Ninth week lessons

Quadratic Equations (continued)

(Divided into 3 lectures of 50 minutes each)

Lecture – 25 (50 minutes)

a) Nature of roots of a quadratic equation.
b) Relation between roots and coefficients.
c) Class works

A. Nature of roots of a quadratic equation.

The nature of the roots of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ depends upon the value of the expression $\Delta = b^2 - 4ac$, called the discriminant.
The following cases arise (for a, b, c rational numbers).

Case I : When $\Delta > 0$ and not a perfect square, the roots are unequal and irrational.

Case II When $\Delta > 0$ and a perfect square, the roots are rational and unequal.

Case III when $\Delta = 0$, the roots are rational and equal.
Case IV when $\Delta < 0$, the roots are unequal and imaginary.

Example1:

Write the nature of roots of $2x^2 + 3x - 5 = 0$

Solution:

Comparing the given equation with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = 3$ and $c = -5$.

Now

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (3)^2 - 4.\ 2.\ (-5) \\ &= 9 + 40 \\ &= 49 \\ &= (7)^2 > 0\end{aligned}$$

Here we got the final result as a perfect square and positive. So it indicates that the given equation must give two roots which are rational and unequal.

Example2:

Write the nature of roots of $2x^2 + 3x - 4 = 0$

Solution:

Comparing the given equation with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = 3$ and $c = -4$.

Now

$\Delta = b^2 - 4ac$

$= (3)^2 – 4. 2. (-4)$
$= 9 + 32$
$= 41 > 0.$

Here we got the final result as a non-perfect square and positive. So it indicates that the given equation must give two roots which are irrational and unequal.

Example3:

Write the nature of roots of $2x^2 + 3x + 5 = 0$

Solution:

Comparing the given equation with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = 3$ and $c = 5$.

Now

$\Delta = b^2 - 4ac$
$= (3)^2 – 4. 2. (5)$
$= 9 - 40$
$= - 31< 0.$

Here we got the final result as a negative. So it indicates that the given equation must give two roots which are imaginary and unequal.

Example4:

Write the nature of roots of $2x^2 - 4x + 2 = 0$

Solution:

Comparing the given equation with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = -4$ and $c = 2$.

Now

$\Delta = b^2 - 4ac$
$= (- 4)^2 – 4. 2. (2)$
$= 16 - 16$
$= 0$

Here we got the final result zero. So it indicates that the given equation must give two roots which are rational and equal. That is, the roots are repeated.

Example 5:

If the equation $x^2 + (k+2)x + 2k = 0$ has equal roots, find the value of k.

Solution:

comparing the given equation with $ax^2+bx+c=0$

$a = 1$, $b = k + 2$, $c = 2k$

since the roots are equal;

$b^2 - 4ac = 0$

or $(k + 2)^2 - 4.1.2k = 0$

or $k^2 - 4k + 4 = 0$

or $(k - 2)^2 = 0$

hence $k = 2$.

B. Relation between roots and coefficients.

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad \text{(i)}$$

On solving, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If α and β be two roots of the equation (i) such that

$\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

then sum of the roots

$$= \alpha + \beta$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b}{a}$$

$$= \frac{-(coefficient\ \ of\ \ x)}{coefficient\ \ of\ \ x^2}$$

and the product of the roots

$$= \alpha.\beta$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{c}{a}$$

$$= \frac{constant\ \ term}{coefficient\ of\ \ x^2}$$

Example:
Without finding the actual roots of $2x^2 + 3x - 5 = 0$, find its sum of roots and product of roots.
Solution:

comparing to $ax^2 + bx + c = 0$, we get a = 2, b = 3 and c = -5.
So,

sum of roots = $-\frac{b}{a}$
= -3/ 2.

product of roots = $\frac{c}{a}$
= -5/2

[Note: you can verify the result by actual calculation of roots]

D. class works.

1. Write the nature of roots of the following quadratic equations.
   i) $x^2 - 2x + 5 = 0$ ii) $x^2 - 2x - 3 = 0$
   ii) $x^2 - 2x - 5 = 0$ iv) $x^2 - 2x + 1 = 0$
2. Find the sum and product of roots of $3x^2 - 2x - 3 = 0$
3. Calculate the sum and product of roots of $x^2 - 2x - 5 = 0$

Lecture – 26 (50 minutes)

a) Formation of a quadratic equations.
b) Quadratic inequalities and their solution sets.
c) Class works

A. Formation of quadratic equations.

Suppose $ax^2 + bx + c = 0$ ($a \neq 0$) is the required quadratic equation and α and β are the given roots.

Then the required equation may be written as

x^2 - (sum of the roots)x + product of the roots = 0

Thus a quadratic equation can be formed when sum and product of the roots are known.

Example1:

Form a quadratic equation whose roots are 2 and 3.

Solution:

sum of roots = 2 + 3 = 5

product of roots = 2.3 = 6

so

required quadratic equation is

x^2 -(sum of roots)x + product of roots = 0

or x^2 - 5x + 6 = 0.

Example 2:

Form a quadratic equation whose roots are twice the roots of $3x^2 + 5x + 1 = 0$.

Solution:

comparing the given equation with $ax^2 + bx + c = 0$, we get a = 3, b = 5 and c = 1.

suppose α and β are the roots of the given equation. Then

$$\text{sum of roots}(\alpha + \beta) = -\frac{b}{a}$$
$$= -5/3.$$

$$\text{product of roots}(\alpha.\beta) = \frac{c}{a}$$
$$= -1/3$$

According to the question, the roots of the new equation(i.e. required equation) are 2α and 2β.

For this

$$\text{sum of roots} = 2\alpha + 2\beta$$
$$= 2(\alpha + \beta)$$
$$= 2.\ (-5/3)$$
$$= -10/3$$

$$\text{product of roots} = 2\alpha.2\beta$$
$$= 4.\ \alpha\beta$$
$$= 4.\ (-1/3)$$
$$= -4/3.$$

Hence the required equation is

x^2 – (sum of roots)x + product of roots = 0.

or $x^2 - (-10/3)x + (-4/3) = 0$

or $3x^2 + 10x - 4 = 0$.

B. Quadratic inequalities and their solution sets.

If we have a quadratic inequality, we can often solve it if the algebraic expression can be rewritten in factorial form. The following example may make the concept clear.

Example:

Solve the inequality $x^2 - 3x + 2 \leq 0$.

Solution:

we have $x^2 - 3x + 2 \leq 0$.

or $(x-2)(x-1) \leq 0$.

From this we can say that the number line is divided into three parts cutting at $x = 1$ and $x = 2$. So the possible inequalities are $x<1$, $1<x<2$ and $x>2$. One or two of them is the required solution of the given inequality. We need to add equal sign to the required solution because there is $\leq$ sign which contains equality also.

For $x < 1$, the value of $x - 2$ is always –ve and $x-1$ is also –ve. ... ... [a]

So, the product of $x - 2$ and $x - 1$ is +ve

For $1<x<2$, the value of $x - 2$ is always –ve and $x-1$ is +ve. ... ... [b]

So, the product of $x - 2$ and $x - 1$ is -ve

For $x>2$, the value of $x - 2$ is always +ve and $x-1$ is also +ve. ... ... [c]

So, the product of $x - 2$ and $x - 1$ is +ve

Since we need the product negative or equal to zero [$(x-2)(x-1) \leq 0$]; the required solution is of type [b] with equality.

Hence the required solution is $1 \leq x \leq 2$.

This can be summarized int he following table:

	$x<1$	$1<x<2$	$x>2$
$x-2$	-ve	-ve	+ve
$x-1$	-ve	+ve	+ve
$(x-1)(x-2)$	+ve	-ve	+ve

For $(x-1)(x-2)<0$, the middle column gives the solution area.

Hence for $(x-1)(x-2)\leq 0$, the solution set is $1 \leq x \leq 2$.

C. Class works.

1. Form a quadratic equation whose roots are squares of the roots of $x^2 - 2x - 3 = 0$
2. Form a quadratic equation whose roots are thrice the roots of $x^2 - 2x - 5 = 0$
3. Solve the inequalities given below.

   i) $x^2 - 3x + 2 \leq 0$ ii) $x^2 - 3x - 4 \geq 0$ iii) $2x^2 - 3x - 2 \leq 0$

Lecture – 27 (50 minutes)

a) Exercise of the problems related to quadratic equations.
b) Assignment

A. Exercise of the problems related to quadratic equations.

1. Determine the nature of the roots of the equations

   i) $x^2 - 4x + 3 = 0$ ii) $2x^2 - 6x + 7 = 0$

   iii) $x^2 - 4x + 1$

2. For what values of a will the equation $x^2 - (3a - 1)x + 2(a^2 - 1) = 0$ have equal roots?

3. If the roots of the quadratic equation $9x^2 - 6x + k = 0$ are equal find the value of k.

4. Find the value of k so that the equation $2x^2 + kx - 15 = 0$ has one root 3.
5. Form a quadratic equation whose roots are the reciprocals of the roots of $2x^2 + 5x + 4 = 0$.

B. Assignment.

1. Find the value of k so that the equation $3x^2 + kx - 2 = 0$ has roots whose sum is 6.
2. If $2x^2 + (4 - k)x - 17 = 0$ has roots equal but opposite in sign, find the value of k.
3. Find the value of k so that the equation $3x^2 + 7x + 6 - k = 0$ has one root equal to zero.
4. If the roots of $ax^2 + bx + c = 0$ be in the ration 3:4, prove that $12b^2 = 49ac$.
   [Hints: take one root 3α and other 4α. Then using sum of roots = -b/a, find the value of α and put that value in the relation of product of roots.]
5. If one root of the equation $x^2 - px + q = 0$ be twice the other, show that $2p^2 = 9q$.
6. For what values of m, the equation $x^2 - mx + m + 1 = 0$ may have its roots in the ratio 2:3?
7. What are the roots of $ax^2 + bx + c = 0$.
8. If the roots of $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ are equal show that ad = bc.
9. Form the quadratic equation in which roots are 3, -2.
10. Find the quadratic equation whose roots are the reciprocals of the roots of $3x^2 - 5x - 2 = 0$

11. Form the quadratic equation whose roots will be
    a) m times the roots of $x^2 - px + q = 0$.
    b) greater by h than the roots of $x^2 - px + q = 0$.
    c) the reciprocals of the roots of the equation $ax^2 + bx + c = 0$.
12. If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$