Extensions to the basic scheme

3.1 Random walks with drift

We return to the set-up of 2.1, but now want to include the possibility that the position of the random walk at a time t, X_t, has a given non-zero expectation. For simplicity, consider first the case where the marker drifts, on average, with constant rate μ, such that

$$\mathbb{E}(X_{t+\Delta t} - X_t) = \mu \Delta t. \tag{3.1}$$

We will look at two ways to achieve this: one of making the walk asymmetric by attaching a higher probability to up- or down-moves, hence creating a bias; and one of making the nodes drift themselves. Finally, we bring the two together by a scheme that is motivated by the latter approach, but, by projeting the moves onto a fixed lattice, results in a scheme similar to the former one.

3.1.1 Biased moves

First, we define a random walk again by

$$X_{t_m+\Delta t} = X_{t_m} + Z_m \Delta x,$$

with $X_0 = 0$, $\Delta x > 0$, Z_m i.i.d., as in 2.1, but now allowing unequal probability for the direction of moves,

$$\xi_m = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } q, \\ 0 & \text{with probability } 1-p-q, \end{cases}$$

where $0 \leq p, q$ and $p + q \leq 1$. To match (3.1), one needs

$$\mathbb{E}(X_{\Delta t}) = \mu \Delta t. \tag{3.2}$$

The variance is again (see 2.1.1) normalised by

$$Var(X_{\Delta t}) = \mathbb{E}(X_{\Delta t}^2) - \mathbb{E}(X_{\Delta t})^2 = \Delta t. \tag{3.3}$$

It is left as an exercise to work out the details of the following steps. The above two conditions define p and q as

$$p = \frac{1}{2}\frac{\Delta t}{\Delta x^2} + \frac{1}{2}\mu\frac{\Delta t}{\Delta x} + \frac{1}{2}\mu^2\frac{\Delta t^2}{\Delta x^2}, \quad q = p - \mu\frac{\Delta t}{\Delta x}. \tag{3.4}$$

Note that $0 \le p, q$ and $p + q \le 1$ only if

$$\begin{aligned} |\mu|\Delta x + \mu^2\Delta t &\le 1 \text{ and} \\ \frac{\Delta t}{\Delta x^2} + \mu^2\frac{\Delta t^2}{\Delta x^2} &\le 1, \end{aligned}$$

respectively. The first condition is not critival because Δt and Δx are small, the second one is not much more critical than in the symmetric case $\mu = 0$ because if $\Delta t \le \Delta x^2$, and Δx small, then the second term is also small.

The inductive formula for the probability density $u_n^m\Delta x = \mathbb{P}(X_{m\Delta t} = n\Delta x)$ (see 2.1.2 and 2.1.3) then reads

$$u_n^{m+1} = pu_{n-1}^m + (1 - p - q)u_n^m + qu_{n+1}^m, \tag{3.5}$$

with its continuous limit the drift-diffusion equation

$$\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{3.6}$$

The inital density is again $u(x, 0) = \delta(x)$ and the solution to (3.6) is then given by

$$u_\mu(x, t) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-(x-\mu t)^2/2t}, \tag{3.7}$$

which is seen to be a shift of the drift-less solution u to the heat equation, $u_\mu(x, t) = u(x - \mu t, t)$.

Inserting p and q from (3.4) into (3.5), an explicit finite difference scheme for (3.6) is given by

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} + \mu\frac{u_{n+1}^m - u_{n-1}^m}{2\Delta x} = \frac{1}{2}(1 + \Delta t\mu^2)\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\Delta x^2}. \tag{3.8}$$

Neglecting the Δt term (why is this justified?), this slightly simplifies to

$$\delta_t^+ u_n^m + \mu\delta_x u_n^m = \delta_x^2 u_n^m \tag{3.9}$$

The second term

$$\delta_x u_n^m = \frac{u_{n+1}^m - u_{n-1}^m}{2\Delta x}$$

discretises the first spatial derivative and is a *central first difference*, which is of second order accurate in Δx.

The scheme (3.9) defines a random walk if

$$|\mu|\Delta x \le 1$$

in addition to the standard condition on $\Delta t \le \Delta x^2$. A more subtle point is that even if Δt and Δx are chosen in such a way, this random walk will *not* satisfy (3.2) and (3.3). This does not preclude convergence of the finite difference solution from (3.9) to the exact solution of (3.6). In fact, many of the numerical schemes we will look at later will not exactly track the moments of the underlying random walks, or have an underlying random walk in the first place. What is relevant is that the scheme approximates the PDE in the limit of small Δt and Δx, in a sense to be made precise later.

3.1.2 Moving trees and grids

As an alternative to the previous approach, a deterministic drift can be incorporated in a shift of the nodes of the tree over time. This leads to

$$X_{t_m+\Delta t} = X_{t_m} + \mu \Delta t + \xi_m \Delta x,$$

where as originally $0 \leq p = q \leq 1/2$. The moves around the mean are symmetric. At time $t = t_m = m\Delta t$, the marker moves from $X_t = x$ to $X_{t+\Delta t} = x + \mu\Delta t + \Delta x$ with probability p, to $x + \mu\Delta t - \Delta x$ with equal probability p, and to $x + \mu\Delta t$ with probability $1 - 2p$.

Defining $u_n^m \Delta x = \mathbb{P}(X_{m\Delta t} = n\Delta x + \mu m \Delta t)$,

$$u_n^{m+1} = p u_{n-1}^m + (1-2p) u_n^m + p u_{n+1}^m. \tag{3.10}$$

This finite difference scheme is the same as for the heat equation, as the drift is already incorporated in the coordinate system. One confirms by Taylor expansion that for $\Delta t, \Delta x \to 0$, $\Delta t/\Delta x^2 = 2p$, the limiting PDE is again (3.6), if u_n^m is seen as approximation to $u(n\Delta x + \mu m \Delta t, m\Delta t)$.

Another twist on this is to solve the heat equation *without* drift with finite differences on a *fixed* grid up to time $t = T$, but to evaluate the numerical solution at time T at the point $x - \mu T$ to get an approximation to $u_\mu(x, t)$ which solves (3.6). If $x - \mu T$ is not a grid point, one can take the closest neighbour or interpolate more accurately from its neighbours. This leads to so-called *Lagrangian* schemes discussed in the next section.

3.1.3 Lagrangian coordinates and upwind differencing

Following through the reasoning of the previous sections, we note that the solution to

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$$

is the solution to the heat equation, shifted along a path

$$\xi(t) = x + \mu\,(t - t_m).$$

We have anchored the path to go through x at a time point t_m. Along this coordinate,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\xi(t), t) = \frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x},$$

so integrating in time leads to

$$u(x, t_m) - u(x - \mu\Delta t, t_{m-1}) = \int_{t_{m-1}}^{t_m} \frac{\mathrm{d}}{\mathrm{d}t} u(\xi(t), t)\,\mathrm{d}t = \frac{1}{2} \int_{t_{m-1}}^{t_m} \frac{\partial^2 u}{\partial x^2}(\xi(t), t)\,\mathrm{d}t.$$

Approximating the x-derivative with central differences, using shorthand

$$\delta_x^2 u(x, t) = \frac{u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t)}{\Delta x^2},$$

leads to

$$u(x, t_m) - u(x - \mu\Delta t, t_{m-1}) = \frac{1}{2} \int_{t_{m-1}}^{t_m} \left[\delta_x^2 u(\xi(t), t) + O(\Delta x^2)\right]\,\mathrm{d}t.$$

At this point, a decision has to be made about approximation of the integral. Note that $\xi(t)$ will not coincide with a grid point for most t, but we have the freedom to choose e.g. $\xi(t_m) = x$, and then can approximate the time integral by $\delta_x u(x, t_m)\Delta t$ with an error of $O(\Delta t^2)$. This is effectively the implicit Euler scheme and leads to

$$\frac{u(x,t_m) - u(x-\mu\Delta t, t_{m-1})}{\Delta t} = \frac{1}{2}\delta_x^2 u(x,t_m) + O(\Delta x^2) + O(\Delta t).$$

The problem with practical implementation of the scheme on a fixed grid is that if x, $x - \Delta x$, $x + \Delta x$ are gridpoints, then $x - \mu\Delta t$ still is not. But $u(x - \mu\Delta t, t_{m-1})$ can be reconstructed from grid values at t_{m-1} by interpolation, say Iu^{m-1}.

So if

$$-pu_{n-1}^m + (1+2p)u_n^m - pu_{n+1}^m = u_n^{m-1}$$

is the implicit scheme for the heat equation (i.e. without drift!), then

$$-pu_{n-1}^m + (1+2p)u_n^m - pu_{n+1}^m = Iu^{m-1}(x_n - \mu\Delta t)$$

is the corresponding scheme in Lagrangian coordinates for the equation with drift.

For the interpolation, use e.g. piecewise linear,

$$Iu^{m-1}(x) = \sum_n u_n^{m-1}\widehat{\Phi}_n(x)$$

with “hat functions” $\widehat{\Phi}_n$ defined as

$$\widehat{\Phi}_n = \begin{cases} 0 & x \le x_{n-1} \text{ or } x \ge x_{n+1}, \\ 1 + \frac{x-x_n}{\Delta x} & x_{n-1} \le x \le x_n, \\ 1 - \frac{x-x_n}{\Delta x} & x_n \le x \le x_{n+1}. \end{cases}$$

Then $Iu^{m-1}(x_n) = u_n^{m-1}$ for all n, and Iu^{m-1} is linear in intervals $[x_n, x_{n-1}]$.

The interpolation point $x_n - \mu\Delta t$ lies left of x_n if $\mu > 0$ and right of x_n if $\mu < 0$. It lies in $[x_{n-1}, x_{n+1}] = [x_n - \Delta x, x_n + \Delta x]$ if $\mu\Delta t \le \Delta x$. In this case, the resulting finite difference scheme is equivalent to

$$\begin{aligned} \frac{u_n^{m+1} - u_n^m}{\Delta t} + \mu\frac{u_n^m - u_{n-1}^m}{\Delta x} &= \frac{1}{2}\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\Delta x^2}, \\ \delta_t^+ u_n^m + \mu\delta_x^- u_n^m &= \frac{1}{2}\delta_x^2 u_n^m, \end{aligned}$$

if $\mu \ge 0$, and to

$$\begin{aligned} \frac{u_n^{m+1} - u_n^m}{\Delta t} + \mu\frac{u_{n+1}^m - u_n^m}{\Delta x} &= \frac{1}{2}\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\Delta x^2}, \\ \delta_t^+ u_n^m + \mu\delta_x^+ u_n^m &= \frac{1}{2}\delta_x^2 u_n^m, \end{aligned}$$

if $\mu \le 0$. It is interesting to note that the discretisation of the first derivative in this case uses one-sided differences δ_x^- and δ_x^+ respectively, taking into account the direction of the drift. Because of its physical interpretation of going up against the drift (“wind”), it is traditionally called *upwind* differencing.

For this piecewise linear interpolation, the overall error is of order $O(\Delta x)$ in spite of second order of the finite difference δ_x^2. This can be fixed by higher order interpolation, e.g. cubic splines, at the expense of losing monotonicity.

As an aside, we record the relations

$$\begin{aligned}\delta_x^2 &= \delta_x^+\delta_x^- = \delta_x^-\delta_x^+,\\ \delta_x &= \frac{1}{2}\left(\delta_x^+ + \delta_x^-\right).\end{aligned}$$

3.2 Weighted timestepping schemes and Crank-Nicolson

3.2.1 Combining schemes

We now switch the focus on improving the accuracy of the time discretisations. The explicit and implicit schemes are "one-sided" approxiamtions to the time-derivative from opposite directions, both resulting in first order accuracy. This begs the question whether a combination of the two might give better, possibly second order accurate solutions. We first try alternating the steps, which leaves two possibilities.

If the first step is explicit, and the second implicit,

$$\begin{aligned}\frac{u_n^m - u_n^{m-1}}{\Delta t} &= \frac{1}{2}\frac{u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{\Delta x^2},\\ \frac{u_n^{m+1} - u_n^m}{\Delta t} &= \frac{1}{2}\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\Delta x^2},\end{aligned}$$

the resulting scheme is

$$\delta_t u = \frac{u_n^{m+1} - u_n^{m-1}}{2\Delta t} = \frac{1}{4}\frac{u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{\Delta x^2} + \frac{1}{4}\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\Delta x^2}.$$

The scheme jumps level m altogether, so can leave it out and re-write it with half the step-size,

$$\frac{u_n^m - u_n^{m-1}}{\Delta t} = \frac{1}{4}\frac{u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{\Delta x^2} + \frac{1}{4}\frac{u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}}{\Delta x^2}. \tag{3.11}$$

This is the very popular *Crank-Nicolson scheme*. It is implicit. One would expect that at the very least that the scheme inherits the accuracy from the individual sub-steps, but given its symmetry we hope to get an improvement. It is also clear that if both sub-steps preserve the defining properties of a probability distribution, then the overall scheme will preserve them.

If we switch the order of explicit and implicit step, i.e. the first sub-step is implicit, and the second one explicit,

$$\begin{aligned}\frac{u_n^m - u_n^{m-1}}{\Delta t} &= \frac{1}{2}\frac{u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}}{\Delta x^2},\\ \frac{u_n^{m+1} - u_n^m}{\Delta t} &= \frac{1}{2}\frac{u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}}{\Delta x^2},\end{aligned}$$

the resulting scheme is

$$\frac{u_n^{m+1} - u_n^{m-1}}{2\Delta t} = \frac{1}{2}\frac{u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}}{\Delta x^2}. \tag{3.12}$$

Recall that m here is an odd number by the alternating definition of the scheme, and (3.12) alone does not define the solution. If instead we apply (3.12) in every timestep m, the resulting method is known as the *Leapfrog scheme*. Note that it is only defined for $m > 1$. In the first step, one has to provide an approximation to u^1 from the initial condition u^0. One possibility is to use the explicit Euler scheme. The scheme is then explicit. Unfortunately, we will see later that the scheme is always unstable.

We therefore focus the discussion now on the Crank-Nicolson scheme (3.11).

3.2.2 Method of lines and the θ-scheme

To bring some order into this growing zoo of timestepping schemes, we can think of the problem more systematically by taking as stepping stone the semi-discrete system

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n(t) = \frac{1}{2} \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{\Delta x^2} \tag{3.13}$$

where $n \in \mathbb{Z}$ and $u_n(0) = u_n^0$ a given initial value. We have seen in 2.4, specifically (2.36), that this is the limit of the explicit and implicit finite difference schemes if we let $\Delta t \to 0$. Coming from the continuous equations, we can think of it as letting time continuous, but discretising x. We track the solution continuously along lines (x_n, t), which leads to the name ("vertical") *method of lines*. (A "horizontal" method of lines would fix Δt and consider continuous x.) The structure of these equations is that of a coupled systems of ordinary differential equations. The explicit Euler scheme, implicit Euler scheme, Crank-Nicolson scheme etc are all *timestepping* schemes for this system of ODEs. For simplicity, we omit boundary conditions but one could truncate by setting $u_{-N}(t) = u_N(t) = 0$.

Integrating (3.13) over a time interval $[m\Delta t, (m+1)\Delta t]$,

$$u_n((m+1)\Delta t) - u_n(m\Delta t) = \frac{1}{2} \int_{m\Delta t}^{(m+1)\Delta t} \frac{1}{\Delta x^2} \left[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)\right] \mathrm{d}t.$$

Instead of approximating a derivative, we need to now approximate an integral. The left-sided rectangle rule,

$$\frac{1}{\Delta t} \int_a^b u(t)\,\mathrm{d}t = u(a) + O(\Delta t),$$

leads to the explicit scheme. The right-sided rectangle rule,

$$\frac{1}{\Delta t} \int_a^b u(t)\,\mathrm{d}t = u(b) + O(\Delta t),$$

gives the implicit scheme. The second order accuate trapezium rule,

$$\frac{1}{\Delta t} \int_a^b u(t)\,\mathrm{d}t = \frac{u(a) + u(b)}{2} + O(\Delta t^2),$$

motivates the *Crank-Nicolson* scheme.

Slightly more generally, we can embed these schemes in a family

$$\frac{u_n^m - u_n^{m-1}}{\Delta t} = \frac{\theta}{2} \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\Delta x^2} + \frac{(1-\theta)}{2} \frac{u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{\Delta x^2} \tag{3.14}$$

with a parameter $\theta \in [0, 1]$. This *θ-scheme* has as its special cases, for

- $\theta = 0$ the explicit Euler scheme, for
- $\theta = 1/2$ the Crank-Nicolson scheme, and for
- $\theta = 1$ the implicit Euler scheme.

The explicit scheme has as prime virtue its explicitness and resulting implementational ease, the implicit scheme its unfaltering stability, and the Crank-Nicolson scheme's distinguishing feature is its second order accuracy. None of the schemes with differing θ have any of these advantages and are not of much practical relevance in their own right. We will see later that timesteps with varying θ can be used as sub-steps of useful combined schemes, in a similar spirit to Crank-Nicolson. Moreover, seeing these three examples as special cases of a bigger scheme allows as a unified analysis later on, and to some extent unified implementation.

3.2.3 Implementation and numerical tests

Rearrange the θ-scheme for the heat equation (3.14) into

$$-\frac{\theta\lambda}{2}u_{n+1}^{m} + (1+\theta\lambda)u_n^m - \frac{\theta\lambda}{2}u_{n-1}^{m} = \frac{(1-\theta)\lambda}{2}u_{n+1}^{m-1} + (1-(1-\theta)\lambda)u_n^{m-1} + \frac{(1-\theta)\lambda}{2}u_{n+1}^{m-1}$$

for $-N < n < N$, and add boundary conditions $u_{-N}^m = u_{-N}^m = 0$. If u_{-N}^m and u_{-N}^m are eliminated from the system, this reads in matrix-vector form

$$\underbrace{\begin{pmatrix} b & c & & \dots & 0 \\ a & b & c & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{pmatrix}}_{:=K_1} \begin{pmatrix} u_{-N+1}^m \\ u_{-N+2}^m \\ \vdots \\ u_{N-2}^m \\ u_{N-1}^m \end{pmatrix} = \underbrace{\begin{pmatrix} B & C & & \dots & 0 \\ A & B & C & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & A & B & C \\ 0 & \dots & 0 & A & B \end{pmatrix}}_{:=K_2} \begin{pmatrix} u_{-N+1}^{m-1} \\ u_{-N+2}^{m-1} \\ \vdots \\ u_{N-2}^{m-1} \\ u_{N-1}^{m-1} \end{pmatrix} \tag{3.15}$$

with

$$\begin{aligned} a &= -\theta\frac{\lambda}{2} \\ b &= 1+\theta\lambda \\ c &= -\theta\frac{\lambda}{2} \end{aligned}$$

and

$$\begin{aligned} A &= (1-\theta)\frac{\lambda}{2} \\ B &= 1-(1-\theta)\lambda \\ C &= (1-\theta)\frac{\lambda}{2} \end{aligned}$$

where $\lambda = \Delta t/\Delta x^2$. In each timestep, a linear system $K_1 u^m = K_2 u^{m-1}$ with tridiagonal K_1 has to be solved for u^m, similar to the implicit Euler method. This leads to the following algorithm.

3.3 Forward and backward equations

The upshot of the previous sections is that we can use finite difference schemes to approximate the transition densities of diffusion processes. The question arises if we can deduce further quantities of interest. An important example are expectations of functions of the process, not least because the majority of option pricing equations are of this form.

With the setup as previously, denote for a function G and fixed end time T

$$V_n^m = \mathbb{E}(G(X_T)|X_{t_m} = x_n).$$

One way to compute this is to use the density $U_n^m = \mathbb{P}(X_{t_m} = x_n)$, then

$$\begin{aligned} V_n^m &= \sum_{k\in\mathbb{Z}} G(x_k)\mathbb{P}(X_T = x_k|X_{t_m} = x_n) \\ &= \sum_{k\in\mathbb{Z}} G(x_k)U_{k-n}^{M-m}. \end{aligned} \tag{3.16}$$

Another approach is via

$$\begin{aligned} V_n^m &= \mathbb{E}(G(X_T)|X_{t_m} = x_n) = \sum_{k\in\mathbb{Z}} \mathbb{E}(G(X_T)|X_{t_{m+1}} = x_k)\cdot\mathbb{P}(X_{t_{m+1}} = x_k|X_{t_m} = x_n) \\ &= pV_{n-1}^{m+1} + (1-2p)V_n^{m+1} + pV_{n+1}^{m+1}. \end{aligned} \tag{3.17}$$

In fact, one gets the same formula by using the induction (2.7) for U_n^m into (3.16), which is left as an exercise. The key observation, and the main difference to the scheme for U_n^m is that we do not know any initial value U_n^0, but this is exactly what we want to compute. The one time at which we do know V_n^m is at $t_m = t_M = T$, because then

$$V_n^M = \mathbb{E}(G(X_T)|X_T = x_n) = G(x_n). \tag{3.18}$$

This leads to a backward induction for V_n^m, with m running back from $M-1$ to 0, using the terminal condition (3.18) at t_M.

Given the backward nature of the problem, the continuous-time limit is the *backward heat equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0, \tag{3.19}$$

$$v(x,T) = G(x). \tag{3.20}$$

This follows from an analysis similar to the forward case.

It is well-known that the *backward* heat equation (3.19) with prescribed *initial* data is *ill-posed*, ie the solution does not depend continuously on the data. In the same way, the *forward* heat equation with prescribed *terminal* data is ill-posed. It is impossible to construct a probability distribution at an earlier time from the current distribution. The *backward* heat equation with prescribed *terminal* data however is *well-posed* in the same way the *forward* heat equation with prescribed *initial* data is well-posed. This is easily seen after noticing that time reversal $t \leftrightarrow T-t$ transforms the one into the other.

Remark 3.3.1. *Note that in spite of its resemblence of the implicit scheme for the heat equation, (3.17) is an explicit scheme in the present context. The implicit Euler scheme for the backward heat equation instead reads*

$$\frac{v_n^{m+1} - v_n^m}{\Delta t} + \frac{1}{2}\frac{v_{n+1}^m - 2v_n^m + v_{n-1}^m}{\Delta x^2} = 0,$$

where now v^m is implicitly defined in terms of v^{m+1}.

It is for this reason that the terms "forward" and "backward" Euler scheme are avoided throughout. For the (forward) heat equation, backward Euler is the implicit and therefore more stable scheme, whereas for the backward heat equation one would need the forward scheme. We therefore stick to the terminology "explicit" and "implicit" which is unambiguous.