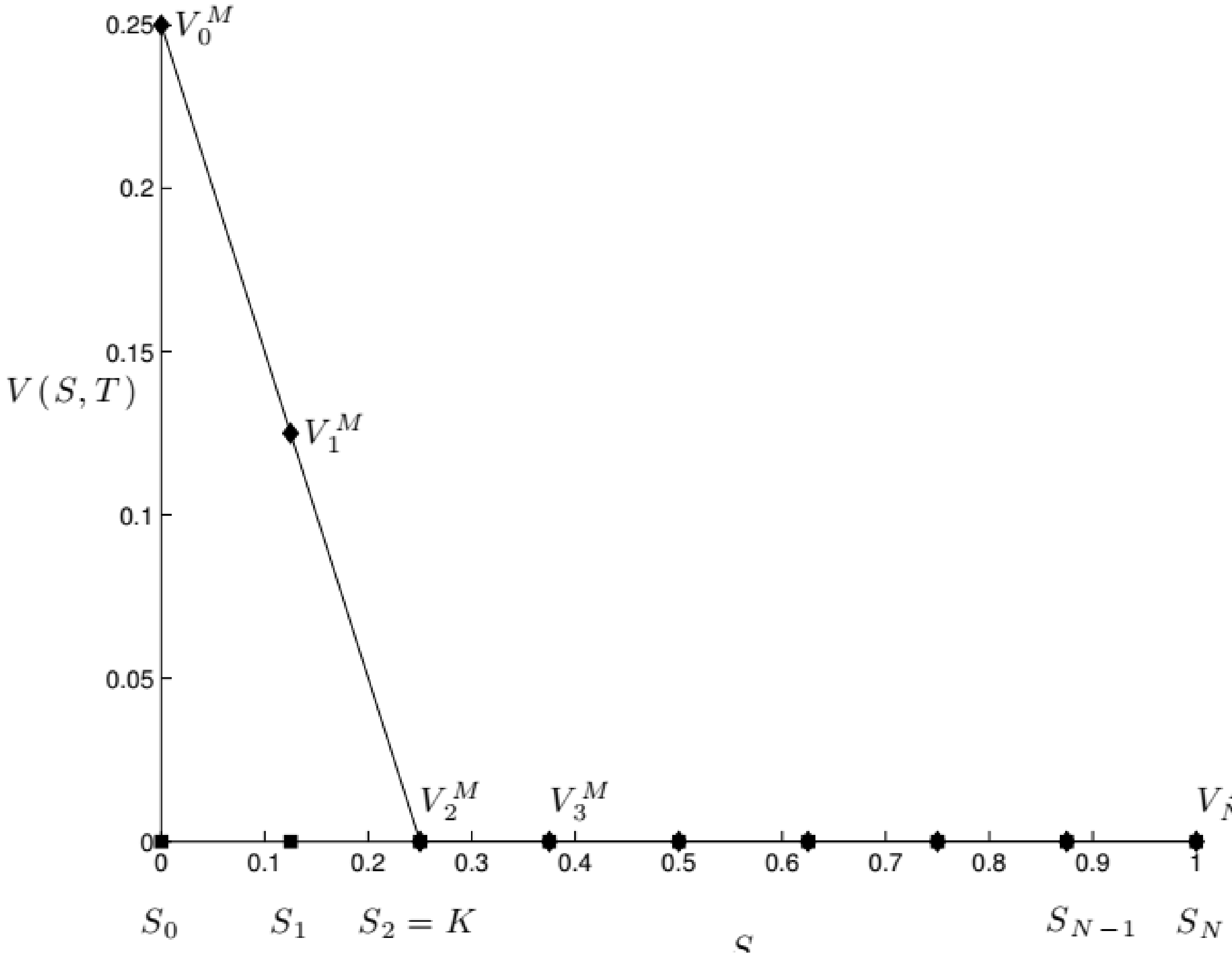

Figure 6.7: Pointwise approximation to the put payoff.

Remark 6.4.1. *The "linear boundary condition" is motivated by the goal to use a generic boundary condition which approximates the problem on the infinite axis well by one on a finite interval. This approximate boundary condition is necessary because the PDE on a finite interval is* not fully specified without boundary conditions. *An idea which sounds to good to be true is therefore the one proposed in [Tavella and Randall, 2000], to "apply the pricing equation itself as a boundary condition". Exercise 6 in 6.5 shows that – inevitably – specific boundary conditions on the solution* are *implied by this approach, although this is not transparent from the outset. There is also evidence in [Wincliff et al., 2004] that this way of applying boundary conditions results in unstable schemes.*